MARCO BURZIO DAVIDE CARLO DEMARIA

Istituto di Geometria, Università di Torino

# THE FIRST NORMALIZATION THEOREM FOR REGULAR HOMOTOPY OF FINITE DIRECTED GRAPHS.(*)

RIASSUNTO.- *Dati uno spazio topologico normale* $S$ *ed un grafo finito ed orientato* $G$, *si dimostra che ogni funzione regolare di* $S$ *in* $G$ *è omotopa ad una funzione completamente regolare, vale a dire priva di singolarità.*

INTRODUCTION.- Keeping on [2] and using the results obtained there (see Background), we prove that every regular function from a normal (**) topological space $S$ to a finite directed graph $G$ is homotopic to a *completely regular function*, i.e. *without singularities* (see Theorem 12). (The first normalization theorem).

In order to define the singularities, we consider particular subset of the graph $G$. Precisely, we say that a subset of $G$ is *headed* (resp. *tailed*) if it includes a vertex which is a predecessor (resp. successor) of all the others; while it is *totally headed* (resp. *totally tailed*) if all its subsets are headed (resp. tailed). (See Definition 1).

---

(*) Work performed under the auspices of the *Consiglio Nazionale delle Ricerche* (CNR, GNASA), Italy

(**) Consequently, we distinguish between normal space and $T_4$-space, according to whether it is a $T_2$-space or not.

We note that a totally headed set is also totally tailed and vice-versa. (See Proposition 4).

Then, a $n$-tuple $v_1,\ldots,v_n$ of the graph $G$ is called a *singularity* for an o-regular (resp. $o^*$-regular) function $f$, if it is non-headed (resp. non-tailed) and if the intersection $\overline{f^{-1}(v_1)} \cap \ldots \cap \overline{f^{-1}(v_n)}$ is non-empty. (See Definition 5). (In particular, in a finite undirected graph there are only singular couples).

Moreover, we give the first normalization theorem for regular functions from a pair of topological spaces $S,S'$ to a pair of graphs $G,G'$, where $S$ is a normal topological space and $S'$ a closed subspace of $S$. (See Theorem 15). At least, in similar conditions, we prove that two homotopic completely regular functions are also completely homotopic. (See Theorem 16).

The previous results and, particularly, the first normalization theorem in its different statements will be used in the next papers in order to prove that:

1) *If $S$ is a paracompact topological space, there is a bijection between the sets of homotopy classes* $Q(S,G)$ *and* $Q^*(S,G)$. (Duality theorem).
2) *The homotopy groups of a finite directed graph* $G$ *are isomorphic to the classical homotopy groups of the poljhedron of a suitable simplicial complex associated with* $G$.

As concerns 1), we note that the first normalization theorem allows us to identify the sets $Q(S,G)$ and $Q^*(S,G)$ of regular homotopy classes with the ones $Q_c(S,G)$ and $Q^*_c(S,G)$ of completely regular homotopy classes. Consequently, the duality theorem follows from a bijection between $Q_c(S,G)$ and $Q^*_c(S,G)$, as we prove in a paper near to appear.

As concerns 2), to obtain the above-mentioned isomorphisms, we can now anticipate that we will associate with $G$ the simplicial complex, whose simplexes are the totally headed subsets of $G$.

## 0) *Background*(*)

Let $G$ *be a finite directed graph.*
If $v,w$ are two vertices of $G$, we use the symbol $v \to w$ (resp. $v \not\to w$) to denote that $vw$ is(resp. is not) a directed edge of $G$. if $v \to w$, we call $v$ a *predecessor* of $w$ and $w$ a *successor* of $v$.
If, for all $v,w \in G$, we have $(v \to w) \Longleftrightarrow (w \to v)$, the graph is called *undirected*.

Let $S$ be *a topological space.*

Given a function $f : S \to G$ from $S$ to $G$, we denote by capital letter $V$ the set of the $f$-counterimages of $v \in G$, and if we must display the function $f$, we put $V^f = f^{-1}(v)$.

A function $f : S \to G$ is called *o-regular* (resp. *$o^*$-regular*) if, for all $v,w \in G$ such that $v \neq w$ and $v \not\to w$, it is $V \cap \bar{W} = \phi$ (resp. $\bar{V} \cap W = \phi$). (See Definition 3).

