1. INTRODUCTION

It is not often that we can formulate evolution equations pertinent to nonlinear phenomena admitting exact special solutions.

Among these equations the following are, for instance, of great interest both in physical applications and from a mathematical point of view (see review articles [1] , [2] and [3]):

i) The Korteweg-de Vries equation [4,5]

(1.1) $$u_t + u_x u + \beta u_{xxx} = 0$$

and its modified form [6]

(1.2) $$u_t + u_x u^2 + \beta u_{xxx} = 0,$$

where $u=u(x,t)$, β is the dispersive parameter [7] and subscripts indicate partial derivatives.

ii) The so-called sine-Gordon equation [8,1]

(1.3) $$u_{tx} = a \sin \sigma u,$$

and Liouville's equation [9] :

(1.4) $$u_{tx} = a\, e^{\sigma u},$$

where a and σ are constants.

All the aforesaid nonlinear differential equations afford exact self-similar solutions, i.e. solutions of the form $u=u(\xi)$, where $\xi = x + v t$ and v is a (real) constant.

When suitable asymptotic conditions on $u(\xi)$ are fulfilled, $u(\xi)$ is usually called a solitary wave solution [1] and plays a central role in many branches of science, as for example solid state and plasma physics, and biological systems ([1,3] and [10]).

Besides the abovementioned equations, in recent years other interesting, but analytically more intractable, evolution equations referred to nonlinear phenomena have been introduced [3], as the modified sine-Gordon equation

$$(1.5) \qquad u_{tx} = a \sin \sigma u + bu + c,$$

which is pertinent to the so-called massive Schwinger model (see [11] and references quoted therein).

In most cases, where (1.5) is a particular example, exact self-similar solutions for nonlinear evolution equations cannot be obtained in terms of known functions. However such solutions may be given sometimes provided that of course new functions are defined. But this is a ticklish question, since introducing new functions is generally satisfying only if their use goes beyond the specific context we are concerned with, that is at present the problem of finding special solutions of certain nonlinear differential equations.

Adopting this philosophy, in this paper we have introduced the new function

$$(1.6) \qquad \psi(\alpha,\beta,\gamma;x) = \int_x^\infty dt \; t^{\alpha-1} \left[1 - \left(1 - \frac{e^{-t}}{t^\beta}\right)^\gamma\right],$$

where α, β and γ are free parameters, which arises in a natural way when looking for an exact self-similar solution of the nonlinear wave equation

$$(1.7) \qquad u_{tx} = a\, e^{-u} + bu^{\beta-1},$$

a, b and β being constants, It should be remarked that (1.7) may be regarded as an extended form of Licuville's equation (1.4).

Notice that from (1.6) one has also

$$(1.8)\qquad \psi(\alpha,\beta,\gamma;x) = x^{\alpha}\int_1^{\infty} dz\ z^{\alpha-1}\{1-\left[1-\frac{e^{-xz}}{(xz)^{\beta}}\right]^{\gamma}\},$$

and

$$(1.9)\qquad \psi(\alpha,\beta,\gamma;x) = x^{\alpha}\int_0^{\infty} dy(1+y)^{\alpha-1}\{1-\left[1-\frac{e^{-x(1+y)}}{x^{\beta}(1+y)^{\beta}}\right]^{\gamma}\}.$$

These formulae will prove helpful later.

One of the main characteristics of the function (1.6), whicn has strongly affected the present investigation, is that of covering a series of both known and new special functions and certain functional relations connected with them. As for instance, when $\gamma = 1$ (1.6) reduces to the in - complete Gamma function [12] and for $\gamma = -1$, $\beta = 0$ (1.6) becomes the so-called Debye function (see §6). In other words, studying the properties of the function (1.6) means providing insight into the properties of <u>a whole family</u> of (old and new) special functions of physical and mathematical interest. Specifically, the aim of this work is to derive some of most significant relations concerning the function (1.6), laying stress on what these become when the parameters α, β and γ are suitably specialized.

For simplicity's sake, here we have assumed that α,β,γ and x are real, being understood that the latter is non-negative. Furthermore, we have restricted ourselves to consider only real values of $\psi(\alpha,\beta,\gamma;x)$. This implies that $e^{-t} < t^{\beta}$ for any t verifying $x \leq t < +\infty$. To this end, we show below that the integral (1.6) can also be extended to the interval $(0, +\infty)$ open on the left. We have the following

PROPOSITION 1.1. Suppose that $-e < \beta < 0$ and $\alpha > -|\beta|$. Then the integral

$$(1.10)\qquad \int_0^\infty dt\, t^{\alpha-1}\left[1-\left(1-\frac{e^{-t}}{t^{\beta}}\right)^{\gamma}\right],$$

exists for any (real) value of the parameter γ.

