Proof. - By definition of strongly regular function there is no singular couple of vertices. Besides, by Proposition 6, there does not exist any non-headed minimal n-tuple with $n > 2$, then there are no proper singularities of f. Hence, by Remark to Proposition 8, f is c. regular.

DEFINITION 7. - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite directed graph,* G' *a subgraph of* G *and* $f : S,S' \to G,G'$ *a function from the pair* S,S' *to the pair* G,G'. *The function* f *is called* completely o-regular (*resp.* completely o^*-regular) *or simply* c.o-regular (*resp.* c.o^*-regular), *if both* $f : S \to G$ *and its restriction* $f' : S' \to G'$ *are c.o-regular* (*resp.* c.o^**-regular*).

REMARK. - If S'' is a subspace of S', G'' a subgraph of G including G', $f : S,S' \to G,G'$ a c.o-regular (resp. c.o^*-regular) function, then also the functions $f : S,S'' \to G,G'$, $f : S,S' \to G,G''$ and $f : S,S'' \to G,G''$ are c.o-regular (resp. c.o^*-regular).

PROPOSITION 10. - *Every strongly regular function from a pair of topological spaces* S,S' *to a pair of finite undirected graphs* G,G' *is also c.regular.*

3) *The first normalization theorem.*

PROPOSITION 11. - *Let* S *be a normal topological space,* G *a finite directed graph,* $f : S \to G$ *an o-regular function from* S *to* G *and* $X=\{v_1,\dots,v_n\}$ *a singularity of* f. *Then there exists an o-regular function* g *from* S *to* G, *which is o-homotopic to* f *and such that:*

i) X *is not a singularity of* g;

ii) all the singularities of g *are also singularities of* f.

Proof. - i) Since X is a singularity of f, by Definition 5 and Proposition 7, it follows $V_1^f \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f} = \phi$ and $\overline{V_1^f} \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f} \neq \phi$. If we put $\overline{V_2^f} \cap \ldots \cap \overline{V_n^f} = Y$, by R.1 there exists an o-regular function g from S to G, which is o-homotopic to f and such that $\overline{V_1^g} \cap Y = \phi$. Now, by Proposition 8, $v_1, \ldots, v_n \notin f(\overline{V^f} \cap Y)$. Since, from the definitions of function $g^{(i,j)}$ (see [2], Proof of Lemma 6), only the couterimages of elements of $f(\overline{V_1^f} \cap Y)$ are increased, it follows:

$$Y = \overline{V_2^f} \cap \overline{V_3^f} \cap \ldots \cap \overline{V_n^f} \supseteq \ldots \supseteq \overline{V_2^{g^{(i,j)}}} \cap \ldots \cap \overline{V_n^{g^{(i,j)}}} \supseteq \ldots \supseteq \overline{V_2^g} \cap \ldots \cap \overline{V_n^g} ;$$

hence $(\overline{V_1^g} \cap Y = \phi) \implies (\overline{V_1^g} \cap \overline{V_2^g} \cap \ldots \cap \overline{V_n^g} = \phi)$.

ii) If, for a non-headed subset $\{w_1, \ldots, w_m\}$ of G, we have $\overline{W_1^g} \cap \ldots \cap \overline{W_m^g} \neq \phi$ by R.2, it follows $\overline{W_1^f} \cap \ldots \cap \overline{W_m^f} \neq \phi$, i.e. $\{w_1, \ldots, w_m\}$ is also a singularity of f.

THEOREM 12. - (The first normalization theorem). *Let* S *be a normal topological space,* G *a finite directed graph and* f *an o-regular function from* S *to* G. *Then there exists a completely o-regular function, o-homotopic to the function* f.

Proof. - Let v_1, v_2 be a singular couple of f. By Proposition 11, we construct an o-regular function g, which is o-homotopic to f and such that $\overline{V_1^g} \cap \overline{V^g} = \phi$. Now if w_1, w_2 is another couple, which is a singular set of *g* (and then of *f*), by repeating the argument, we can remove also this singularity. Hence, by a finite number of steps, we eliminate, at first, all the proper singular couples, then, all the proper singular terns, ect., and at last, all the proper singular *n*-tuples. Since the number of vertices is finite, the argument comes to an end and, by Remark to Proposition 8,ever singularity is eliminated. Hence we obtain the assertion.

REMARK. - If we just limite ourselves to eliminate the singular couples, we obtain the Weak normalization theorem: *Under the assumptions of Theorem every regular function is homotopic to a strongly regular function.* (See [2] Theorem 10).

If we now consider functions between pairs, we can obtain, similarly to t proof of Proposition 11, the following results by R.3 and R. 4:

PROPOSITION 13. - *Let* S *be a normal topological space,* S' *a closed subspace of* S,G *a finite directed graph,* G' *a subgraph of* G, f : S,S' - *an o-regular function,* f' : S' → G' *the restriction of* f : S → G *to* S' $X' = \{u_1, \ldots, u_m\}$ *a singularity of* f'. *Then there exists an o-regular function* g : S,S' → G,G',*which is o-homotopic to* f *and such that:*

i) X' *is not a singularity of* g';
ii) all the singularities of g' *are also singularities of* f';
iii) all the singularities of g *are also singularies of* f;
iv) all the singularities of g *with a non-empty support in* S' *are of the same type for* f, *i.e.* $(\overline{V_1^g} \cap \ldots \cap \overline{V_n^g} \cap S' \neq \phi) \Longrightarrow (\overline{V_1^f} \cap \ldots \cap \overline{V_n^f} \cap$

PROPOSITION 14. - *Under the assumptions of Proposition 13, let* $X = \{v_1, \ldots$ *be a singularity of* f *with a non-empty support in* S',*i.e.* $\overline{V_1^f} \cap \ldots \cap \overline{V_n^f} \cap$

Then there exists an o-regular function g : S,S' → G,G' *which is o-homotop to* f *and such that:*

i) $\overline{V_1^g} \cap \ldots \cap \overline{V_n^g} \cap S' = \phi$;

ii) conditions ii), iii), iv) of Proposition 13 are true.