That is equivalent to saying:
$(v \neq w, V \cap \bar{W} \neq \phi$ and $f$ o-regular) $\Longrightarrow v \to w$

$(v \neq w,\ V \cap \bar{W} \neq \phi$ and $f$ $o^*$-regular) $\Rightarrow w \to v$.

A function $f : S \to G$ is called strongly *o-regular* (resp. *strongly $o^*$-regular*) if:
i) $f$ is o-regular (resp. $o^*$-regular);

ii) for all $v,w \in G$ such that $v \neq w$, $v \not\to w$ and $w \not\to v$ it follows $\bar{V} \cap \bar{W} = \phi$. (See Definition 4).

Let $I = [0,1]$ be the unit interval in $R^1$. Two o-regular (resp. $o^*$-regular) functions $f,g : S \to G$ are called *o-homotopic* (resp. *$o^*$-homotopic*) if there exists an o-regular (resp. $o^*$-regular) function $F : S \times I \to G$ such that, for all $x \in S$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. The function $F$ is called

(*) The references are ralative to [2]

an *o-homotopy* (resp. *$o^*$-homotopy*) between $f$ and $g$. (See Definition 5).

The previous o-homotopy (resp. $o^*$-homotopy) relation is an equivalence relation in the set of o-regular (resp. $o^*$-regular) functions from $S$ to $G$. We denote by $Q(S,G)$ (resp. $Q^*(S,G)$ *the set of the o-homotopy* (resp. *$o^*$-homotopy*) *classes* of o-regular (resp. $o^*$-regular) functions.

The graph $G^*$ with the same vertices of $G$ and such that $(u \to v$ in $G) \Longleftrightarrow$ $\Longleftrightarrow (v \to u$ in $G^*)$ is called the *dually directed graph* as regards $G$. (See Definition 6). Hence, we have:

DUALITY PRINCIPLE. - *Every true proposition in which appear the concepts of o-regularity, $o^*$-regularity, strongly o-regularity, strongly $o^*$-regularity, o-homotpoty, $o^*$-homotopy,* Q(S,G), $Q^*$(S,G), *remains true if the concepts of o-regularity and $o^*$-regularity (strongly o-regularity and strongly $o^*$-regularity, o-homotopy and $o^*$-homotopy,* Q(S,G) *and* $Q^*$(S,G) *are interchanged throught the statement of the proposition.*

Moreover, let $S'$ be *a subspace* of $S$ and $G'$ *a subgraph* of $G$. A function $f$ from the pair $S,S'$ to the pair $G,G'$ is called *o-regular* (resp. *$o^*$-regular, strongly o-regular, strongly $o^*$regular*) if both the function $f : S \to G$ and its restriction $f' = f_{/S'} : S' \to G'$ are o-regular (resp. $o^*$-regular, etc.) functions. (See Definition 7).

Two o-regular (resp. $o^*$-regular) functions $f,g : S,S' \to G,G'$ are called *o-homotopic* (resp. *$o^*$-homotopic*), if there exists an o-regular (resp. $o^*$-regular) function $F : SxI,S'xI \to G,G'$, such that for all $x \in S$, $F(x,0)=f(x)$ and $F(x,1) = g(x)$. The function $F$ is called an *o-homotopy* (resp. *$o^*$-homotopy*) between $f$ and $g$. (See Definition 8).

In [2] we proved the following results:

R.1.- *Let* f *be an o-regular function from a normal topological space* S *to a finite directed graph* G *and* Y *a closed subset of* S. *Then, if for* $a \in G$ *we have* $A^f \cap Y = \phi$ and $\overline{A^f} \cap Y \neq \phi$ *there exists an o-regular function* $g : S \to G$, *which is o-homotopic to* f *and such that* $A^g \cap Y = \phi$ (See Lemma 6).

R.2 - *In the construction of* R.1, *if there exist* n *vertices* $p_1, \ldots, p_n \in G$, *such that* $\overline{P_1^f \cap \ldots \cap P_n^f} = \phi$ *then also it follows* $P_1^g \cap \ldots \cap P_n^g = \phi$. (See Corollary 7).