Proof. As we have previously said, here we are interested only in dealing with real values of the function (1.6). In order that this occurs for any γ, we should require that

$$(1.11)\qquad t^{-\beta}\, e^{-t} < 1.$$

Since the function $t^{-\beta}e^{-t}$ takes its maximum value at $t = -\beta > 0$, the limitation (1.11) implies that

$$(1.12)\qquad -e < \beta < 0.$$

Furthermore, since

$$t^{\alpha}\left[1-\left(1-t^{|\beta|}e^{-t}\right)^{\gamma}\right] \sim O(t^{\alpha+|\beta|}).$$

as $t \to 0^+$, the assertion is proved.

PROPOSITION 1.2. When $\beta = 0$, the integral (1.10) exists for $\gamma > 0, \alpha > 0$ and for $\gamma < 0, \alpha > |\gamma|$.

Proof. The first part of the lemma is obvious. The second part arises from

$$t^{\alpha}(1-e^{-t})^{-|\gamma|} \sim t^{\alpha-|\gamma|}\left[1+O(t)\right]$$

as $t \to 0^+$.

The extension of the function (1.6) and its basic relations to complex variables will be given elsewhere.

Let us close this section with a brief remark. As a wide class of known functions can be interpreted in the light of group theory (see, for example, [13]), so one might investigate whether the same happens for the function (1.6). We shall be concerned with this challenging prospect in the near future.

2. SELF-SIMILAR SOLUTIONS FOR THE NONLINEAR WAVE EQUATION $u_{tx} = ae^{-u} + bu^{\beta-1}$.

Consider the nonlinear partial differential equation (1.7) in 1 + 1 space-time coordinate system, where $u = u(x,t)$ and a, b and $\beta \neq 0$ are real parameters.

We shall look for self-similar solutions of (1.7). In doing so, let us put $u = u(\xi)$ in (1.7) where $\xi = x + vt$. Then (1.7) transforms into the ordinary differential equation (reduced form of (1.7)):

$$\frac{1}{2} v u_\xi^2 = -ae^{-u} + \frac{b}{\beta} u^\beta + c, \tag{2.1}$$

c being an integration constant.

By choosing $c = 0$ and

$$\frac{1}{2a} v = k > 0, \tag{2.2}$$

$$\frac{b}{\beta a} = 1, \tag{2.3}$$

(2.1) yields

$$(2.4)\qquad \lambda(\xi - \xi_o) = \pm \int_{u_o}^{u} dt\ t^{-\frac{\beta}{2}} \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}} ,$$

where $u_o \equiv u(\xi_o)$, $\lambda = k^{-\frac{1}{2}}$ and ξ_o is a constant. (In the following, we shall select the positive sign in front of the integral (2.4)).

First let us deal with the case $\beta \neq 2$. Using a simple trick, (2.4) reads

$$(2.5)\qquad \lambda\xi = \frac{2}{2-\beta}\ u^{\frac{2-\beta}{2}} + \int_{u}^{\infty} dt\ t^{-\frac{\beta}{2}} \left[1-\left(1-\frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}}\right] + \text{const.}$$

Taking account of (1.6), (2.5) can be written as

$$(2.6)\qquad \lambda\xi = \frac{2}{2-\beta}\ u^{\frac{2-\beta}{2}} + \psi\left(\frac{2-\beta}{2}, \beta, -\tfrac{1}{2}; u\right) + \text{const.}$$

At this stage it is instructive to treat some special cases of (1.7), namely:

a) Case $\beta = 2$.

Equations (1.7) and (2.1) become respectively:

$$(2.7)\qquad u_{tx} = ae^{-u} + bu ,$$

and

$$(2.8)\qquad \tfrac{1}{2}\ vu_{\xi}^{2} = -ae^{-u} + \tfrac{1}{2}\ bu^{2} + c.$$

From (2.8) we obtain for $c = 0$:

$$(2.9)\qquad \lambda(\xi - \xi_o) = \int_{u_o}^{u} dt\ t^{-1} \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{-\frac{1}{2}} ,$$

where $b = 2a$ and ξ_o is a constant.

