Chapter 4

BV functions and parabolic problems: the first characterization

This chapter is entirely devoted to functions of bounded variation and sets of finite perimeter. We have collected several results related to these functions, from the classical ones present in literature to a new characterization of such functions. This chapter is organized as follows: in the first section we recall definitions, basic properties and classical results for functions of bounded variation and sets of finite perimeter.
In the second one we extend classical definitions and properties to functions with possibly weighted bounded variation on Ω and finally, in the last section we give a first characterization for such class of functions in terms of the short-time behavior of $T(t)$.

4.1 The space BV: definitions and preliminary results

First we give a brief introduction to the definition of BV functions in non-weighted Euclidean domains (complete discussions and proofs can be found in [5] and [20]). These are integrable functions whose weak first-order distributional derivatives are finite Radon measures. Throughout this chapter we denote by Ω a generic open set of $\mathbf{R}^n$. The classical integration by parts formula shows that if $f \in C^1(\Omega)$ and $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$, then

$$\int_\Omega f \mathrm{div}\varphi \, dx = -\int_\Omega \varphi \cdot Df \, dx.$$

The definition of Sobolev functions is based upon a generalization of the integration by parts formula. A locally summable function $g : \Omega \mapsto \mathbf{R}^n$ is called a weak derivative of f

if for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^n)$,

$$\int_\Omega f \mathrm{div}\varphi \, dx = -\int_\Omega \varphi \cdot g \, dx.$$

If $|g|$ is integrable, then f belongs to the Sobolev space $W^{1,1}(\Omega)$.

Definition 4.1.1. *Let $f \in L^1(\Omega)$; we say that f is a* function of bounded variation in Ω *if there exists a vector-valued Radon measure $\mu_f = (\mu_f^1, \dots, \mu_f^n)$ on Ω with $|\mu_f|(\Omega)$ finite such that for all $\varphi \in C_c^\infty(\Omega, \mathbf{R}^n)$,*

$$\int_\Omega f \mathrm{div}\varphi dx = -\int_\Omega \varphi \cdot d\mu_f = -\sum_{i=1}^n \int_\Omega \varphi_i d\mu_f^i(x). \tag{4.1}$$

The vector space of all functions of bounded variation is denoted by $BV(\Omega)$.

By (4.1) it follows that a BV function f belongs to the Sobolev space $W^{1,1}(\Omega)$ if and only if μ_f is absolutely continuous with respect to the Lebesgue measure on Ω. In this case $\mu_f = \nabla f dx$ (see [20, Sec 5.1]), where ∇f denotes the density of μ_f with respect to dx provided by the Besicovitch differentiation Theorem 1.4.10 and coincides with the approximate gradient of u. According to the notation adopted in the Sobolev case we denote by Df the distributional derivative measure μ_f. The following proposition leads to the current working definition for BV functions.

Proposition 4.1.2. *Let $f \in L^1(\Omega)$. Then $f \in BV(\Omega)$ if and only if*

$$|Df|(\Omega) = \sup\{\int_\Omega f \mathrm{div}\varphi \, dx : \ \varphi \in C_c^1(\Omega, \mathbf{R}^n), \|\varphi\|_{L^\infty(\Omega)} \leq 1\} < \infty.$$

The space BV is a Banach space if endowed with the norm

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + |\mu_f|(\Omega) \tag{4.2}$$

but the norm-topology is too strong for many applications. Indeed, continuously differentiable functions are not dense in $BV(\Omega)$. For example let $\Omega := \mathbf{R}$, $f := \chi_{(1,2)} \in L^1(\mathbf{R})$ and consider $\{f_k\}$ a sequence of smooth functions obtained by convolution. Then f_k does not converge to f with respect to the norm (4.2). In fact Df_k is absolutely continuous with respect the Lebesgue measure whereas Df is singular with respect the Lebesgue measure, being $Df = \delta_1 - \delta_2$ a measure concentrated on two points. Therefore

$$|Df_k - Df|(\Omega) = |Df_k|(\Omega) + |Df|(\Omega) \geq |Df|(\Omega) \geq 1.$$

This is true because $|\lambda - \mu| = |\lambda| + |\mu|$ for mutually singular measures λ, μ.

An important application of BV function theory is the study of sets of finite perimeter introduced by R. Caccioppoli in [10]; a detailed analysis of these sets was carried on by E. De Giorgi (see [16]) and H. Federer (see [21] and the references there).

4.1.1 Sets of finite perimeter

Given a subset $E \subset \mathbf{R}^n$, we denote by $|E|$ its Lebesgue measure, and by $\mathcal{H}^{n-1}(E)$ its $(n-1)$-dimensional Hausdorff measure.

Definition 4.1.3. *Let E be a measurable subset of $\mathbf{R}^n$. The perimeter of E in Ω is the variation of χ_E in Ω, i.e.*

$$\mathcal{P}(E,\Omega) = \sup\left\{ \int_{\Omega\cap E} \mathrm{div}\varphi\, dx : \ \varphi \in C^1_c(\Omega,\mathbf{R}^n), \|\varphi\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{4.3}$$

We say that E is a set of finite perimeter in Ω if $\mathcal{P}(E,\Omega) < \infty$.

When $\Omega = \mathbf{R}^n$, $\mathcal{P}(E,\mathbf{R}^n)$ will be simply denoted by $\mathcal{P}(E)$. The class of sets of finite perimeter in Ω contains all sets E with C^1 boundary inside Ω such that $\mathcal{H}^{n-1}(\Omega\cap\partial E) < \infty$. Indeed, by the Gauss-Green theorem, for these sets E we have

$$\int_E \mathrm{div}\varphi\, dx = -\int_{\partial E} \langle \varphi, \nu_E\rangle d\mathcal{H}^{n-1} \qquad \forall \varphi \in C^1_c(\Omega,\mathbf{R}^n) \tag{4.4}$$

where ν_E is the inner unit normal to E. Using this formula the supremum in (4.3) can be easily computed and it turns out that $\mathcal{P}(E,\Omega) = \mathcal{H}^{n-1}(\Omega\cap\partial E)$

The theory of sets of finite perimeter is closely connected to the theory of BV functions. First of all we notice that if $E \subset \mathbf{R}^n$ has finite measure in Ω, that is $\chi_E \in L^1(\Omega)$, then by Proposition 4.1.2, E has finite perimeter in Ω if and only if the characteristic function χ_E belongs to $BV(\Omega)$; in this case $\mathcal{P}(E,\Omega)$ coincides with $|D\chi_E|(\Omega)$, the total variation in Ω of the distributional derivative of χ_E.
The variational measure $D\chi_E$ can be used to define a measure theoretic boundary denoted by $\mathcal{F}E$ and called *reduced boundary* of E, defined as follows.

Definition 4.1.4. *(Reduced boundary) Let E be a measurable subset of $\mathbf{R}^n$ with finite perimeter in Ω. We define*

$$\mathcal{F}E = \left\{ x \in \mathrm{supp}\,|D\chi_E| \cap \Omega : \exists \lim_{\varrho\to 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} = \nu_E(x), \ \textit{and}\ |\nu_E(x)| = 1 \right\}. \tag{4.5}$$

The function $\nu_E : \mathcal{F}E \to \mathbf{S}^{n-1}$ is called the generalized inner normal *to E,*

By the Besicovitch differentiation theorem (see Theorem 1.4.10) we know that $|D\chi_E|$ is concentrated on $\mathcal{F}E$ and $D\chi_E = \nu_E|D\chi_E|$. De Giorgi proved that $\mathcal{F}E\cap\Omega$ is a countably $(n-1)$- rectifiable set (i.e. $\mathcal{F}E = \bigcup_{h\in\mathbf{N}} K_h \cup N_0$ with $\mathcal{H}^{n-1}(N_0) = 0$ and K_h compact subsets of C^1 manifolds M_h, see Definition 1.4.14) and that

$$D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \mathcal{F}E. \tag{4.6}$$

These results imply that the classical Gauss-Green formula can be rewritten for sets of finite perimeter in Ω in the form

$$\int_{E\cap\Omega} \mathrm{div}\varphi\, dx = -\int_{\mathcal{F}E} \langle \varphi, \nu_E\rangle d\mathcal{H}^{n-1} \qquad \forall \varphi \in C^1_c(\Omega,\mathbf{R}^n). \tag{4.7}$$

Observe that in (4.7) the inner normal and the boundary have to be thought in a measure theoretic sense and not in the topological one.
Another important result due to De Giorgi is a blow-up property for points of the reduced boundary (see [16] for the original reference).