R.3.- *Let* f *be an o-regular function from* S,S' *to* G,G', *where* S *is a normal topological space,* S' *a closed subspace of* S, Y *a closed subset of* S',G *a finite directed graph and* G' *a subgraph of* G. *Then, if for* $a \in G$ *we have* $A^f \cap Y = \phi$ *and* $\overline{A^f} \cap Y = \phi$, *there exists an o-regular function* $g : S,S' \to G,G'$, *which is o-homotopic to* f *and such that* $\overline{A^g} \cap Y = \phi$. (See Lemma 11).

R.4.- *In the construction of* R. 3, *if there exist* n *vertices* $p_1, \ldots, p_n \in G$ *and* m *vertices* $q_1, \ldots, q_m \in G'$, such that $\overline{P_1^f} \cap \ldots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}} \cap \ldots \cap \overline{Q_m^{f'}} = \phi$, *then also it follows that* $\overline{P_1^g} \cap \ldots \cap \overline{P_n^g} \cap \overline{Q_1^{g'}} \cap \ldots \cap \overline{Q_m^{g'}} = \phi$. *While, from* $\overline{P_1^f} \cap \ldots \cap \overline{P_n^f} \cap \ldots = \phi$, *it results* $P_1^g \cap \ldots \cap P_n^g \cap S' = \phi$. (See Corollary 12).

By Duality Principle, the results dual to the previous ones are also true for $o^*$-regular functions.

## 1) *Headed and totally headed subsets of a graph*

DEFINITION 1.- *Let* G *be a directed graph and* X *a non-empty subset of* G. *A vertex of* X *is called a* head *(resp. a* tail) *of* X *in* G, if it is a *predecessor (resp. a successor) of all the other vertices of* $X$. *We denote by* $H_G(X)$ *(resp.* $T_G(X)$) *or, more simply, by* $H(X)$ *(resp.* $T(X)$) *the set of the heads (resp. tails) of* $X$ *in* $G$. *Then* $X$ *is called* headed *(resp.* tailed) *if* $H(X) \neq \phi$ *(resp.* $T(X) \neq \phi$), *otherwise,* $X$ *is called* non-headed *(resp.* non-tailed).

*Finally* $X$ *is called* totally headed (*resp.* totally tailed), *if all the non-empty subsets of* $X$ *are headed (resp. tailed).*

REMARK 1. - If $X$ is a singleton, we agree to say that $H(X) = T(X) = X$, then $X$ is totally headed and also totally tailed. If $X$ is a pair, $X$ headed $\Longleftrightarrow$ $X$ totally headed $\Longleftrightarrow$ $X$ tailed $\Leftrightarrow$ $X$ totally tailed.

REMARK 2. - This definition and the following ones can be extended to undirected graphs. (See Proposition 6).

REMARK 3. - The concepts of head and tail (headed and tailed subset, etc.) are dual to each other.

DEFINITION 2. - A *non-headed (resp. non-tailed) subset* $X$ *is called* minimal *if all its non-empty proper subsets are headed (resp. tailed).*

DEFINITION 3. - A *finite directed graph* $G$ *is called* almost complete *if the set of its vertices is totally headed.*

REMARK. - A complete finite undirected graph is also almost complete.

PROPOSITION 1. - A *finite directed graph* $G$ *is almost complete iff the diagram* (*) *of the relation* ( $\rightarrow$ ) *includes the diagram of a totally ordered relation* ( $<$ ) *in* $G$.

---

(*) We use the term *diagram* rather than *graph* because *graph* is already used in another sense.

*Proof.* - i) Since $G$ is almost complete, we can choose a vertex $v_1 \in G$, which is a predecessor of all the other vertices of $G$, as the first one; then a vertex $v_2 \in G - \{v_1\}$, predecessor of all the other vertices of $G-\{v_1\}$, as the second one; and so on.

ii) Since the diagram of the relation ( $\to$ ) includes the diagram of a totally ordered relation ( $<$ ) in $G$, we can totally order the vertices of $G$. Then every vertex of $G$ is a predecessor of the vertices subsequent in the order relation.
Hence $G$ is almost complete.