Following the same procedure previously used, we are led to the expression

$$(2.10) \qquad \lambda\xi = \ell n\, u + \int_u^\infty dt\, t^{-1} \left[1 - \left(1 - \frac{e^{-t}}{t^2}\right)^{-\frac{1}{2}} \right] + \text{const.}$$

Equation (2.10) reads also

$$(2.11) \qquad \lambda\xi = \ell n\, u + \psi(0,2,-\tfrac{1}{2}\,;\, u) + \text{const,}$$

where $\psi(0,2,-\frac{1}{2};\, u)$ is defined by (1.6).

Remark 2.1. An explicit solution of the form (2.11) holds also for the equation

$$(2.12) \qquad w_{tx} = p\, e^{-w} + q\, w + r\,,$$

p, q and r being constants.

In fact, by carrying out the substitution $w = u - \frac{r}{q}$, (2.12) transforms into (2.7), where $a = p\, e^{\frac{r}{q}}$ and $b = q$.

b) Case $\beta = 1$.

Equations (1.7) and (2.1) read respecitvely

$$(2.13) \qquad u_{tx} = a\, e^{-u} + b$$

and

$$(2.14) \qquad \tfrac{1}{2}\, v\, u_\xi^2 = -a\, e^{-u} + bu + c.$$

The change of variable $u = w - \frac{c}{b}$ transforms (2.14) into the equation

$$(2.15) \qquad \tfrac{1}{2}\, v\, w_\xi^2 = -a\, e^{\frac{c}{b}}\, e^{-w} + bw.$$

Choosing then $c = b \ln \frac{b}{a}$, from (2.15) one has

$$(2.16) \qquad h\, w_{\xi}^{2} = w - e^{-w},$$

where $h = \frac{v}{2b}$.

Equation (2.16) provides

$$(2.17) \qquad \mu(\xi - \xi_{o}) = \int_{w_{o}}^{w} dt\; t^{-\frac{1}{2}} \left(1 - \frac{e^{-t}}{t}\right)^{-\frac{1}{2}},$$

where $\mu = h^{-\frac{1}{2}}$.

From (2.17) one gets

$$(2.18) \qquad \lambda\xi = 2w^{\frac{1}{2}} + \int_{w}^{\infty} dt\; t^{-\frac{1}{2}} \left[1 - \left(1 - \frac{e^{-t}}{t}\right)^{-\frac{1}{2}}\right] + \text{const},$$

where $w = u + \ln \frac{b}{a}$.

Finally, taking account of (1.6), (2.18) can be written as

$$(2.19) \qquad \lambda\xi = 2w^{\frac{1}{2}} + \psi(\tfrac{1}{2}, 1, -\tfrac{1}{2}; w) + \text{const}.$$

c) <u>Case</u> $b = 0$.

In this case (1.7) specializes to Liouville's equation (1.4), whilst (2.1) becomes

$$(2.20) \qquad k\, u_{\xi}^{2} = -e^{-u} + 1,$$

where k is given by (2.2) and c has been chosen equal to a.

Equation (2.20) yields

$$\lambda(\xi - \xi_o) = \int_{u_o}^{u} dt\ (1 - e^{-t})^{-\frac{1}{2}} . \tag{2.21}$$

The integral on the right of (2.21) can be expressed in terms of the function (1.6) as follows

$$\int_{u_o}^{u} dt\ (1 - e^{-t})^{-\frac{1}{2}} = u + \psi(1,0,-\tfrac{1}{2};u) - u_o - (1,0,-\tfrac{1}{2};u_o) . \tag{2.22}$$

Using (2.22), from (2.21) we obtain

$$\lambda\xi = u + \psi(1,0,-\tfrac{1}{2};u) + \text{const.} \tag{2.23}$$

The function $\Psi(1,0\ -\tfrac{1}{2};u)$ can be explicitly determined in terms of elementary functions. In fact, since

$$\int_{u_o}^{u} dt(1-e^{-t})^{-\frac{1}{2}} = -2\ \ell n\left[1-(1-e^{-u})^{\frac{1}{2}}\right] - u + 2\ \ell n\left[1-(1-e^{-u_o})^{\frac{1}{2}}\right] + u_o , \tag{2.24}$$

from (2.22) and (2.24) we obtain

$$\psi\ (1,0,-\tfrac{1}{2};u) = -2\ell n\left[1-(1-e^{-u})^{\frac{1}{2}}\right] - 2u + \text{const}, \tag{2.25}$$

where the constant on the right is given by

$$2 \lim_{t \to +\infty} \left\{ t + \ell n\left[1-(1-e^{-t})^{\frac{1}{2}}\right]\right\} = -2\ \ell n\ 2. \tag{2.26}$$

