Therefore. the use of the following integral representation for the incomplete Γ-function [14]

$$\mu^{\nu}\Gamma(\nu,\mu x) = \int_{x}^{\infty} dt\ t^{\nu-1} e^{-\mu t} ,$$

for $x > 0$ and $\mathrm{Re}\,\mu>0$, leads us to the expression

$$(3.3)\quad \psi(\alpha,\beta,\gamma;x) = -\frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!}\, n^{n\beta-\alpha}\,\Gamma(\alpha-n\beta,nx).$$

Obviously, in the special case $\gamma = m$, where m is a positive integer, the expression (3.3) reduces to a finite sum of incomplete Γ-functions, specifically:

$$(3.4)\quad \psi(\alpha,\beta,m;x) = -\sum_{n=1}^{m} (-1^{n}) \binom{m}{n} n^{n\beta-\alpha}\,\Gamma(\alpha-n\beta,nx).$$

4. A RECURRENCE RELATION.

The following recurrence relation holds:

$$(4.1)\quad (1-\frac{\gamma\beta}{\alpha})\,\psi(\alpha,\beta,\gamma;x) = -\frac{1}{\alpha}\,x^{\alpha}\left[1-(1-\frac{e^{-x}}{x^{\beta}})^{\gamma}\right] +$$

$$+\frac{\gamma}{\alpha}\left[\psi(\alpha+1,\beta,\gamma;x) - \psi(\alpha+1,\beta,\gamma-1;x)-\beta\psi(\alpha,\beta,\gamma-1;x)\right] ,$$

for $\alpha \neq 0$.

In fact, from (1.6) we can write

$$\psi(\alpha,\beta,\gamma;x) = \int_{x}^{\infty} dt\ t^{\alpha-1}\left[1-(1-\frac{e^{-t}}{t^{\beta}})(1-\frac{e^{-t}}{t^{\beta}})^{\gamma-1}\right],$$

which yields

$$(4.2) \quad \int_x^\infty dt\ t^{\alpha-\beta-1} e^{-t} \left(1 - \frac{e^{-t}}{t^\beta}\right)^{\gamma-1} = \psi(\alpha,\beta,\gamma;x) - \psi(\alpha,\beta,\gamma-1;x).$$

Furthermore, integrating by parts we obtain from (1.6):

$$(4.3) \quad \psi(\alpha,\beta,\gamma;x) = -\frac{1}{\alpha} x^\alpha \left[1 - \left(1 - \frac{e^{-t}}{x^\beta}\right)^\gamma\right] +$$

$$+ \frac{\gamma}{\alpha} \int_x^\infty dt\ t^{\alpha-\beta-1} (t+\beta) e^{-t} \left(1 - \frac{e^{-t}}{t^\beta}\right)^{\gamma-1},$$

for $\alpha \neq 0$.

Now using (4.2), the integral on the right hand side of (4.3) can be expressed as

$$(4.4) \quad \int_x^\infty dt\ t^{\alpha-\beta-1} (t+\beta)\, e^{-t} \left(1 - \frac{e^{-t}}{t^\beta}\right)^{\gamma-1} =$$

$$= \psi(\alpha+1,\beta,\gamma;x) - \psi(\alpha+1,\beta,\gamma-1;x) + \beta\left[\psi(\alpha,\beta,\gamma;x) - \psi(\alpha,\beta,\gamma-1;x)\right]$$

Finally, inserting (4.4) into (4.3) we get the recurrence formula (4.1).

Notice that for $\gamma = 1$, the relation (4.1) gives for any value of α and β:

$$(4.5) \quad \Gamma(\alpha-\beta+1,x) = x^{\alpha-\beta-1} e^{-x} + (\alpha-\beta)\Gamma(\alpha-\beta,x),$$

which is the well-known recurrence relation for the incomplete Γ-function [15].