REMARK. - By ordering the vertices of $G$ as in b) of Proposition 1, we say that the order relation ( $<$ ) of $G$ is compatible with the relation ( $\to$ ) of $G$.

PROPOSITION 2. - *Let* G *be an almost complete graph. Then the dually directed graph* $G^*$ *is also almost complete.*

*Proof.* - Let ( $<$ ) be a totally order relation, compatible with the relation ( $\to$ ) of $G$. Then the dual order relation ( $>$ ) is compatible with the relation ($\leftarrow$) of the dually directed graph $G^*$.

DEFINITION 4. - *Let* G *be a directed graph and* X *a subset of* G. *We call* maximal subgraph induced by X *the subgraph of* G *consisting of those directed edges of* G, *whose vertices are in* X.

PROPOSITION 3. - *A subset* X *of* G *is totally headed iff the maximal subgraph induced by* X *is almost complete.*

PROPOSITION 4. - *A subset* X *of* G *is totally headed iff it is totally tailed.*

*Proof.* - By Remark 3 to Definition 1 and by Proposition 2,3 we have:

$X$ totally headed in $G$ $\iff$ the maximal subgraph induced by $X$ is almost complete $\iff$ the dually directed graph of the maximal subgraph induced by $X$ is almost complete $\iff$ $X$ is totally headed in $G^*$ $\iff$ $X$ is totally tailed in $G$.

PROPOSITION 5. - *A subset* X *of* G *is non-headed minimal iff it is non-tailed minimal.*

*Proof.* - Since all the subsets of $X$ are totally headed, by Proposition 4, they are also totally tailed. If we assume that $X$ is tailed, then, by Definition 1, it is totally tailed. Hence, by Proposition 4, it is also totally headed. Contradiction.

REMARK. - Then *almost complete graph, totally headed subset, non-headed minimal subset are selfdual concepts*, while it does not follow for headed or tailed subset.

PROPOSITION 6. - *In an undirected graph there does not exist any non-headed minimal* n-*tuple* X *with* n > 2.

*Proof.*- If all the pairs of vertices of $X$ are headed (i.e. they are vertices of edges), then the maximal subgraph induced by $X$ is complete. Hence it is also totally headed.

EXAMPLES.

1) Let $G=\{a,b,c,d,e\}$ be the graph with the edges $a \to b$, $a \to c$, $a \to d$, $b \to d$, $b \to e$, $c \to d$. Then the subset $\{a,b,e\}$ is non-headed and non-tailed, but it is not minimal non-headed (i.e.minimal non-tailed); $\{a,b,c\}$ is headed and non-tailed; $\{b,c,d\}$ is non-headed and tailed; $\{a,b,c,d\}$ is headed and tailed, but not totally headed (tailed).

2) The graphs $G = \{u,v,w\}$ with the edges $u \to v,\ u \to w,\ v \to w$ and $G' = \{q,r,s,t\}$ with edges $q \to r,\ q \to s,\ q \to t,\ r \to s,\ r \to t,\ s \to t$ are examples of almost complete graphs. Moreover, the sets $\{u,v,w\}$, $\{q,r,s,t\}$ are examples of totally headed (i.e. totally tailed) subsets. Their compatible orders are, respectively, $u < v < w$, $q < r < s < t$.

3) In the graphs $G = \{f,g,h\}$ with the edges $f \to g,\ g \to h,\ h \to f$ and $G' = \{\ell,m,n,p\}$ with the edges $\ell \to m,\ \ell \to n,\ m \to n,\ m \to p,\ n \to \ell,\ n \to p,\ p \to \ell,\ p \to m$ the sets $\{f,g,h\}$ and $\{\ell,m,n,p\}$ are examples of non-headed minimal (i.e. non-tailed minimal) subsets.