Theorem 4.1.5. *(De Giorgi) For any $x \in \mathcal{F}E$ the following properties hold*

(i) the sets $E^x_\rho = (E-x)/\rho$ locally converge in measure in $\mathbf{R}^n$ to the half space $H_{\nu_E(x)}$ orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$ as $\rho \to 0^+$

$$H_{\nu_E(x)} = \{y \in \mathbf{R}^n : \ \langle \nu_E(x), y-x\rangle \geq 0\};$$

(ii) $\mathcal{L}^n \llcorner E^x_\rho \xrightarrow{w^*_{loc}} \mathcal{L}^n \llcorner H_{\nu_E(x)}$ as $\rho \to 0^+$, i.e.

$$\lim_{\rho\to 0^+} \int_{\Omega\cap E^x_\rho} \phi(y)dy = \int_{H_{\nu_E(x)}} \phi(y)dy \quad \forall \phi \in C_c(\mathbf{R}^n).$$

Now we examine the density properties of sets of finite perimeter.

Definition 4.1.6. *Let E be a measurable subset of $\mathbf{R}^n$. For every $\alpha \in [0,1]$ we denote by E^α the set of points of $\mathbf{R}^n$ where E has density α, that is*

$$E^\alpha = \left\{ x \in \mathbf{R}^n : \exists \lim_{\varrho\to 0} \frac{|E\cap B_\varrho(x)|}{|B_\varrho(x)|} = \alpha \right\}; \tag{4.8}$$

The essential boundary is then defined as $\partial^* E = \mathbf{R}^n \setminus (E^0 \cup E^1)$, i.e., the set of points where the density of E is neither 0 nor 1.

Theorem 4.1.7. *(Federer) Let E be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{1/2} \subset \partial^* E \quad \textit{and} \quad \mathcal{H}^{n-1}(\Omega \setminus (E^0 \cup \mathcal{F}E \cup E^1)) = 0$$

In particular, $\mathcal{H}^{n-1}$- a.e. $x \in \partial^* E \cap \Omega$ belongs to $\mathcal{F}E$.

4.2 Weighted BV functions

A natural way to extend the definition of functions of bounded variation in the weighted Euclidean case on Ω is described here. Given a symmetric positive definite matrix $P = (p_{ij})^n_{i,j=1}$, and a function $f \in L^1(\Omega)$, we define the weighted total variation, by setting

$$|Df|_P(\Omega) = \sup\left\{ \int_\Omega f \mathrm{div}\psi dx : \psi \in C^1_c(\Omega, \mathbf{R}^n), \|P^{-1/2}\psi\|_\infty \leq 1 \right\} \tag{4.9}$$

and say that f *has finite total weighted variation*, if $|Df|_P(\Omega) < +\infty$. Thus, as in the classical case we denote by BV_P as the space of L^1 functions that have finite weighted total variation. Notice that if P has entries $p_{ij} \in C^1(\Omega)$, then the total variation can be equivalently defined by

$$|Du|_P(\Omega) = \sup\left\{\int_\Omega u\mathrm{div}(P^{1/2}\phi)dx : \phi \in C_c^1(\Omega, \mathbf{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1\right\}.$$

Of course, if P is the identity matrix then $|Df|_P$ reduces to the classical definition of total variation for an L^1 function and in this case we write $f \in BV(\Omega)$ and drop the P everywhere. The space $BV_P(\Omega)$ turns out to be a Banach space with the norm

$$\|f\|_{BV_P} = \|f\|_{L^1(\Omega)} + |Df|_P(\Omega).$$

In a similar way, a set E is said to have finite weighted perimeter if $|D\chi_E|_P(\Omega) < +\infty$. In this case, its total variation measure is the perimeter of E and it is denoted also by $\mathcal{P}_P(E, \Omega) = |D\chi_E|_P(\Omega)$.

Henceforth, we assume that P is a symmetric μ elliptic matrix i.e., there exists $\mu \geq 1$ such that $\mu^{-1}|\xi|^2 \leq \langle P(x)\xi, \xi\rangle \leq \mu|\xi|^2$ for all $\xi \in \mathbf{R}^n$ and all $x \in \Omega$. We also assume that the coefficients $p_{ij} \in C_b(\overline{\Omega})$, then, the seminorms $|Df|(\Omega)$ and $|Df|_P(\Omega)$ are equivalent, more precisely

$$\frac{1}{\sqrt{\mu}}|Df|(\Omega) \leq |Df|_P(\Omega) \leq \sqrt{\mu}|Df|(\Omega),$$

where μ is the ellipticity constant of P and this immediately implies that $BV(\Omega) = BV_P(\Omega)$ with equivalence of the norms.

We also notice that if f is regular, then the equality

$$|Df|_P(\Omega) = \int_\Omega |Df(x)|_P dx$$

holds, where $|Df(x)|_P = |P^{1/2}Df(x)| = \langle PDf(x), Df(x)\rangle^{1/2}$.

Remark 4.2.1. *(Lower semicontinuity of the total variation)* It is useful to notice that $|D\cdot|_P(\Omega)$ is lower semicontinuous with respect to the convergence in $L^1_{\mathrm{loc}}(\Omega)$. Indeed for any $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_\infty \leq 1$ the integral $\displaystyle\int_\Omega f\mathrm{div}\varphi\, dx$ is continuous with respect to the L^1-norm of f, hence $|Df|_P$, as the supremum of continuous functionals, is lower semicontinuous.

As in the unweighted case, the norm topology is in some respects too strong, since for instance smooth functions are not dense with respect to it. Nevertheless, a classical weaker approximation result is given by the Anzellotti-Giaquinta theorem, see e.g. [5, Theorem 3.9]. It states that for every $f \in BV(\Omega)$ there exists a sequence of functions $(f_k)_k \subset C^\infty(\Omega) \cap BV(\Omega)$ such that

$$\|f - f_k\|_{L^1(\Omega)} \to 0, \quad \int_\Omega |Df_k|dx \to |Df|(\Omega);$$

Such a sequence is said to converge *in variation* to f.
The Anzellotti-Giaquinta theorem can be adapted also to the case of weighted BV functions as follows: given a matrix Q, we define

$$C_Q(\Omega) = \left\{ f \in C^\infty(\Omega) \cap C^1(\overline{\Omega}); \langle QDf, \nu \rangle = 0 \text{ on } \partial\Omega \right\}, \tag{4.10}$$

and the following approximation result holds. We point out that we shall use this proposition in order to approximate a function in $BV(\Omega)$ with functions in the domain of A_1 which verify a condition on $\partial\Omega$.