THEOREM 15. - (The first normalization theorem between pairs). *Let* S *be a normal topological space,* S' *a closed subspace of* S , G *a finite directed graph,* G' *a subgraph of* G *and* f : S,S' → G,G' *an o-regular funct Then there exists a completely o-regular function* k : S,S' → G,G', *o-homoto to the function* f.

Proof. - By using Propositions 13,14 we proceed as in the proof of Theore 12. So,at first, we can construct an o-regular function *h : S,S' → G,G'*, which is o-homotopic to *f* and such that:

1) *h' : S' → G'* is a c. o-regular function;

2) every singularity of *h* has an empty support in *S'*.

Hence the singularities of *h* have the support in the open set *S-S'*. Ther in order to obtain the c.o-regular function *k : S,S' → G,G'*, we use Theorem But now we choose the closed neighbourhoods $W^{(i,j)}$, which we employed in the proof of R. 2 (see [2], Lemma 6), such that they are disjoined from S'. Then *k* is the sought function.

REMARK 1. - By using Theorem 20 (Extension theorem) and Corollary 21 of [2], we have two other ways for proving this theorem or, more exactly, for obtaining the previous funtion *h*.

The first way consists in constructing an o-regular function *g:S,S → G,G'*, which is o-homotopic to *f* and such that its restriction *g' : S' → G'* is

c.o-regular, and then by taking an extension h of g.
The second way lies in constructing an extension $g : S,U \to G,G'$ of f, where U is a closed neighbourbood of S', and then an o-regular function $h : S,U \to G,G'$, such that its restriction $\hat{h} : U \to G'$ is c.o-regular.

. REMARK 2. - If we just limit ourselves to eliminate the singular couples of vertices, we obtain the *Weak normalization theorem between pairs*. (See [2] Theorem 16).

THEOREM 16. - (The first normalization theorem for homotopies). *Let* SxI *be a normal topological space,* S' *a closed subspace of* S , G *a finite directed graph,* G' *a subgraph of* G, f,g : S → G (*resp.* f,g : S,S' → (*two o-homotopic completely o-regular functions. Then between the functions* f *and* g *there also exists an o-homotopy, which is a completely o-regular function.* (See [2], Theorem 17).

Proof. - Let $F : SxI \to G$ be an o-homotopy between f and g. We defi the homotopy $J : SxI \to G$, given by:

$$J(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in \left[0, \frac{1}{3}\right] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ g(x) & \forall x \in S, \quad \forall t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

If we call J_1, J_2, J_3 the restrictions of J respectively to $Sx[0,\frac{1}{3}]$, $Sx[\frac{1}{3}, \frac{2}{3}]$, $Sx[\frac{2}{3},1]$, it follows that J is o-regular since the function J_1, J_2, J_3 are such. Moreover, J_1 and J_3 are also c.o-regular, in fact a

singularity of J_1, for example, implies directly a singularity of f .
Consequently, also the restriction of J to $Sx\{[0,\frac{1}{3}] \cup [\frac{2}{3},1]\}$ is c.o-regular. By Theorem 12 (resp. Theorem 15), we can replace J with a c.o-regula function K which coincides with J on $Sx\{0\}$ and $Sx\{1\}$, by choosing the closed neighbourboods $W^{(i,j)}$ (resp. $L^{(i,j,k)}$), which we employed in the proof of R.2 (resp. R. 4), disjoined from the closed sets $Sx\{0\}$ and $Sx\{1$

FINAL REMARKS.

i) We can generalize the foregoing results to the case of n closed subspac $S_1,\dots,S_n$ of S and of n subgraphs $G_1,\dots,G_n$ of G such that S_j is a subspace of S_i and G_j a subgraph of G_i, $\forall i,j = 1,\dots,n, j > i$. (See [2 § 8 b)).

For example, in the case similar to Theorem 15, in order to construct a c.o-regular function $k : S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ o-homotopic to a given function $f : S,S_1,\dots,S_n \to G,G_1,\dots,G_n$, at first, we construct a function h^1 which is o-homotopic to f and such that:

1) its restriction $h^1_n : S^1_n \to G_n$ is c.o-regular;

2) every singularity of $h^1 : S \to G$ and of the restrictions $h^1_i : S_i \to G_i$, $\forall i = 1,\dots,h-1$, has an empty support in S_n.

Then, by choosing the closed neighbourhoods, which we employ, disjoined from S_n , we construct a function h^2 which is o-homotopic to h^1 anc such that:

1) its restriction $h^2_{n-1} : S_{n-1} \to G_{n-1}$ is c.o-regular;

2) every singularity of $h^2 : S \to G$ and of the restrictions $h^2_i : S_i \to G_i$, $\forall i = 1,\dots,n-2$, has an empty support in S_{n-1}.

And so on.

ii) The previous propositions and theorems can be translated by duality for o^*-regular functions.

iii) A further generalization can be obtained by asking that the spaces S or SxI are T_3+T_4 spaces rather than normal. (See [2], Lemma 23).

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