## 2) *Singularities of a regular function.*

PROPOSITION 7. - *Let* S *be a topological space,* G *a finite directed graph,* $f : S \to G$ *an o-regular function from* S *to* G *and* $X = \{v_1, v_2, \dots, v_n\}$ *a non-headed subset of* G ($n \geqslant 2$). *Then it holds:*

$$v_1^f \cap \overline{v_2^f} \cap \dots \cap \overline{v_n^f} = \phi;$$

$$\overline{v_1^f} \cap v_2^f \cap \dots \cap \overline{v_n^f} = \phi$$

.................

$$\overline{v_1^f} \cap \dots \cap \overline{v_{n-1}^f} \cap v_n^f = \phi$$

*Proof.* - Since $X$ is a non-headed subset, there is no vertex $v_i$, which is a predecessor of all the other $n-1$ vertices. Then, for every $i = 1,\dots,n$ let $w_i$ be a vertex such that $v_i \not\to w_i$. From o-regularity of $f$ it is $V_i \cap \overline{w}_i = \phi$. Since $w_i$ is one of the vertices $v_1,\dots,v_{i-1},v_{i+1},\dots,v_n$, it follows $\overline{v_1^f} \cap \dots \cap \overline{v_{i-1}^f} \cap v_i^f \cap \overline{v_{i+1}^f} \cap \dots \cap \overline{v_n^f} = \phi$.

DEFINITION 5. - *Let* S *be a topological space,* G *a finite directed*

*graph,* $f : S \to G$ *an o-regular (resp.* $o^*$*-regular) function from* $S$ *to* $G$ *and* $X = \{v_1, v_2, \ldots, v_n\}$ *a* n-*tuple of vertices of* $G$ *with* $n \geq 2$.
*Then* $X$ *is called a* singularity *of* $f$ *or a* singular set *of* $f$ *if:*

*i)* $X$ *is non-headed (resp. non-tailed);*

*ii)* $\overline{V_1^f} \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f} \neq \phi$ .

*Moreover,* $X$ *is called a* proper singularity *of* $f$ *if i) is replaced by:*

*i')* $X$ *is non-headed minimal (i.e. non-tailed minimal).*

*Finally, the closed set* $\overline{V_1^f} \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f}$ *is called the* support *of the singularity.*

PROPOSITION 8. - *If* $X = \{v_1, v_2, \ldots, v_n\}$ *is a singularity of* $f$, *then* $X$ *has an empty intersection with the image of its support, i.e.* $f(\overline{V_1^f} \cap \ldots \cap \overline{V_n^f}) \cap X =$

*Proof.* - It follows from Proposition 7.

REMARK. - Since every non-headed (non-tailed) subset of $G$ includes a non-headed minimal (non-tailed minimal) subset of $G$, every singularity includes a proper singularity, Hence, every singular couple is a proper singularity.

DEFINITION 6. - *Let* $S$ *be a topological space,* $G$ *a finite directed graph and* $f : S \to G$ *an o-regular (resp.* $o^*$*-regular) function from* $S$ *to* $G$. *The function* $f$ *is called* completely o-regular (*resp.* completely $o^*$-regular) *or simply* c.o-regular (*resp.* c.$o^*$-regular), *if there are no singularities of* $f$.

We note that Definitions 5,6 can be extended to undirected graphs. Then it follows:

PROPOSITION 9. - *Let* $S$ *be a topological space and* $G$ *a finite undirected graph. Then a strongly regular function* (see [5] , Definition 3) $f : S \to G$ *from* $S$ *to* $G$ *is also c.regular.*

*Proof*. - By definition of strongly regular function there is no singular couple of vertices. Besides, by Proposition 6, there does not exist any non-headed minimal $n$-tuple with $n > 2$, then there are no proper singularities of $f$. Hence, by Remark to Proposition 8, $f$ is c. regular.

DEFINITION 7. - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite directed graph,* G' *a subgraph of* G *and* $f : S,S' \to G,G'$ *a function from the pair* S,S' *to the pair* G,G'. *The function* f *is called* completely o-regular (*resp.* completely $o^*$-regular) *or simply* c.o-regular (*resp.* c.$o^*$-regular), *if both* $f : S \to G$ *and its restriction* $f' : S' \to G'$ *are c.o-regular (resp. c.$o^*$-regular).*

REMARK. - If $S''$ is a subspace of $S'$, $G''$ a subgraph of $G$ including $G'$, $f : S,S' \to G,G'$ a c.o-regular (resp. c.$o^*$-regular) function, then also the functions $f : S,S'' \to G,G'$, $f : S,S' \to G,G''$ and $f : S,S'' \to G,G''$ are c.o-regular (resp. c.$o^*$-regular).