Proposition 4.2.2. *Let Ω, $P = (p_{ij})_{i,j=1}^n$ be as above, and let $Q = (q_{ij})_{i,j=1}^n$ be an elliptic matrix with $q_{ij} \in C_b^1(\overline{\Omega})$. Then, for every $f \in BV_P(\Omega)$ there exists a sequence of functions $(f_k)_k \subset C_Q(\Omega)$ such that*

$$\lim_{k\to\infty} \|f - f_k\|_{L^1(\Omega)} = 0, \quad \lim_{k\to\infty} \int_\Omega |Df_k|_P dx = |Df|_P(\Omega).$$

Proof. The proof goes as the classical one, except that we have to modify the usual approximation sequence in a neighborhood of the boundary of Ω.
Fix $\varepsilon > 0$; since $f \in BV(\Omega)$, there exist functions $\{f_k\}_k \in C^\infty(\Omega) \cap BV(\Omega)$ such that

$$f_k \to f \text{ in } L^1(\Omega)$$

$$\int_\Omega |Df_k|\, dx \to |Df|(\Omega) \text{ as } k \to \infty.$$

We can find $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ the set $\Omega^\delta = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \delta\}$ satisfies

$$\|f_k\|_{L^1(\Omega\setminus\Omega^\delta)} \leq \varepsilon, \qquad \int_{\Omega\setminus\Omega^\delta} |\nabla f_k| dx \leq \varepsilon \qquad \forall k \in \mathbf{N}. \tag{4.11}$$

The assumption on the regularity on $\partial\Omega$ is used to modify the approximating sequence to make it constant in the direction $Q\nu$. Indeed, for every $x \in \Omega \setminus \Omega^\delta$ there is the projection on $\partial\Omega^\delta$, say $P_Q(x)$, such that x may be written $x = (1-t)P_Q(x) + \delta t Q(P_Q(x))\nu(P_Q(x))$ for some $t \in [0,1)$ ($\nu(y)$ is the outer normal to $\partial\Omega^\delta$ in y). This is possible since the map $\psi : \partial\Omega^\delta \times [0,\varepsilon) \to \Omega$, $\psi(y,t) = y + tQ(y)\nu(y)$ defines, for sufficiently small $\varepsilon > 0$, a diffeomorphism on its image, and then we can define $P_Q(x) = \pi_1(\psi^{-1}(x))$ for any $x \in \psi(\partial\Omega^\delta \times [0,\varepsilon))$, where $\pi_1 : \partial\Omega^\delta \times [0,\varepsilon) \to \partial\Omega^\delta$ is given by $\pi_1(y,t) = y$.
Let us modify the functions f_k in the following way

$$\tilde{f}_k(x) := \begin{cases} f_k(P_Q(x)) & x \in \Omega \setminus \Omega^\delta \\ f_k(x) & x \in \Omega^\delta. \end{cases}$$

Then, choosing δ sufficiently small, we have that

$$\Big| \int_\Omega |\nabla \tilde{f}_k| dx - \int_\Omega |\nabla f_k| dx \Big| \leq \varepsilon \tag{4.12}$$

Finally, for every $\tau < \frac{\delta}{2}$ we can define the approximants as follows

$$g_k^\tau(x) := \begin{cases} f_k(P_Q(x)) & x \in \Omega \setminus \Omega^{\frac{\delta}{2}} \\ (\rho_\tau * \tilde{f}_k)(x) & x \in \Omega^{\frac{\delta}{2}} \setminus \Omega^{\frac{3}{2}\delta} \\ f_k(x) & x \in \Omega^{\frac{3}{2}\delta}. \end{cases}$$

where ρ_τ is the standard mollifier. Then $g_k^\tau \in C_c^\infty(\overline{\Omega})$, $(\nabla g_k^\tau, Q\nu) = 0$ in $\partial\Omega \quad \forall k \in \mathbf{N}$. Finally with a standard procedure of diagonalization we can find a sequence $\{g_k^{\tau(k)}\} \subset \{g_k^\tau\}$ such that

$$\lim_{k\to\infty} \|g_k^{\tau(k)} - f\|_{L^1(\Omega)} \le 3\,\varepsilon\,, \qquad \Big|\int_\Omega |\nabla g_k^{\tau(k)}| dx - |Df|(\Omega)\Big| \le 3\,\varepsilon\,.$$

Now, let $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \le 1$. Then taking into account (4.11) and (4.12) we have

$$\begin{aligned}
\int_\Omega g_k^{\tau(k)} \mathrm{div}\varphi\; dx &= \int_{\Omega\setminus\Omega^{\frac{\delta}{2}}} f_k(P_Q(x))\mathrm{div}\varphi\; dx + \int_{\Omega^{\frac{\delta}{2}}\setminus\Omega^{\frac{3}{2}\delta}} (\rho_{\tau_k} * \tilde{f}_k)\mathrm{div}\varphi\; dx \\
&\quad + \int_{\Omega^{\frac{3}{2}\delta}} f_k \mathrm{div}\varphi\; dx \\
&\le 2\varepsilon\|\varphi\|_{W^{1,\infty}} + \int_{\Omega^{\frac{3}{2}\delta}} (f_k - f)\mathrm{div}\varphi\; dx + \int_\Omega f\mathrm{div}\varphi\; dx \\
&\quad - \int_{\Omega\setminus\Omega^{\frac{3}{2}\delta}} f\mathrm{div}\varphi\; dx \\
&\le 3\varepsilon\|\varphi\|_{W^{1,\infty}} + |Df|_P(\Omega)
\end{aligned}$$

and so

$$\int_\Omega |Dg_k^{\tau(k)}|_P\; dx \le |Df|_P(\Omega) + 3\varepsilon\|\varphi\|_{W^{1,\infty}}.$$

This estimate and Remark 4.2.1 complete the proof. ☐

Remark 4.2.3. A particular case of Proposition 4.2.2 is given when $Q = A$; in this case we have that $C_A(\Omega) \subset D(A_1)$ (it is a core for A_1, i.e. it is dense in $D(A_1)$ for the graph norm $\|\cdot\|_{L^1(\Omega)} + \|A_1\cdot\|_{L^1(\Omega)}$), and then the weighted BV functions can be approximated in variation via functions in the domain of the operator A_1.

There are several other useful properties connecting BV functions to sets of finite perimeter such as the coarea formula. Next we state a weighted version of it, a particular case of (see [13, Lemma 2.4]). We relate the weighted variation measure of f and the weighted perimeter of its level sets.
For $f : \Omega \to \mathbf{R}$ and $t \in \mathbf{R}$, define

$$E_t = \{x \in \Omega : \;\; f(x) > t\}.$$

Lemma 4.2.4. *If $f \in BV(\Omega)$, the mapping*

$$t \in \mathbf{R} \mapsto \mathcal{P}_P(E_t, \Omega)$$

is $\mathcal{L}^1$-measurable.

Proof. Since $f \in L^1(\Omega)$, the mapping $(x, t) \mapsto \chi_{E_t}(x)$ is $\mathcal{L}^n \times \mathcal{L}^1$- measurable, and thus, for each $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$, the function

$$t \mapsto \int_\Omega \chi_{E_t} \mathrm{div}\varphi\; dx$$

is $\mathcal{L}^1$-measurable. Let D denote any countable dense subset of $C_c^1(\Omega, \mathbf{R}^n)$. Then

$$t \mapsto \mathcal{P}(E_t, \Omega) = \sup\left\{ \int_{E_t} \mathrm{div}\varphi \; dx; \; \varphi \in D, |\varphi| \le 1 \right\}$$

is $\mathcal{L}^1$-measurable since it is the supremum of a countable family of measurable functions.

□

Theorem 4.2.5. *Let $f \in BV(\Omega)$. Then E_t has finite perimeter for $\mathcal{L}^1$ a.e. $t \in \mathbf{R}$ and*

$$|Df|_P(\Omega) = \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt. \tag{4.13}$$

Conversely, if $f \in L^1(\Omega)$ and

$$\int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt < \infty$$

then $f \in BV(\Omega)$.

Proof. Let $\varphi \in C_c^1(\Omega, \mathbf{R}^n)$, $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \le 1$. Then

$$\int_\Omega f \mathrm{div}\varphi \; dx = \int_{\mathbf{R}} \Big(\int_{E_t} \mathrm{div}\varphi \; dx \Big) \; dt. \tag{4.14}$$

Indeed, suppose first $f \ge 0$, so that

$$f(x) = \int_0^\infty \chi_{E_t}(x) \; dt \qquad \text{a.e. } x \in \Omega.$$

Thus

$$\begin{aligned} \int_\Omega f \mathrm{div}\varphi \; dx &= \int_\Omega \Big(\int_0^\infty \chi_{E_t}(x)\, dt \Big) \mathrm{div}\varphi(x)\, dx \\ &= \int_0^\infty \Big(\int_\Omega \chi_{E_t}(x) \mathrm{div}\varphi(x)\, dx \Big) \, dt \\ &= \int_0^\infty \Big(\int_{E_t} \mathrm{div}\varphi \, dx \Big) \, dt. \end{aligned}$$