Now let us go back to (2.23). In view of (2.25) we have

$$(2.27) \qquad A\, e^{\lambda(\xi - \xi_o)} = \frac{e^{-u}}{\left[1-(1-e^{-u})^{\frac{1}{2}}\right]^2} ,$$

where

$$(2.28) \qquad A = \frac{e^{-u_o}}{\left[1-(1-e^{-u_o})^{\frac{1}{2}}\right]^2} .$$

Finally, by means of simple calculatinns, (2.27) allows us to obtain the following expression of u in terms of ξ :

$$(2.29) \qquad u = \ell n \frac{\left[1 + 2Ae^{\lambda(\xi - \xi_o)}\right]^2}{4\, Ae^{\lambda(\xi - \xi_o)}\left[1+e^{\lambda(\xi - \xi_o)}\right]} .$$

3. <u>SERIES REPRESENTATION OF</u> $\psi(\alpha,\beta,\gamma;x)$ <u>IN TERMS OF INCOMPLETE GAMMA FUNCTIONS</u>.

Let us consider the binomial expansion

$$(3.1) \qquad \left(1 - \frac{e^{-t}}{t^{\beta}}\right)^{\gamma} = 1 + \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} \frac{e^{-nt}}{t^{n\beta}} ,$$

where the series on the right is uniformly convergent for $t \geqslant x$, x being any fixed number such that $e^{-x} < x^{\beta}$.

We can thus write

$$(3.2) \ \psi(\alpha,\beta,\gamma;x) = - \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} \int_x^{\infty} dt\, t^{\alpha-n\beta-1} e^{-nt} .$$

Therefore. the use of the following integral representation for the incomplete Γ-function [14]

$$\mu^{\nu}\Gamma(\nu,\mu x) = \int_{x}^{\infty} dt\ t^{\nu-1} e^{-\mu t} ,$$

for $x > 0$ and $\mathrm{Re}\mu>0$, leads us to the expression

$$(3.3) \quad \psi(\alpha,\beta,\gamma;x) = -\frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} n^{n\beta-\alpha}\Gamma(\alpha-n\beta,nx).$$

Obviously, in the special case $\gamma = m$, where m is a positive integer, the expression (3.3) reduces to a finite sum of incomplete Γ-functions, specifically:

$$(3.4) \quad \psi(\alpha,\beta,m;x) = -\sum_{n=1}^{m} (-1^{n})\binom{m}{n} n^{n\beta-\alpha}\Gamma(\alpha-n\beta,nx).$$

4. A RECURRENCE RELATION.

The following recurrence relation holds:

$$(4.1) \quad (1-\frac{\gamma\beta}{\alpha})\ \psi(\alpha,\beta,\gamma;x) = -\frac{1}{\alpha}x^{\alpha}\left[1-(1-\frac{e^{-x}}{x^{\beta}})^{\gamma}\right] +$$

$$+\frac{\gamma}{\alpha}\left[\psi(\alpha+1,\beta,\gamma;x) - \psi(\alpha+1,\beta,\gamma-1;x)-\beta\psi(\alpha,\beta,\gamma-1;x)\right] ,$$

for $\alpha \neq 0$.

In fact, from (1.6) we can write

$$\psi(\alpha,\beta,\gamma;x) = \int_{x}^{\infty} dt\ t^{\alpha-1}\left[1-(1-\frac{e^{-t}}{t^{\beta}})(1-\frac{e^{-t}}{t^{\beta}})^{\gamma-1}\right],$$

which yields

$$(4.2)\quad \int_x^\infty dt\ t^{\alpha-\beta-1}e^{-t}\left(1-\frac{e^{-t}}{t^\beta}\right)^{\gamma-1} = \psi(\alpha,\beta,\gamma;x) - \psi(\alpha,\beta,\gamma-1;x).$$

Furthermore, integrating by parts we obtain from (1.6):

$$(4.3)\quad \psi(\alpha,\beta,\gamma;x) = -\frac{1}{\alpha}x^\alpha\left[1-\left(1-\frac{e^{-t}}{x^\beta}\right)^\gamma\right] +$$

$$+\frac{\gamma}{\alpha}\int_x^\infty dt\ t^{\alpha-\beta-1}(t+\beta)e^{-t}\left(1-\frac{e^{-t}}{t^\beta}\right)^{\gamma-1},$$

for $\alpha \neq 0$.