PROPOSITION 10. - *Every strongly regular function from a pair of topological spaces* S,S' *to a pair of finite undirected graphs* G,G' *is also c.regular.*

## 3) The first normalization theorem.

PROPOSITION 11. - *Let* S *be a normal topological space,* G *a finite directed graph,* $f : S \to G$ *an o-regular function from* S *to* G *and* $X=\{v_1,\dots,v_n\}$ *a singularity of* f. *Then there exists an o-regular function* g *from* S *to* G, *which is o-homotopic to* f *and such that:*

*i)* X *is not a singularity of* g;

*ii)* *all the singularities of* g *are also singularities of* f.

*Proof.* - i) Since $X$ is a singularity of $f$, by Definition 5 and Proposi tion 7, it follows $V_1^f \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f} = \phi$ and $\overline{V_1^f} \cap \overline{V_2^f} \cap \ldots \cap \overline{V_n^f} \neq \phi$. If we put $\overline{V_2^f} \cap \ldots \cap \overline{V_n^f} = Y$, by R.1 there exists an o-regular function $g$ from $S$ to $G$, which is o-homotopic to $f$ and such that $\overline{V_1^g} \cap Y = \phi$. Now, by Proposition 8, $v_1, \ldots, v_n \notin f(\overline{V^f} \cap Y)$. Since, from the definitions of function $g^{(i,j)}$ (see [2], Proof of Lemma 6), only the couterimages of elements of $f(\overline{V_1^f} \cap Y)$ are increased, it follows:

$$Y = \overline{V_2^f} \cap \overline{V_3^f} \cap \ldots \cap \overline{V_n^f} \supseteq \ldots \supseteq \overline{V_2^{g^{(i,j)}}} \cap \ldots \cap \overline{V_n^{g^{(i,j)}}} \supseteq \ldots \supseteq \overline{V_2^g} \cap \ldots \cap \overline{V_n^g} ;$$

hence $(\overline{V_1^g} \cap Y = \phi) \implies (\overline{V_1^g} \cap \overline{V_2^g} \cap \ldots \cap \overline{V_n^g} = \phi)$.

ii)If, for a non-headed subset $\{w_1, \ldots, w_m\}$ of $G$, we have $\overline{W_1^g} \cap \ldots \cap \overline{W_m^g} \neq \phi$ by R.2, it follows $\overline{W_1^f} \cap \ldots \cap \overline{W_m^f} \neq \phi$ , i.e. $\{w_1, \ldots, w_m\}$ is also a singula rity of $f$.

THEOREM 12. - (The first normalization theorem). *Let* S *be a normal topological space,* G *a finite directed graph and* f *an o-regular function from* S *to* G *. Then there exists a completely o-regular function, o-homotopic to the function* f.

*Proof.* - Let $v_1, v_2$ be a singular couple of $f$. By Proposition 11, we construct an o-regular function $g$, which is o-homotopic to $f$ and such tha $\overline{V_1^g} \cap \overline{V^g} = \phi$ . Now if $w_1, w_2$ is another couple, which is a singular set of *g* (and then of *f*), by repeating the argument, we can remove also this singularity. Hence, by a finite number of steps, we eliminate, at first, all the proper singular couples, then, all the proper singular terns, ect., and at last, all the proper singular $n$-tuples. Since the number of vertices is finite, the argument comes to an end and, by Remark to Proposition 8,ever singularity is eliminated. Hence we obtain the assertion.

REMARK. - If we just limite ourselves to eliminate the singular couples, we obtain the Weak normalization theorem: *Under the assumptions of Theorem every regular function is homotopic to a strongly regular function.* (See [2] Theorem 10).