Similarly if $f \le 0$,

$$f(x) = \int_{-\infty}^0 (\chi_{E_t}(x) - 1)\, dt,$$

whence

$$\begin{aligned} \int_\Omega f \mathrm{div}\varphi \; dx &= \int_\Omega \Big(\int_{-\infty}^0 (\chi_{E_t}(x) - 1)\, dt \Big) \mathrm{div}\varphi(x)\, dx \\ &= \int_{-\infty}^0 \Big(\int_\Omega (\chi_{E_t}(x) - 1) \mathrm{div}\varphi(x)\, dx \Big) \, dt \\ &= \int_{-\infty}^0 \Big(\int_{E_t} \mathrm{div}\varphi \, dx \Big) \, dt. \end{aligned}$$

For the general case, write $f = f^+ - f^-$ and (4.14) is proved. From (4.14) we see that for all φ as above,

$$\int_\Omega f \operatorname{div}\varphi \, dx \leq \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) dt.$$

Hence

$$|Df|_P(\Omega) \leq \int_{\mathbf{R}} \mathcal{P}_P(E_t, \Omega) \, dt. \tag{4.15}$$

Now, we prove that assertion (4.13) holds for all $f \in BV_P(\Omega) \cap C^\infty(\Omega)$.
Let

$$m(t) := \int_{\Omega \setminus E_t} |Df|_P \, dx = \int_{\{f \leq t\}} |Df|_P \, dx.$$

Then the function m is non decreasing, and thus m' exists for a.e. $t \in \mathbf{R}$, with

$$\int_{\mathbf{R}} m'(t) \, dt \leq \int_\Omega |Df|_P \, dx. \tag{4.16}$$

Now, fix $t \in \mathbf{R}$, $r > 0$, and define $\eta : \mathbf{R} \to \mathbf{R}$ this way:

$$\eta(s) = \begin{cases} 0 & \text{if } s \leq t\,, \\ \frac{s-t}{r} & \text{if } t \leq s \leq t + r\,, \\ 1 & \text{if } s \geq t + r. \end{cases}$$

Then

$$\eta'(s) = \begin{cases} \frac{1}{r} & \text{if } t < s < t + r\,, \\ 0 & \text{if } s < t \text{ or } s > t + r. \end{cases}$$

Hence, for all $\varphi \in C^1_c(\Omega, \mathbf{R}^n)$ with $\|P^{-1/2}\varphi\|_{L^\infty(\Omega)} \leq 1$

$$-\int_\Omega \eta(f(x)) \operatorname{div}\varphi \, dx = \int_\Omega \eta'(f(x)) Df \cdot \varphi \, dx = \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \varphi \, dx.$$

Now,

$$\begin{aligned} \frac{m(t+r) - m(t)}{r} &= \frac{1}{r} \left[\int_{\Omega \setminus E_{t+r}} |Df|_P \, dx - \int_{\Omega \setminus E_t} |Df|_P \, dx \right] \\ &= \frac{1}{r} \int_{E_t \setminus E_{t+r}} |Df|_P \, dx \\ &\geq \frac{1}{r} \int_{E_t \setminus E_{t+r}} Df \cdot \varphi \, dx \\ &= -\int_\Omega \eta(f(x)) \operatorname{div}\varphi \, dx \end{aligned}$$

For those t such that $m'(t)$ exists, we then let $r \to 0$:

$$m'(t) \geq -\int_{E_t} \operatorname{div}\varphi \, dx \quad \text{a.e. } t \in \mathbf{R}.$$

Taking the supremum over all φ as above:

$$\mathcal{P}_P(E_t, \Omega) \leq m'(t),$$

and recalling (4.16) we find

$$\int_{\mathbf{R}} \mathcal{P}_P(E_t,\Omega) \leq \int_{\Omega} |Df|_P\, dx = |Df|_P(\Omega).$$

This estimate and (4.15) complete the proof for $f \in BV(\Omega) \cap C^{\infty}(\Omega)$. Finally, fix $f \in BV_P(\Omega)$ and choose $\{f_k\}_{k\in\mathbf{N}}$ as in Proposition 4.2.2. Then

$$f_k \to f \qquad \text{in } L^1(\Omega) \text{ as } k \to \infty.$$

Define

$$E_t^k = \{x \in \Omega : \ f_k(x) > t\}.$$

Now,

$$\int_{\mathbf{R}} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\, dt = \int_{\min\{f(x),f_k(x)\}}^{\max\{f(x),f_k(x)\}} dt = |f_k(x) - f(x)|,$$

consequently

$$\int_{\Omega} |f_k(x) - f(x)|\, dx = \int_{\mathbf{R}} (\int_{\Omega} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\, dx)\, dt.$$

Since $f_k \to f$ in $L^1(\Omega)$, there exists a subsequence which, upon reindexing by k if necessary, satisfies

$$\chi_{E_t^k} \to \chi_{E_t} \quad \text{in } L^1(\Omega), \quad \text{a.e.}\, t \in \mathbf{R}.$$

Then by the lower semicontinuity of the the total variation,

$$\mathcal{P}_P(E_t,\Omega) \leq \liminf_{k\to\infty} \mathcal{P}_P(E_t^k,\Omega).$$

Thus Fatou's Lemma implies

$$\begin{aligned}
\int_{\mathbf{R}} \mathcal{P}_P(E_t,\Omega)\, dt &\leq \liminf_{k\to\infty} \mathcal{P}_P(E_t^k,\Omega) \\
&= \lim_{k\to\infty} |Df_k|_P(\Omega) \\
&= |Df|_P(\Omega)
\end{aligned}$$

This calculation and (4.15) complete the proof. □

Remark 4.2.6. The coarea formula is true for Borel sets. If $f \in BV(\Omega)$ the set E_t has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbf{R}$ and

$$|Df|(B) = \int_{\mathbf{R}} |D\chi_{E_t}|(B)\, dt, \qquad Df(B) = \int_{\mathbf{R}} D\chi_{E_t}(B)\, dt$$

for any Borel set $B \subset \Omega$.

For the weighted total variation also the following continuity property under uniform convergence holds.

Proposition 4.2.7. *Let $P = (p_{ij})_{i,j=1}^n$ be a symmetric μ-elliptic matrix valued function and let $(P_{(k)})_{k\in\mathbf{N}}$ be a sequence of matrices valued functions uniformly convergent to P. Then, for every $f \in L^1(\Omega)$ the following holds:*

$$\lim_{k\to+\infty} |Df|_{P_{(k)}}(\Omega) = |Df|_P(\Omega). \tag{4.17}$$

PROOF. We denote by $c_k = \|P^{-1/2} - P_{(k)}^{-1/2}\|_\infty$; by the uniform convergence, we have that $c_k \to 0$ as $k \to +\infty$; moreover, we may assume that the $P_{(k)}$ are $(\mu + 1/k)$-elliptic, that is

$$\frac{1}{\mu + 1/k}|\xi|^2 \leq |P_{(k)}^{1/2}\xi|^2 \leq (\mu + 1/k)|\xi|^2,$$

or, simply defining $w = P_{(k)}^{1/2}\xi$,

$$\frac{1}{\sqrt{\mu + 1/k}}|w| \leq |P_{(k)}^{-1/2}w| \leq \sqrt{\mu + 1/k}|w|.$$

Then, if $\psi \in C_c^1(\Omega, \mathbf{R}^n)$ with $\|P_{(k)}^{-1/2}\psi\|_\infty \leq 1$, we get

$$\begin{aligned}
\|P^{-1/2}\psi\|_\infty &\leq \|P_{(k)}^{-1/2}\psi\|_\infty + \|(P^{-1/2} - P_{(k)}^{-1/2})\psi\|_\infty \\
&\leq \|P_{(k)}^{-1/2}\psi\|_\infty + c_k\|\psi\|_\infty \\
&\leq \|P_{(k)}^{-1/2}\psi\|_\infty + c_k\sqrt{\mu + 1/k}\|P_{(k)}^{-1/2}\psi\|_\infty \\
&\leq 1 + c_k\sqrt{\mu + 1/k}.
\end{aligned}$$

By definition of weighted variation, we get

$$\int_\Omega f \mathrm{div}\psi dx \leq (1 + c_k\sqrt{\mu + 1/k})|Df|_P(\Omega)$$

whence

$$|Df|_{P_{(k)}}(\Omega) \leq (1 + c_k\sqrt{\mu + 1/k})|Df|_P(\Omega).$$

With a similar computation, we also get

$$|Df|_P(\Omega) \leq (1 + c_k\sqrt{\mu})|Df|_{P_{(k)}}(\Omega),$$

and then (4.17) follows by letting $k \to +\infty$. □

4.3 A first characterization of BV functions

In this last section we show some connections between the total variation of a generic function $u_0 \in L^1$ and the short time behavior of the solution of a parabolic problem with initial datum u_0. More precisely we connect the total variation of u_0 to the L^1 norm of the gradient of such solution. This result is strictly linked with the original definition given by E. De Giorgi [15] of functions of bounded variation which is recalled in the following paragraph.