Now using (4.2), the integral on the right hand side of (4.3) can be expressed as

$$(4.4)\quad \int_x^\infty dt\ t^{\alpha-\beta-1}(t+\beta)\,e^{-t}\left(1-\frac{e^{-t}}{t^\beta}\right)^{\gamma-1} =$$

$$= \psi(\alpha+1,\beta,\gamma;x) - \psi(\alpha+1,\beta,\gamma-1;x) + \beta\left[\psi(\alpha,\beta,\gamma;x) - \psi(\alpha,\beta,\gamma-1;x)\right]$$

Finally, inserting (4.4) into (4.3) we get the recurrence formula (4.1).

Notice that for $\gamma = 1$, the relation (4.1) gives for any value of α and β:

$$(4.5)\quad \Gamma(\alpha-\beta+1,x) = x^{\alpha-\beta-1}e^{-x} + (\alpha-\beta)\Gamma(\alpha-\beta,x),$$

which is the well-known recurrence relation for the incomplete Γ-function [15].

5. SOME INTEGRALS INVOLVING $\psi(\alpha,\beta,\gamma, ;x)$.

In this section, we derive some integrals involving the function (1.6).

The following theorems hold:

THEOREM 5.1. *Let α_2, β and γ be (real) arbitrary parameters. Then the following relation holds:*

$$(5.1) \qquad \int_x^\infty dt\ t^{\alpha_1-1}\ \psi(\alpha_2,\beta,\gamma;t) \quad =$$

$$= -\frac{1}{\alpha_1}\left[x^{\alpha_1}\psi(\alpha_2,\beta,\gamma;x) - \psi(\alpha_1+\alpha_2,\beta,\gamma;x)\right],$$

for $\alpha_1 \neq 0$ *and* $x > 0$ *such that* $e^{-x} < x^\beta$.

Proof. The proof of (5.1) is easily obtained by integration by parts, and using the fact that

$$\lim_{t\to+\infty} t^{\alpha_1}\psi(\alpha_2,\beta,\gamma;t) = 0,$$

for $\alpha_1 > 0$.

Remark 5.2. From (5.1) one obtains for $\gamma = 1$:

$$(5.2) \quad \int_x^\infty dt\ t^{\alpha_1-1}\ \Gamma(\alpha_2-\beta,t) = -\frac{1}{\alpha_1}\left[x^{\alpha_1}\Gamma(\alpha_2-\beta,x) - \Gamma(\alpha_1+\alpha_2-\beta,x)\right],$$

which produces the well-known relation [16] for the incomplete Γ-function:

$$(5.3) \quad \int_0^\infty dt\ t^{\alpha_1-1}\Gamma(\alpha_2-\beta,t) = \frac{1}{\alpha_1}\ \Gamma(\alpha_1+\alpha_2-\beta),$$

for $\alpha_1 > 0$ and $\alpha_1+\alpha_2>\beta$.

Using THEOREM 5.1, on the basis of PROPOSITIONS 1.1 and 1.2 we are led to the following

COROLLARY 5.3. *Let* α *and* β *be such that* $-e < \beta < 0$ *and* $\alpha > -|\beta|$. *Then*

$$(5.4)\qquad \psi(\alpha,\beta,\gamma;0) = \int_0^\infty dt\ \psi(\alpha-1,\beta,\gamma;t) \quad ,$$

for any value of γ.

COROLLARY 5.4. *The relation (5.4) holds also when* $\beta = 0$, *provided that* $\gamma > 0,\ \alpha > 0$ *or* $\gamma < 0,\ \alpha > |\gamma|$.