If we now consider functions between pairs, we can obtain, similarly to t proof of Proposition 11, the following results by R.3 and R. 4:

PROPOSITION 13. - *Let* S *be a normal topological space,* S' *a closed subspace of* S,G *a finite directed graph,* G' *a subgraph of* G, f : S,S' - *an o-regular function,* $f' : S' \to G'$ *the restriction of* $f : S \to G$ *to* S' $X' = \{u_1,\dots,u_m\}$ *a singularity of* f'. *Then there exists an o-regular function* $g : S,S' \to G,G'$,*which is o-homotopic to* f *and such that:*

*i)* X' *is not a singularity of* g';

*ii) all the singularities of* g' *are also singularities of* f';

*iii) all the singularities of* g *are also singularies of* f;

*iv) all the singularities of* g *with a non-empty syppont in* S' *are of the same type for* f, *i.e.* $(\overline{V_1^g} \cap \dots \cap \overline{V_n^g} \cap S' \neq \phi) \Longrightarrow (\overline{V_1^f} \cap \dots \cap \overline{V_n^f} \cap$

PROPOSITION 14. - *Under the assumptions of Proposition 13, let* $X=\{v_1,\dots$ *be a singularity of* f *with a non-empty support in* S',*i.e.* $\overline{V_1^f} \cap \dots \cap \overline{V_n^f} \cap$

*Then there exists an o-regular function* $g : S,S' \to G,G'$ *which is o-homotop to* $f$ *and such that:*

*i)* $\overline{V_1^g} \cap \ldots \cap \overline{V_n^g} \cap S' = \phi$ ;

*ii) conditions ii), iii), iv) of Proposition 13 are true.*

THEOREM 15. - (The first normalization theorem between pairs). *Let* $S$ *be a normal topological space,* $S'$ *a closed subspace of* $S$ , $G$ *a finite directed graph,* $G'$ *a subgraph of* $G$ *and* $f : S,S' \to G,G'$ *an o-regular funct Then there exists a completely o-regular function* $k : S,S' \to G,G'$, *o-homoto to the function* $f$.

*Proof.* - By using Propositions 13,14 we proceed as in the proof of Theore 12. So,at first, we can construct an o-regular function $h : S,S' \to G,G'$, which is o-homotopic to $f$ and such that:

1) $h' : S' \to G'$ is a c. o-regular function;

2) every singularity of $h$ has an empty support in $S'$.

Hence the singularities of $h$ have the support in the open set $S-S'$. Ther in order to obtain the c.o-regular function $k : S,S' \to G,G'$, we use Theorem But now we choose the closed neighbourhoods $W^{(i,j)}$, which we employed in the proof of R. 2 (see [2], Lemma 6), such that they are disjoined from S'. Then $k$ is the sought function.

REMARK 1. - By using Theorem 20 (Extension theorem) and Corollary 21 of [2], we have two other ways for proving this theorem or, more exactly, for obtaining the previous funtion $h$.

The first way consists in constructing an o-regular function $g:S,S \to G,G'$, which is o-homotopic to $f$ and such that its restriction $g' : S' \to G'$ is

c.o-regular, and then by taking an extension $h$ of $g$.
The second way lies in constructing an extension $g : S,U \to G,G'$ of $f$, where $U$ is a closed neighbourbood of $S'$, and then an o-regular function $h : S,U \to G,G'$, such that its restriction $\hat{h} : U \to G'$ is c.o-regular.

. REMARK 2. - If we just limit ourselves to eliminate the singular couples of vertices, we obtain the *Weak normalization theorem between pairs*. (See [2] Theorem 16).

THEOREM 16. - (The first normalization theorem for homotopies). *Let* SxI *be a normal topological space,* S' *a closed subspace of* S , G *a finite directed graph,* G' *a subgraph of* G, f,g : S → G (*resp.* f,g : S,S' → ( *two o-homotopic completely o-regular functions. Then between the functions* f *and* g *there also exists an o-homotopy, which is a completely o-regular function.* (See [2], Theorem 17).