Consider the heat semigroup $(W(t))_{t\geq 0}$ in $\mathbf{R}^n$. We show how it is linked to the definition of function with bounded variation originally given by De Giorgi (see [15]). For a given function $f \in L^1(\mathbf{R}^n)$, we consider the function

$$\begin{aligned} W(t)f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} f(x+\sqrt{2t}y)e^{-\frac{|y|^2}{2}}\,dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} f(y)e^{-\frac{|x-y|^2}{4t}}dy \\ &= (G_t * f)(x) \end{aligned}$$

where $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ denotes the Gauss-Weierstrass kernel. By using simple tools of analysis one can easily prove that $W(t)f(x) \to f(x)$ almost everywhere and also in $L^1(\mathbf{R}^n)$ as $t \to 0^+$. The operator $W(t)$ is also contractive, thus $\|W(t)f\|_{L^1(\mathbf{R}^n)} \leq \|f\|_{L^1(\mathbf{R}^n)}$ for any $f \in L^1(\mathbf{R}^n)$ and any $t > 0$. Moreover, if the function g is regular, then $DW(t)g(x) = W(t)Dg(x)$. Finally, since $W(t+s)f(x) = W(s)W(t)f(x)$, using the previous property for $g(x) = W(t)f(x)$, we get

$$\int_{\mathbf{R}^n} |DW(t+s)f(x)|\,dx = \int_{\mathbf{R}^n} |W(s)(DW(t)f)(x)|\,dx \leq \int_{\mathbf{R}^n} |DW(t)f(x)|\,dx.$$

This computation shows that the function

$$t \mapsto \int_{\mathbf{R}^n} |DW(t)f(x)|\,dx$$

is monotone decreasing for every $f \in L^1(\mathbf{R}^n)$ and then it is well defined the quantity:

$$\mathcal{I}[f] = \lim_{t\to 0} \int_{\mathbf{R}^n} |DW(t)f(x)|\,dx, \tag{4.18}$$

that a priori can be finite or not. De Giorgi called $\mathcal{I}[f]$ the total variation of f in $\mathbf{R}^n$ and he defined the space $BV(\mathbf{R}^n)$ as the space of functions such that $\mathcal{I}[f] < \infty$.

In Theorem 4.3.4 we prove that (4.18) still holds in Ω, when the left hand side reduces to (4.3) and $T(t)$ is the semigroup generated by the second order uniformly elliptic operator $(A_1, D(A_1))$. More in detail we prove that

$$|Du_0|_P(\Omega) = \lim_{t\to 0} \int_\Omega |DT(t)u_0|_P\,dx, \tag{4.19}$$

for every $u_0 \in L^1(\Omega)$, where $|D\cdot|_P(\Omega)$ is defined in (4.9).

Remark 4.3.1. Notice that, since $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^1(\Omega)$, then by the lower semicontinuity of the total variation with respect to the L^1 convergence we obtain

$$|Du_0|_P(\Omega) \leq \liminf_{t\to 0} \int_\Omega |DT(t)u_0|_P dx \tag{4.20}$$

for every u_0 in $L^1(\Omega)$. Therefore in order to prove (4.19) it is sufficient to prove

$$\limsup_{t\to 0} \int_\Omega |DT(t)u_0|_P \, dx \le |Du_0|_P(\Omega).$$

Now observe that, for functions in the domain of the operator A_1, (4.19) is true. Actually for these functions the result is stronger than (4.19), indeed the following equality holds

$$\lim_{t\to 0} \|DT(t)u_0 - Du_0\|_{L^1(\Omega)} = 0.$$

This can be easily seen if we take into account that, by Remark 3.0.6, $D(A_1)$ is continuously embedded in $W^{1,1}(\Omega)$, i.e., there exists $k = k(\Omega, \mu, M_1) > 0$ such that $u_0 \in D(A_1)$ implies $u_0 \in W^{1,1}(\Omega)$ and

$$\|u_0\|_{W^{1,1}(\Omega)} \le k(\|u_0\|_{L^1(\Omega)} + \|A_1 u_0\|_{L^1(\Omega)}); \tag{4.21}$$

Furthermore $T(t)A_1u_0 = A_1T(t)u_0$ and by the strong continuity of $T(t)$ in $L^1(\Omega)$ we get

$$\begin{aligned} \|DT(t)u_0 - Du_0\|_{L^1(\Omega)} &\le k\left(\|T(t)u_0 - u_0\|_{L^1(\Omega)} + \|A_1T(t)u_0 - A_1u_0\|_{L^1(\Omega)}\right) \\ &= k\left(\|T(t)u_0 - u_0\|_{L^1(\Omega)} + \|T(t)A_1u_0 - A_1u_0\|_{L^1(\Omega)}\right). \end{aligned}$$

Example 1. Another simple case in which the existence of the limit as $t \to 0$ of $\int_\Omega |DT(t)u_0(x)| \, dx$ is guaranteed is when Ω is convex and $A = P = I$, $B = c = 0$, i.e., $(T(t))_{t\ge 0}$ is the heat semigroup generated by the Neumann Laplacian and the total variation is the classical (non-weighted) one. In this case, it is easily seen that $F(t) = \|DT(t)u_0\|_{L^1(\Omega)}$ is decreasing (as is the case if $\Omega = \mathbf{R}^n$), *provided that* Ω *is convex.* In fact, in this case computations significantly simplify and go as follows, where we set $u(t,x) = (T(t)u_0)(x)$ and $F(t) = \int_\Omega |Du| \, dx$,

$$\begin{aligned} F'(t) &= \int_\Omega \partial_t |Du| \; dx = \int_\Omega \frac{1}{|Du|} \langle Du, D\partial_t u\rangle \; dx = \int_\Omega \frac{1}{|Du|} \sum_{i,k} D_i u D_i D^2_{kk} u \; dx \\ &= \int_{\partial\Omega} \frac{1}{|Du|} \sum_{i,k} D_i u D^2_{ik} u \nu_k \, d\mathcal{H}^{n-1} - \int_\Omega \sum_{i,k} D_k \frac{D_i u}{|Du|} D^2_{ik} u \; dx \\ &= -\int_{\partial\Omega} \frac{1}{|Du|} \langle D\nu Du, Du\rangle \, d\mathcal{H}^{n-1} + \int_\Omega \frac{1}{|Du|} \left[\left| D^2 u \frac{Du}{|Du|} \right|^2 - \mathrm{Tr}\, (D^2 u)^2 \right] \; dx \le 0 \end{aligned}$$

where we have taken into account the Neumann boundary conditions and the fact that if Ω is convex then all the curvatures (i.e., the eigenvalues of the matrix $D\nu$) are non-negative. This estimate and (4.20) allow us to conclude.
The monotonicity is not true in general also when $\mathcal{A} = \Delta$; if Ω is not convex F may not be non-increasing. In [22, Theorem 2.16] there is an example with Ω non convex and $F'(0) > 0$.

Before stating the main result, we recall an useful boundary trace theorem whose proof can be found in [1, Theorem 5.3.6].