THEOREM 5.5. *Assuming all the hypotheses of Theorem 5.1, then the following transform holds:*

$$\int_x^\infty dt\ e^{-nt}\ \psi(\alpha,\beta,\gamma;t) = \frac{1}{n}\ e^{-nx}\ \psi(\alpha,\beta,\gamma;x)$$

$$(5.5)\qquad - \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(\alpha+n\beta,\beta,\gamma+k;x) +$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k+j+1} \binom{n}{k}\binom{k}{j} j^{(j-n)\beta-\alpha}\ \Gamma\left(-(j-n)\beta+\alpha, jx\right),$$

where n is a positive integer.

Proof. Consider the function $e^{-nt}\ \psi(\alpha,\beta,\gamma;t)$, n being a positive integer, and integrate by parts from $x > 0$ to infinity. One has

$$\int_x^\infty dt\ e^{-nt}\ \psi(\alpha,\beta,\gamma;t) = \frac{1}{n}\ e^{-nx}\ \psi(\alpha,\beta,\gamma;x)$$

$$- \frac{1}{n} \int_x^\infty dt\ t^{\alpha-1}\ e^{-nt} \left[1-\left(1-\frac{e^{-t}}{t^\beta}\right)^\gamma\right] .$$

(5.6)

Now, by using the relation

$$(5.7)\qquad \frac{e^{-nt}}{t^{n\beta}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1-\frac{e^{-t}}{t^{\beta}}\right)^k ,$$

Eq. (5.6) becomes

$$(5.8)\qquad \int_x^\infty dt\, e^{-nt}\, \psi(\alpha,\beta,\gamma;t) = \frac{1}{n} e^{-nx}\, \psi(\alpha,\beta,\gamma;x)$$

$$- \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Big[\psi(\alpha+n\beta,\beta,\gamma+k;x) - \psi(\alpha+n\beta,\beta,k;x)\Big]$$

Since k is a nonnegative integer, we may express $\psi(\alpha+n\beta,\beta,k;x)$ as a finite sum of incomplete Γ-functions, namely

$$(5.9)\qquad \psi(\alpha+n\beta,\beta,k;x) = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} j^{(j-n)\beta-\alpha}\, \Gamma((n-j)\beta+\alpha, jx) .$$

Inserting (5.9) into (5.8), one achieves the result (5.5).

THEOREM 5.6

"<u>Suppose that the conditions</u> $-e < \beta < 0$ <u>and</u> $\alpha > -|\beta|$ <u>are valid. Then one has</u>

$$(5.10)\qquad \int_0^\infty dt\{-e^{-t}\psi(\alpha,\beta,\gamma;t) - \psi(\alpha-1,\beta,\gamma;t) + \psi(\alpha+\beta-1,\beta,\gamma;t)$$

$$-\psi(\alpha+\beta-1,\beta,\gamma+1;t)\} = \Gamma(\alpha),$$

<u>for any value of the parameter</u> γ".

Proof. By putting n=1 in (5.5), we obtain

$$\int_x^\infty dt\ e^{-t}\ \psi(\alpha,\beta,\gamma;t) = e^{-x}\ \psi(\alpha,\beta,\gamma;x)-\psi(\alpha+\beta,\beta,\gamma;x) +$$

$$+\psi(\alpha+\beta,\beta,\gamma+1;x)-\Gamma(\alpha,x). \tag{5.11}$$

In virtue of Proposition 1.1 the relation (5.11) is valid also when x=0. Using then (5.4) the assertion is proved.

6. SOME FUNCTIONS AND RELATIONS CONNECTED WITH THE ψ-FUNCTION.

a) "Case" $\gamma = 0$.

Obviously one has $\psi(\alpha,\beta,0;x) = 0$.

b) "Case" $\gamma = 1$.

For $\gamma = 1$ the function (1.6) specializes to the incomplete Γ-function. In fact, we have

$$\psi(\alpha,\beta,1;x) = \int_x^\infty dt\ t^{\alpha-\beta-1} e^{-t} = \Gamma(\alpha-\beta,x). \tag{6.1}$$

c) "Case" $\gamma = n$ (positive integer).

As we have already noted (see Sec.3), the function (1.6) can be expressed as a finite sum of incomplete Γ-functions.

d) "Case" $\gamma = -1$, $\alpha = n+1$, $\beta = 0$.