*Proof.* - Let $F : SxI \to G$ be an o-homotopy between $f$ and $g$. We defi the homotopy $J : SxI \to G$, given by:

$$J(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in \left[0, \frac{1}{3}\right] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ g(x) & \forall x \in S, \quad \forall t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

If we call $J_1,J_2,J_3$ the restrictions of $J$ respectively to $Sx[0,\frac{1}{3}]$, $Sx[\frac{1}{3},\frac{2}{3}]$ , $Sx[\frac{2}{3},1]$, it follows that $J$ is o-regular since the function $J_1,J_2,J_3$ are such. Moreover, $J_1$ and $J_3$ are also c.o-regular, in fact a

singularity of $J_1$, for example, implies directly a singularity of $f$ .
Consequently, also the restriction of $J$ to $S\times\{[0,\frac{1}{3}] \cup [\frac{2}{3},1]\}$ is c.o-regular. By Theorem 12 (resp. Theorem 15), we can replace $J$ with a c.o-regula function $K$ which coincides with $J$ on $S\times\{0\}$ and $S\times\{1\}$, by choosing the closed neighbourboods $W^{(i,j)}$ (resp. $L^{(i,j,k)}$), which we employed in the proof of R.2 (resp. R. 4), disjoined from the closed sets $S\times\{0\}$ and $S\times\{1$

FINAL REMARKS.

i) We can generalize the foregoing results to the case of $n$ closed subspac $S_1,\dots,S_n$ of $S$ and of $n$ subgraphs $G_1,\dots,G_n$ of $G$ such that $S_j$ is a subspace of $S_i$ and $G_j$ a subgraph of $G_i$, $\forall i,j =1,\dots,n, j > i$.(See [2 § 8 b)).
For example, in the case similar to Theorem 15, in order to construct a c.o-regular function $k : S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ o-homotopic to a given function $f : S,S_1,\dots,S_n \to G,G_1,\dots,G_n$, at first, we construct a function $h^1$ which is o-homotopic to $f$ and such that:

1) its restriction $h^1_n : S^1_n \to G_n$ is c.o-regular;
2) every singularity of $h^1 : S \to G$ and of the restrictions $h^1_i:S_i \to G_i$, $\forall i = 1,\dots,h-1$, has an empty support in $S_n$.

Then, by choosing the closed neighbourhoods, which we employ, disjoined from $S_n$ , we construct a function $h^2$ which is o-homotopic to $h^1$ anc such that:

1) its restriction $h^2_{n-1} : S_{n-1} \to G_{n-1}$ is c.o-regular;
2) every singularity of $h^2: S \to G$ and of the restrictions $h^2_i : S_i \to G_i$, $\forall i = 1,\dots,n-2$, has an empty support in $S_{n-1}$.

And so on.

ii) The previous propositions and theorems can be translated by duality for $o^*$-regular functions.

iii) A further generalization can be obtained by asking that the spaces $S$ or $S \times I$ are $T_3 + T_4$ spaces rather than normal. (See [2], Lemma 23).

## BIBLIOGRAPHY


[1] BERGE C., *Graphes et hypergraphes*, Dunod, Paris, 1970.

[2] BURZIO M. and DEMARIA D.C., *A normalization theorem for regular homotopy of finite directed graphs*, to appear in Rend. Circ. Mat.Palermo, preprint in Quaderni Ist. Matem. Univ. Lecce, n. 17, 1979.

[3] DEMARIA D.C., *Sull'omotopia e su alcune sue generalizzazioni*, Conf.Semin. Matem. Univ. Bari, n. 144, 1976.

[4] DEMARIA D.C., *Sull'omotopia regolare: applicazioni agli spazi uniformi ed ai grafi finiti*, Conf. Semin.Matem.Univ.Bari, n. 148, 1977.

[5] DEMARIA D.C., *Teoremi di normalizzazione per l'omotopia regoalre dei grafi*, Rend. Semin. Matem. Fis. Milano, XLVI, 1976.

[6] DEMARIA D.C. e GANDINI P.M., *Su una generalizzazione della teoria dell'omtopia*, Rend. Semin. Matem. Univ. Polit. Torino, 34, 1975/76.

[7] GIANELLA G.M., *Su un'omotopia regolare dei grafi*, Rend. Semin. Matem.Univ Polit. Torino, 35, 1976/77.

[8] HILTON P.J., *An introduction to homotopy theory*, Cambridge University Press, 1953.

[9] KOWALSKY H.J., *Topological Spaces*, Academic Press, 1964.