Theorem 4.3.2. *Let Ω be an open subset of $\mathbf{R}^n$ with uniformly C^2 boundary; then the trace operator is continuous from $W^{1,1}(\Omega)$ onto $L^1(\partial\Omega, \mathcal{H}^{n-1})$, that is, there exists $c_\Omega > 0$ such that for every $u \in W^{1,1}(\Omega)$ the trace $v = u_{|\partial\Omega}$ of u on $\partial\Omega$ is well defined and*

$$\|v\|_{L^1(\partial\Omega,\mathcal{H}^{n-1})} \le c_\Omega \|u\|_{W^{1,1}(\Omega)}. \tag{4.22}$$

The following result is a monotonicity estimate for $F(t) = \displaystyle\int_\Omega |DT(t)u_0|dx$ and gives a localized version of (4.19). Here we assume stronger regularity conditions on the coefficient c and recall that

$$M_2 = \max\{\|A\|_{2,\infty}, \|B\|_{2,\infty}, \|c\|_{1,\infty}\}.$$

Without loss of generality, in what follows we take for simplicity the same ellipticity constant μ both for the matrix of the coefficients A of $\mathcal{A}$ and P.

Proposition 4.3.3. *Let $v \in D(A_1)$, where $\mathcal{A}$ is as in* (2.106)-(2.108)*, with coefficients $c \in W^{1,\infty}(\Omega)$. Let $P = (p_{ij})_{i,j=1}^n$ be a non-negative μ-elliptic matrix with $p_{ij} \in W^{1,\infty}(\Omega)$ and $p_{ij} = a_{ij}$ on $\partial\Omega$. Then for every $\eta \in C_b^1(\overline{\Omega})$, η non-negative, there exists a constant*

$$c_5 = c_5(n, \Omega, M_2, \|P\|_{1,\infty}, \|\eta\|_{W^{1,\infty}}, \mu)$$

such that

$$\int_\Omega \eta |DT(t)v|_P \, dx \le \int_\Omega \eta |Dv|_P \, dx + c_5 t^{1-\delta} \|v\|_{W^{1,1}(\Omega)} \tag{4.23}$$

holds for every $t \in (0,1)$, where $\delta \in (1/2, 1)$ is the parameter in (3.20).

Proof. For $v \in D(A_1)$ and $\eta \in C_b^1(\overline{\Omega})$, $\eta \ge 0$, we define the function $F_\eta : (0,1) \to \mathbf{R}$ by

$$F_\eta(t) = \int_\Omega \eta |DT(t)v|_P \, dx.$$

This function is differentiable since $T(t)v$ is regular for every $t > 0$ and the equality

$$\partial_t |DT(t)v|_P = \frac{1}{|DT(t)v|_P} \langle PDT(t)v, D\mathcal{A}T(t)v\rangle$$

holds for a.e. $x \in \Omega$. Moreover, $T(t)v \in D(A_1)$ for every $t > 0$ and then

$$A_1 T(t)v = T\Big(\frac{t}{2}\Big) A_1 T\Big(\frac{t}{2}\Big) v;$$

this implies also that $A_1 T(t)v \in D(A_1)$. Then, thanks to (4.21) and from the fact that

$$\frac{|\langle PDT(t)v, DA_1 T(t)v\rangle|}{|DT(t)v|_P} \le |DA_1 T(t)v|_P,$$

we can differentiate under the integral sign. Denoting by $u(t,x)$ the solution $(T(t)v)(x)$, we obtain

$$
\begin{aligned}
F_\eta'(t) &= \frac{d}{dt}\int_\Omega \eta |Du|_P\,dx = \int_\Omega \frac{\eta}{|Du|_P}\langle PDu, D\mathcal{A}u\rangle\,dx \\
&= \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{p_{ij}D_j u D_i(D_h(a_{hk}D_k u))}{|Du|_P}\,dx \\
&\quad + \sum_{i,j,h=1}^n \int_\Omega \eta \frac{p_{ij}D_j u D_i(b_h D_h u)}{|Du|_P}\,dx + \sum_{i,j=1}^n \int_\Omega \eta \frac{p_{ij}D_j u D_i(cu)}{|Du|_P}\,dx \\
(I_1) \quad &= \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{p_{ij}D_j u(D^2_{ih}a_{hk}D_k u + D_h a_{hk}D^2_{ik}u + D_i a_{hk}D^2_{hk}u)}{|Du|_P}\,dx \\
(I_2) \quad &\quad + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_j u\, a_{hk}D^3_{ihk}u\ dx \\
(I_3) \quad &\quad + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_j u\Big(D_i b_h D_h u + b_h D^2_{ih}u\Big)\,dx \\
(I_4) \quad &\quad + \sum_{i,j,h,k=1}^n \int_\Omega \eta \frac{1}{|Du|_P} p_{ij}D_j u\Big(D_i c\, u + cD_i u\Big)\,dx\,.
\end{aligned}
$$

Notice that there is a constant $k = k(n, M_2, \|\eta\|_{L^\infty}, \|P\|_\infty)$ such that

$$
|I_1| + |I_3| + |I_4| \le k\|u\|_{W^{2,1}(\Omega)}\,.
$$

It remains to estimate I_2; integrating by parts with respect to x_k, we have that

$$
\begin{aligned}
&\sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P} p_{ij}D_j u\, a_{hk}D^3_{ihk}u\,dx \\
(II_1) \quad &= \frac{1}{2}\sum_{i,j,h,k,l,m=1}^n \int_\Omega \frac{\eta}{|Du|^3_P} p_{ij}D_j u\, a_{hk}D^2_{ih}u D_k p_{lm}D_m u D_l u\,dx \\
(II_2) \quad &\quad + \sum_{i,j,h,k,l,m=1}^n \int_\Omega \frac{\eta}{|Du|^3_P} p_{ij}D_j u\, a_{hk}D^2_{ih}u\, p_{lm}D_m u D^2_{kl}u\,dx \\
(II_3) \quad &\quad - \sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P}\Big(D_k p_{ij}D_j u\, a_{hk} + p_{ij}D_j u D_k a_{hk}\Big)D^2_{ih}u\,dx \\
(II_4) \quad &\quad - \sum_{i,j,h,k=1}^n \int_\Omega \frac{\eta}{|Du|_P} p_{ij}D^2_{kj}u\, a_{hk}D^2_{ih}u\,dx \\
(II_5) \quad &\quad - \sum_{i,j,h,k=1}^n \int_\Omega \frac{1}{|Du|_P} p_{ij}D_j u\, a_{hk}D^2_{ih}u\, D_k\eta\,dx \\
(II_6) \quad &\quad + \sum_{i,j,h,k=1}^n \int_{\partial\Omega} \frac{\eta}{|Du|_P} p_{ij}D_j u\, a_{hk}D^2_{ih}u\,\nu_k\,d\mathcal{H}^{n-1}.
\end{aligned}
$$

This implies the existence of a constant $k = k(M_1, \|P\|_{1,\infty}, \|\eta\|_{1,\infty})$, such that

$$|II_1| + |II_3| + |II_5| \leq k \int_\Omega |D^2 u|\, dx\,.$$

where M_1 was so defined

$$M_1 = \max_{i,j}\{\|a_{ij}\|_{W^{2,\infty}(\Omega)}, \|b_i\|_{W^{2,\infty}(\Omega)}, \|c\|_{L^\infty(\Omega)}\}.$$

Notice that for II_2 we have

$$\begin{aligned}\sum_{i,j,k,l,m=1}^n p_{ij} D_j u a_{hk} D^2_{ih} u\, p_{lm} D_m u D^2_{kl} u &= \langle D^2 u\, A\, D^2 u\, P Du, P Du\rangle \\ &= \Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}(P^{1/2} Du), P^{1/2} Du\Big\rangle,\end{aligned}$$

and for II_4 we can write

$$\begin{aligned}\sum_{i,j,h,k=1}^n p_{ij} D^2_{kj} u\, a_{hk} D^2_{ih} u &= \sum_{i,j,h,k,m=1}^n p^{\frac12}_{im} p^{\frac12}_{mj} D^2_{kj} u\, a_{hk} D^2_{ih} u \\ &= \mathrm{Tr}\, \Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}\Big),\end{aligned}$$

where Tr denotes the trace of a matrix. Then

$$\begin{aligned}II_2 + II_4 &= \int_\Omega \frac{1}{|Du|_P}\Big(\Big\langle P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2} \frac{P^{1/2} Du}{|Du|_P}, \frac{P^{1/2} Du}{|Du|_P}\Big\rangle \\ &\quad - \mathrm{Tr}\, \Big(P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}\Big)\Big)\, \eta\, dx \leq 0 \end{aligned} \tag{4.24}$$

since $P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2}$ is positive definite because

$$\Big\langle (P^{1/2} D^2 u\, A\, D^2 u\, P^{1/2})\xi, \xi\Big\rangle = \Big\langle A^{1/2} D^2 u\, P^{1/2}\xi, A^{1/2} D^2 u\, P^{1/2}\xi\Big\rangle.$$