For $\gamma = -1$ the function (1.6) becomes

$$\psi(\alpha,\beta,-1;x) = -\int_x^\infty dt\ \frac{t^{\alpha-1}}{e^t\ t^\beta-1} \quad . \tag{6.2}$$

Putting in (6.2) $\alpha = n + 1$ (n positive integer) and $\beta = 0$ we get

$$\psi(n+1,\ 0,\ -1;x) = -D(n,\ x)\ , \tag{6.3}$$

where

$$D(n,x) = \int_x^\infty dt\ \frac{t^n}{e^t-1} \tag{6.4}$$

is a function introduced by Debye in his theory of specific heat of solids [18]. From now on, we shall call (6.4) <u>the incomplete Debye function.</u>

<u>Remark 6.1</u>. For $x = 0$ and $n \geq 1$ the function (6.3) becomes

$$\psi(n+1,0,-1;0) = -D(n,0) = -\int_0^\infty dt\ \frac{t^n}{e^t-1} = -n!\ \zeta(n+1)\ , \tag{6.5}$$

where $\zeta(z)$ is the Riemann zeta function.

More generally, from (6.2) we deduce that

$$\psi(\alpha,0,-1;0) = -\int_0^\infty dt\ \frac{t^{\alpha-1}}{e^t-1} = -\Gamma(\alpha)\zeta(\alpha), \tag{6.6}$$

for $\alpha > 1$.

<u>Remark 6.2</u>. We shall call <u>generalized incomplete Debye function</u>, the integral

$$\int_x^\infty dt \; \frac{t^{\alpha-1}}{e^t - 1} \quad , \tag{6.7}$$

which appears on the right-hand side of (6.2). Using the symbol $D(\alpha-1,\ x)$ to denote (6.7), we have

$$\psi(\alpha,\ 0,\ -1;\ x) = -D(\alpha,x) \quad , \tag{6.8}$$

from (6.2).

Remark (6.3). Let us point out that one is able to evaluate the sum of the series on the right of (3.3), for any $\gamma = -n$, which is also an arbitrary negative integer, in terms of a combination of incomplete Debye functions and other known functions. In the special case $\gamma = -1$, taking into account (6.8) we obtain

$$D(\alpha-1,\ x) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha,nx)}{n^\alpha} \quad , \tag{6.9}$$

for each $x > 0$, from (3.3).

Furthermore, in view of (6.6) and (6.8), from (3.3) we find the known expansion for the Riemann zeta function:

$$\zeta(\alpha) \equiv -\frac{1}{\Gamma(\alpha)}\,\psi(\alpha,0,-1;0) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad , \tag{6.10}$$

for $\alpha > 1$.

To conclude the case d), we notice that (5.5), for

$\gamma = -1$, $\beta = 0$, $x = 0$ and $\alpha = n + 1$ (n positive integer) provides an integral representation for the finite sum $\sum_{j=1}^{m} \frac{1}{j^{n+1}}$, in terms of the incomplete Debye function (6.4), namely

$$\sum_{j=1}^{m} \frac{1}{j^{n+1}} = \frac{m}{n!}\int_0^{\infty} dt\ e^{-mt} D(n,t).$$

e) "Case" $\gamma = -\frac{1}{2}$, $\beta = 0$.

For $\beta = 0$ and $\gamma = -\frac{1}{2}$, it also exists the integral on the right of (1.6) for any $\alpha > +\frac{1}{2}$ when $x = 0$. Further-more using the series expansion (3.1) one has

$$\psi(\alpha,0,-\tfrac{1}{2};0) = -\Gamma(\alpha) \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{n^{\alpha}} . \tag{6.11}$$

If we now define the function

$$Z(\alpha) = -\frac{1}{\Gamma(\alpha)} \psi(\alpha,0,-\tfrac{1}{2};0), \tag{6.12}$$

the relation (6.11) gives

$$Z(\alpha) = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{n^{\alpha}} ,$$

for $\alpha > \frac{1}{2}$.

f) "Case" $\beta = 0$ and $\alpha > \max(0, -\gamma)$ $(\gamma \neq 0,1,2,\ldots)$.

Both the series on the right of (6.10) and (6.13) can be considered as special cases of the more general series:

$$(6.14) \qquad \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{\Gamma(-\gamma)n!} \quad \frac{1}{n^{\alpha}} ,$$

which converges for any $\alpha > \max(0, -\gamma)$. From (3.3), we deduce that the sum of this series is given by the function

$$- \frac{1}{\Gamma(\alpha)} \quad \psi(\alpha,0,\gamma;0).$$

In order to show that the series (6.14) is convergent, let us determine the asymptotic expansion of $\frac{\Gamma(n-\gamma)}{n!}$ for large n. In doing so, it is enough to recall that [19]

$$(6.15) \qquad \Gamma(az+b) \sim \sqrt{2\pi}\, e^{-az}(az)^{az+b-\frac{1}{2}} ,$$

for $z \to \infty$, $|\arg z| < \pi$ and $a > 0$.

