POINCARE RECURRENCE THEOREM FOR FINITELY ADDITIVE MEASURES E. BARONE , K.P.S. BHASKARA RAO⁽¹⁾

SUMMARY. - In this paper we study the validity of Poincaré recurrence theorem for finitely additive measures.

§ 1.- DEFINITIONS AND PROBLEM

Let X be an arbitrary non empty point set, and T : $X \rightarrow X$ a trasformation on X. If (X,Q,μ) is a charge space, i.e., Q is a field of subsets of X and μ is a nonnegative charge (usually called finitely additive measure) the transformation T is called a measurable transformation if

(1.1)
$$\forall A \in Q : T^{-1}(A) \in Q$$

A measurable trasformation T is said to be measure preserving if

(1.2)
$$\forall A \in (A : \mu(T^{-1}(A) = \mu(A)).$$

If T is a measure preserving transformation and $E \in \mathcal{A}$ then a point x e E is called recurrent if

$$\exists n \in \mathbb{N}^{(2)}$$
 such that $T^n x \in E$

and x is called strongly recurrent if

Tⁿ x є E for infinitely many values of n.

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(2) N is the set $\{1,2,3,\ldots\}$ of positive integers, $\mathbb{N}_{\circ} = \{0,1,2,\ldots\}$ and $\mathbf{Z}'_{1} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Now, the classical Poincaré's recurrence theorem, very usefull in the Ergodic theory, asserts:

If (X) is a σ -field of subsets of a set X, μ is a coutably additive measure (c.a.m.), T is measure preserving, $\mu(X) < +\infty$ and $E \in Q$, then almost every point of E is strongly recurrent.

This theorem is due essentially to H. Poincaré ([1]) p. 67-72) but the first rigorous proof was given in [2] by C. CARATHEODORY.

In this paper we study the validity of Poincaré recurrence theorem for charges.

§ 2.- Results

<u>Theorem 1.</u>— If T is measure preserving μ a charge, $\mu(X) < + \infty$, Q is a σ -field and $E \in \mathcal{G}$, then almost every point of E is recurrent. PROOF.

We consider the set

(2.1)
$$F = \{x \in E : T^n \ x \notin E \ \forall n \in \mathbb{N}\}$$

because of the identity

$$F = E - \prod_{n=1}^{\infty} \{x \in E : T^n \ x \in E\} = E - \prod_{n=1}^{\infty} T^{-n}(E),$$

F is measurable.

Furthermore we have

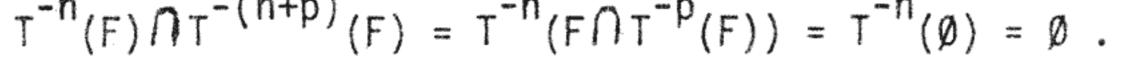
$$F \cap T^{-n}(F) = \emptyset$$

and all the sets

$$F$$
, $T^{-1}(F)$, $T^{-2}(F)$...

are mutually disjoint since

$$-n = (n+n) = n = n$$



Because T is measure preserving we have

$$\mu(T^{-n}(F)) = \mu(F)$$
 for $n = 1, 2, 3, ...$

and so if $\mu(F) > 0$ we would have

$$+\infty = \prod_{n=1}^{\infty} \mu(F) = \prod_{n=1}^{\infty} \mu(T^{-n}(F)) \leq \mu(\prod_{n=1}^{\infty} T^{-n}(F)) \leq \mu(X) < +\infty.$$

This is a contradiction.

It is well known that for $\mu(X) = +\infty$ theorem 1 is not necessarily true even if µ is a c.a.m.

If μ is a c.a.m., we have also the strong recurrence, i.e. almost every point of E is strongly recurrent, but this is not true if μ is only finitely additive.

In fact if for instance $X = \mathbb{R}$ it is well known ([3]) pag. 243) that

there is a charge v on $\mathcal{G}(\mathbb{R})$ satisfying the following conditions:

(i) $0 < v(A) \leq 1$ for all $A \subset \mathbb{R}$ (ii) v(A) = 1 if $[\alpha, \infty] \subset A$ for some $\alpha \in \mathbb{R}$ (iii) v(A) = 0 if A is bounded above (iv) v(A+a) = v(A) for all $A \subset \mathbb{R}$ and $a \in \mathbb{R}$

Now if we consider

T(x) = x-1 and A = [0,1]U[2,3]U[4,5]U... because AUA+1 = $]0, +\infty]$ it follows that v(AUA+1) = 1

and since $A \cap A+1 = \emptyset$ and v(A) = v(A+1)

(by (iv) we have

$$v(A) = \frac{1}{2} .$$

For every $x \in A$ the set $\{n : T^n(x) \in A\}$ is finite and x is not

strongly recurrent.