Finally, for the term II_6, we notice that

$$\begin{aligned}&\sum_{i,j,h,k=1}^n p_{ij} D_j u\, a_{hk} D^2_{ih} u\, \nu_k = \sum_{i=1}^n \Big(\sum_{h,k=1}^n a_{hk} D^2_{ih} u\, \nu_k \sum_{j=1}^n p_{ij} D_j u\Big) \\ &= \sum_{i=1}^n \sum_{h,k=1}^n \Big(D_i\Big(a_{hk} D_h u\, \nu_k\Big) - D_h u D_i (a_{hk}\nu_k)\Big) \sum_{j=1}^n p_{ij} D_j u \\ &= \langle D\langle A Du, \nu\rangle, P Du\rangle - \langle D(A\nu) Du, P Du\rangle = -\langle D(A\nu) Du, P Du\rangle\end{aligned} \tag{4.25}$$

since $P \equiv A$ on $\partial\Omega$. Observe that the regularity of the boundary and the ellipticity of a_{ij} imply that there exists a constant $\tilde c$ depending on $\|A\|_{1,\infty}$ and L (see Definition 1.5.1) such that $|D(A\nu)| \leq \tilde c$. As a consequence, we obtain that

$$\begin{aligned}&\Big|\sum_{i,j,h,k=1}^n \int_{\partial\Omega} \frac{1}{|Du|_P} \eta\, p_{ij} D_j u\, a_{hk} D^2_{ih} u\, \nu_k\, d\mathcal{H}^{n-1}\Big| \\ =\;& \Big|\int_{\partial\Omega} \frac{1}{|Du|_P} \eta\, \langle D(A\nu) Du, P Du\rangle\, d\mathcal{H}^{n-1}\Big| \leq k \int_{\partial\Omega} \eta |Du|_P d\mathcal{H}^{n-1} \\ \leq\;& k\|\eta\|_\infty \sqrt{\mu} \int_{\partial\Omega} |Du| d\mathcal{H}^{n-1} \leq k \int_\Omega \big[|Du| + |D^2 u|\big]\, dx\,,\end{aligned}$$

where $k = k(M_2, L, \mu, \|\eta\|_{L^\infty}, c_\Omega)$, and c_Ω is introduced in (4.22).

Taking now into account that $u(t,x)$ satisfies (3.4) and (3.20), we have proved there is a constant c_5 such that for every $t \in (0,1)$ the inequality

$$F'_\eta(t) = \frac{d}{dt}\int_\Omega \eta |Du|_P\, dx \le c_5 t^{-\delta}\|v\|_{W^{1,1}(\Omega)}.$$

holds. Then, by integration (4.23) follows. □

In the following theorem we show the announced characterization of the space $BV(\Omega)$ in terms of the short-time behavior of $\|DT(t)u_0\|_{L^1(\Omega)}$, analogous to (4.18). Here we may relax the regularity assumption on the coefficients b_i according to Remark 3.0.6.

Theorem 4.3.4. *Assume $\Omega \subset \mathbf{R}^n$ has uniformly C^2 boundary. Let $\mathcal{A}$ be as in Section 2.5 with*

$$a_{ij} \in W^{2,\infty}(\Omega), \qquad b_i, c \in L^\infty(\Omega)$$

and P be a non negative μ-elliptic matrix with $p_{ij} \in C_b(\overline{\Omega})$. If $(T(t))_{t\ge 0}$ is the semigroup generated by $(A_1, D(A_1))$ in $L^1(\Omega)$, then, for every $u_0 \in L^1(\Omega)$, the equality

$$\lim_{t\to 0}\int_\Omega |DT(t)u_0(x)|_P\, dx = |Du_0|_P(\Omega)$$

holds. In particular, u_0 belongs to $BV(\Omega)$ if and only if the above limit is finite.

Proof. We start first assuming that $p_{ij} \in C_b^2(\overline{\Omega})$ and considering the operator $\hat{\mathcal{A}} = \mathrm{div}(ADu)$, i.e., $b_i = c = 0$, $i = 1, \dots n$. We denote by $(\hat{A}_1, D(\hat{A}_1))$ its realization in L^1 (as specified in Section 2.5) and by $\hat{T}$ the generated semigroup. Thanks to (4.20), we have only to prove that

$$\limsup_{t\to 0}\int_\Omega |D\hat{T}(t)u_0(x)|_P\, dx \le |Du_0|_P(\Omega), \tag{4.26}$$

which is trivially satisfied if $u_0 \in L^1(\Omega) \setminus BV(\Omega)$. We then consider $u_0 \in BV(\Omega)$. Fix $\varepsilon > 0$ and consider two open neighborhoods $U \subset V$ of $\partial\Omega$ with disjoint boundaries such that, if we take $S' = \Omega \cap U$ and $S = \Omega \cap \overline{V}$, we get

$$|Du_0|_P(S) < \varepsilon. \tag{4.27}$$

Let then $\eta \in C^2(\Omega)$ be a function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } S', \quad \eta \equiv 0 \text{ on } \Omega \setminus S$$

and define the matrix

$$P_A = \eta^2 A + (1-\eta^2)P.$$

By Proposition 4.2.2 there exists a sequence

$$\begin{aligned}(u_k)_k &\subset \{v \in C^\infty(\Omega) \cap C^1(\overline{\Omega}) : \langle ADv, \nu\rangle = 0 \text{ on } \partial\Omega\}\\ &= \{v \in C^\infty(\Omega) \cap C^1(\overline{\Omega}) : \langle P_A Dv, \nu\rangle = 0 \text{ on } \partial\Omega\} \subset D(A_1)\end{aligned}$$

such that $u_k \to u_0$ in $L^1(\Omega)$ and

$$\lim_{k\to+\infty}\int_\Omega |Du_k|_P\,dx = |Du_0|_P(\Omega).$$

Notice that since P is μ-elliptic we get

$$\int_\Omega |Du_k|dx \le \sqrt{\mu}\int_\Omega |Du_k|_P dx$$

and then there exists $M>0$ such that

$$\|u_k\|_{W^{1,1}(\Omega)} \le M. \tag{4.28}$$

Since $\Omega\setminus S$ is an open set, by lower semicontinuity we have

$$|Du_0|_P(\Omega\setminus S) \le \liminf_{k\to+\infty}\int_{\Omega\setminus S}|Du_k|_P dx$$

and also

$$\int_S |Du_k|_P dx = \int_\Omega |Du_k|_P dx - \int_{\Omega\setminus S}|Du_k|_P dx$$

whence

$$\begin{aligned}\limsup_{k\to+\infty}\int_S |Du_k|_P dx &\le \lim_{k\to+\infty}\int_\Omega |Du_k|_P dx - \liminf_{k\to+\infty}\int_{\Omega\setminus S}|Du_k|_P dx\\ &\le |Du_0|_P(\Omega) - |Du_0|_P(\Omega\setminus S) = |Du_0|_P(S).\end{aligned}$$

This proves that

$$\limsup_{k\to+\infty}\int_S |Du_k|_P dx \le |Du_0|_P(S); \tag{4.29}$$

by the μ-ellipticity of A and P, we get that $|\xi|_A \le \sqrt{\mu}|\xi|_P$ therefore the following holds:

$$\limsup_{k\to+\infty}\int_S |Du_k|_A dx = \limsup_{k\to+\infty}\int_S \langle ADu_k, Du_k\rangle^{1/2}dx \le \mu\limsup_{k\to+\infty}\int_S |Du_k|_P dx,$$

whence by (4.29) and (4.27)

$$\limsup_{k\to+\infty}\int_S |Du_k|_A dx \le \mu\varepsilon. \tag{4.30}$$

We also notice that

$$\begin{aligned}|\xi|_P^2 &= \langle P\xi,\xi\rangle = \langle P_A\xi,\xi\rangle + \langle (P-P_A)\xi,\xi\rangle\\ &= \langle P_A\xi,\xi\rangle + \eta^2\langle (P-A)\xi,\xi\rangle = |\xi|_{P_A}^2 + \eta^2\langle (P-A)\xi,\xi\rangle\end{aligned}$$

and, since P and A are μ-elliptic,

$$|\langle (P-A)\xi,\xi\rangle| \le 2\mu|\xi|^2 \le 2\mu^2|\xi|_A^2, \quad \forall \xi\in\mathbf{R}^n.$$