So the strong version of recurrence theorem is not true, but we can give

a result very near. - 4 -

We need the following definition: a point $x \in E$ is called n-times recurrent (for n = 1, 2, 3, ...) if there are n different values of ke \mathbb{N} such that

Theorem 2. If T is measure preserving, μ is a charge, $\mu(X) < + \infty$ A is a σ -field and $E \in G$, then for each $n \in \mathbb{N}$, almost every point of E is n-times recurrent.

PROOF.

Let $S(1,E) = \{x \in E : T^k(x) \in E \text{ for at least one } k \in \mathbb{N}\}$

Since (see (2.1)

$$F = E - S(1, E)$$

S(1,E) e Q and we have

(2.2)
$$\mu(E) = \mu(S(1,E))$$

define in general We

We can easily recognise that

$$S(1,S(1,E)) = S(2,E)$$

so we have for the same reason of (2.2): $\mu(S(1,E)) = \mu(S(2,E))$

and also

$$\mu(S(2,E)) = \mu(E)$$

In general

S(1,S(n-1,E)) = S(n,E)

and so

$$\mu(S(n,E)) = \mu(E)$$
 for every $n \in \mathbb{N}$.

This means that the set

$$F_n = E - S(n,E) = \{x \in E : x \text{ is not } n-times \text{ ricurrent}\}$$
 is measurable and $\mu(F_n) = 0$

Remark 1.-

In theorem 1 we have used the hypothesis that \mathcal{Q} is a σ -field, in proving that F is measurable. Is this hypothesis essential? We do not know the answer but we give an exemple where if \mathbb{C} is only a field F is not measurable. Let $X = \mathbb{N} \times Z$, $E = \{(m,0); m \in \mathbb{N}_o\} \cup \{(m,-m); m \in \mathbb{N}_o\} \cup \{0,-m\}; m \in \mathbb{N}_o\}$, $T : X \to X'$ be defined by T (n,m) = (n,m+1). Let \mathbb{Q} be the smallest field that contains A and such that T verifies (1.1). Such an \mathbb{Q} is the collection of all finite unions of sets of the form $T^{-n_1}(E) \cap T^{-n_2}(E) \cap \ldots \cap T^{-n_k}(E) \cap T^{-m_1}(E') \cap \ldots \cap T^{-m_k}(E')$ $(E' = X \to E)$

for some integers $n_1, n_2, \ldots, n_k, m_1, \ldots, m_h$ in \mathbb{N}_{\circ} .

Now the set $F = \{x \in E : T^n \times f \in F \text{ for all } n \in \mathbb{N}\} = \{(m, 0); m \in \mathbb{N}_o\}$ is not

an element of CL.

In fact if F was an element of $(\lambda$ there exist $n_1, n_2, \dots, n_k, m_1, \dots, m_h \in \mathbb{N}_o$ such that $F \supset C = T^{-n_1}(E) \cap T^{-n_2}(E) \cap \dots \cap T^{-n_k}(E) \cap T^{-m_1}(E') \cap \dots \cap T^{-m_h}(E')$

and the latter element is nonempty .

Because if $n_i \neq 0$ then $T^{-n_i}(E) \cap F = \emptyset$, it must be $n_1 = n_2 = \dots = n_k = 0$ and $T^{-n_i}(E) \cap \dots \cap T^{-n_k}(E) = E$.

But $E \cap T^{m_1}(E') \cap \ldots \cap {}^{-m_h}(E')$ contains $\{(m,-m) : m \ge p\}$ for some $p \in \mathbb{N}$. Thus F cannot contain $C \ne \emptyset$.

Now by a technique developed in Theorem 2 of [4] we can in fact get a nonnegative charge μ on \mathfrak{A} . such that $\mu(X) = 1$ and $\mu(E) = \frac{1}{2}$, because fon any integer m there is an xeE such that $T^n \times eE$ for all $n \leq m$. This can be even seen directly by the Hahn-Banach Theorem.

Remark 2.

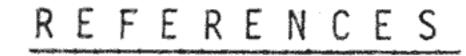
Observing that the proof of Theorem 2 holds for conservative transformations (there does not exist a set F $\epsilon \stackrel{\circ}{\leftarrow}$ with $\mu(F) > 0$ such that the sets

 $F,T^{-1}(F),T^{-2}(F),\ldots$ are pairwise disjoint) one can see that in a charge space a trasformation is conservative iff for every set A of positive charge and for every n, almost every point of A is n-times ricurrent.

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The other aspects of Ergodic Theory for charges are being worked out by the Authors.

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- [3] E. Hewitt K. A. Ross Abstract harmonic Analysis I, Springer-Verlag (1963)
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