We have then obtained that $|\xi|_P \le |\xi|_{P_A} + \mu\sqrt{2}\eta|\xi|_A$ and as a consequence

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |D\hat{T}(t)u_k|_{P_A} dx + \mu\sqrt{2}\int_\Omega \eta|D\hat{T}(t)u_k|_A dx.$$

We can apply Proposition 4.3.3 to both terms in the right hand side in order to obtain, using (4.28), that

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |Du_k|_{P_A} dx + \mu\sqrt{2}\int_\Omega \eta|Du_k|_A dx + (1+\mu\sqrt{2})c_5 M t^{1-\delta}.$$

By definition of P_A, we have that

$$|\xi|^2_{P_A} = \eta^2|\xi|^2_A + (1-\eta^2)|\xi|^2_P, \quad \forall \xi \in \mathbf{R}^n,$$

and then

$$\int_\Omega |Du_k|_{P_A} dx \le \int_\Omega \eta|Du_k|_A dx + \int_\Omega \sqrt{1-\eta^2}|Du_k|_P dx \le \int_S |Du_k|_A dx + \int_\Omega |Du_k|_P dx.$$

We have then obtained the following estimate

$$\int_\Omega |D\hat{T}(t)u_k|_P dx \le \int_\Omega |Du_k|_P dx + (1+\mu\sqrt{2})\int_S |Du_k|_A dx + (1+\mu\sqrt{2})c_5 M t^{1-\delta}. \quad (4.31)$$

Using (4.30), (4.31) and the fact that $\hat{T}(t)u_k \to \hat{T}(t)u_0$ in $L^1(\Omega)$ as $n \to +\infty$, we get

$$\begin{aligned}\int_\Omega |D\hat{T}(t)u_0|_P dx &\le \liminf_{k\to+\infty}\int_\Omega |D\hat{T}(t)u_k|_P dx \le \limsup_{k\to+\infty}\int_\Omega |D\hat{T}(t)u_k|_P dx\\ &\le |Du_0|_P(\Omega) + \mu(1+\mu\sqrt{2})\varepsilon + (1+\mu\sqrt{2})c_5 M t^{1-\delta}\end{aligned}$$

and the result for P regular then follows by letting $t \to 0$, since ε is arbitrary. The case with $p_{ij} \in C_b(\overline{\Omega})$ is a consequence of the approximation result given in Proposition 4.2.7.

Finally, we consider non zero coefficients b_i and c and $\mathcal{A}u = \mathrm{div}(ADu) + \langle B, Du\rangle + cu$ with $b_i, c \in L^\infty(\Omega)$, $i = 1, \dots n$. Notice that the boundary operators associated with A_1 and $\hat{A}_1$ as in (2.110) coincide, and then the set $C_A(\Omega)$ defined in (4.10) is a core both for $(A_1, D(A_1))$ and $(\hat{A}_1, D(\hat{A}_1))$. We denote by $(T(t))_{t\ge 0}$ the semigroup generated by $(A_1, D(A_1))$. Notice that if we define $\hat{u}(t) := \hat{T}(t)u_0$ and $u = T(t)u_0$, with $u_0 \in C_A(\Omega)$, the function $w := \hat{u} - u$ is the solution of the problem

$$\begin{cases} \partial_t w - \mathcal{A}w = \mathcal{E}\hat{u} := -\langle B, D\hat{u}\rangle - c\hat{u} & \text{in } (0,\infty)\times\Omega \\ w(0) = 0 & \text{in } \Omega \\ \langle ADw, \nu\rangle = 0 & \text{in } (0,\infty)\times\partial\Omega\,. \end{cases}$$

Thus, since $w(t) = \int_0^t T(t-s)\mathcal{E}\hat{u}(s)ds$, we get

$$Dw(t) = D(\hat{u} - u)(t) = \int_0^t DT(t-s)\mathcal{E}\hat{u}(s)ds$$

and then using (3.4)

$$\begin{aligned}\|D\hat{T}(t)u_0 - DT(t)u_0\|_{L^1(\Omega)} &\le c_2\|\mathcal{E}\hat{T}(t)u_0\|_{L^1(\Omega)}\int_0^t \frac{1}{\sqrt{t-s}}ds \\ &\le 2c_2\sqrt{t}\left(\|B\|_\infty\|D\hat{T}(t)u_0\|_{L^1(\Omega)} + \|c\|_\infty\|\hat{T}(t)u_0\|_{L^1(\Omega)}\right)\end{aligned} \tag{4.32}$$

Since $\|\hat{T}(t)u_0\|_{L^1(\Omega)} \to \|u_0\|_{L^1(\Omega)}$ and $\limsup_{t\to 0}\|D\hat{T}(t)u_0\|_{L^1(\Omega)}$ is bounded we can conclude that $\lim_{t\to 0}\|D\hat{T}(t)u_0 - DT(t)u_0\|_{L^1(\Omega)} = 0$ and consequently, for $v \in C_A(\Omega)$, it follows

$$\begin{aligned}\limsup_{t\to 0}\int_\Omega |DT(t)v|_P\,dx &\le \limsup_{t\to 0}\int_\Omega |D\hat{T}(t)v|_P\,dx \\ &+ \lim_{t\to 0}\int_\Omega |D\hat{T}(t)v - DT(t)v|_P\,dx = \int_\Omega |Dv|_P\,dx.\end{aligned}$$

The thesis then follows from the density of $C_A(\Omega)$ in $BV_P(\Omega)$ (see Proposition 4.2.2); given $u_0 \in BV_P(\Omega)$, we take a sequence $(u_k) \subset C_A(\Omega)$ approximating u_0 in P-variation. Then, using (4.32) with u_k in place of u_0 and (4.31), we get

$$\begin{aligned}&\int_\Omega |DT(t)u_k|_P dx \le \int_\Omega |D\hat{T}(t)u_k|_P dx + \int_\Omega |DT(t)u_k - D\hat{T}(t)u_k|_P dx \\ \le&(1+2c_2\mu\sqrt{t}\|B\|_\infty)\int_\Omega |Du_k|_P dx \\ &+ (1+\mu\sqrt{2})(1+2c_2\mu\sqrt{t}\|B\|_\infty)\int_S |Du_k|_A dx \\ &+ (1+\mu\sqrt{2})(1+2c_2\mu\sqrt{t}\|B\|_\infty)c_5Mt^{1-\delta} + 2c_2\sqrt{\mu t}\,\|c\|_{L^\infty}\int_\Omega |\hat{T}(t)u_k|dx\end{aligned}$$

and consequently it follows

$$\begin{aligned}&|Du_0|_P(\Omega) \le \liminf_{t\to 0}\int_\Omega |DT(t)u_0|_P dx \le \limsup_{t\to 0}\limsup_{k\to+\infty}\int_\Omega |DT(t)u_k|_P dx \\ \le&\limsup_{t\to 0}\big\{(1+2c_2\mu\sqrt{t}\|B\|_\infty)|Du_0|_P(\Omega) + (1+\mu\sqrt{2})(1+2c_2\mu\sqrt{t}\|B\|_\infty)\varepsilon \\ &+ (1+\mu\sqrt{2})(1+2c_2\mu\sqrt{t}\|B\|_\infty)c_5Mt^{1-\delta} + c_2\sqrt{\mu t}\,\|c\|_{L^\infty}\|u_0\|_{L^1(\Omega)}\big\} \\ =&|Du_0|_P(\Omega) + (1+\mu\sqrt{2})\varepsilon\end{aligned}$$

The result then follows since ε is arbitrary. □