Using (6.15), we thus have

$$(6.16) \qquad \frac{\Gamma(n-\gamma)}{n!} \sim O(n^{-\gamma-1}) ,$$

for large values of n.

Therefore, the convergence of the series (6.14) is assured if $\alpha > -\gamma$.

g) "A functional relation for the polygamma functions".

The properties of the function $\psi(\alpha,\beta,\gamma;x)$ defined according to (1.6) can be usefully exploited in order to re-derive a well-known functional relation for the polygamma functions:

$$(6.17) \qquad \psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ell n \quad \Gamma(x) \ ,$$

where $n = 1,2,3, \ldots$ and $x \neq 0,-1,-2, \ldots$.

More specifically, we will show that

PROPOSITION 6.5. "<u>The following functional relation holds</u>:

$$(6.18) \quad \psi^{(n)}(m+1) = (-1)^n n! \ \{-\zeta(n+1) + \sum_{j=1}^{m} \frac{1}{j^{n+1}}\} \ ,$$

<u>where</u> m <u>is a non-negative integer and</u> $\zeta(n+1)$ <u>is the zeta Riemann function</u>".

In doing so, let us start off with the integral representation [20]

$$(6.19) \quad \psi^{(n)}(m+1) = (-1)^{n+1} \int_0^\infty dt \ \frac{t^n \ e^{-(m+1)t}}{1- \ e^{-t}} \ .$$

Since

$$(6.20) \qquad \frac{d}{dt} \ \psi(\alpha,0,-1;t) = \frac{t^{\alpha-1} \ e^{-t}}{1- \ e^{-t}} \ ,$$

from (6.19) we have

$$(6.21) \quad \psi^{(n)}(m+1) = (-1)^{n+1} \int_0^\infty dt \ e^{-mt} \frac{d}{dt} \ \psi(n+1,0,-1;t) \ ,$$

for $\alpha = n+1$.

Integrating by parts, Eq.(6.21) yields

$$(6.22) \quad \psi^{(n)}(m+1)=(-1)^{n+1}\left[-\psi(n+1,0,-1;0)+m\int_0^\infty dt \ e^{-mt}\psi(n+1,0,-1;t)\right]$$

In virtue of corollary 5.4 and theorem 5.5, the integral on the right of (6.22) reads

$$(6.23)\qquad m\int_0^\infty dt\ e^{-mt}\ \psi(n+1,0,-1;t) =$$

$$= \psi(n+1,0,-1;0) - \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left[\psi(n+1,0,k-1;0)-\psi(n+1,0,k;0)\right].$$

If we set apart the term of (6.23) corresponding to k=0, having in mind that $\psi(n-1,0,0;0) = 0$ and resorting to the recurrence relation (4.1) for $\alpha = n$ and $\gamma = k$, Eq.(6.23) reads

$$(6.24)\qquad m\int_0^\infty dt\ e^{-mt}\psi(n+1,0,-1;t)= \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{n}{k}\ \psi(n,0,k;0).$$

Since (see (3.4))

$$(6.25)\qquad \psi(n,0,k;0) = (n-1)!\ \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{j^n}\ ,$$

Eq. (6.24) becomes

$$(6.26)\qquad m\int_0^\infty dt\ e^{-mt}\ \psi(n+1,0,-1;t) =$$

$$= n!\ \sum_{k=1}^{m} \frac{(-1)^k}{k} \binom{m}{k} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{j^n}\ .$$

By interchanging the summations in (6.26), with the help of the identity

$$\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1} ,$$

we are led to the expression

$$(6.27)\quad m\int_0^\infty dt\ e^{-mt}\psi(n+1,0,-1;t) =$$

$$= -n! \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j^{n+1}} \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1}$$

At this point, we need to show two lemmas, namely:

Lemma 6.5

"Suppose that j and m are positive integers such that $1 \leq j \leq m - 1$. Then one has

$$(6.28)\qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = 0."$$

Proof. Notice that

$$(6.29)\quad \binom{m}{k}\binom{k}{j} = \binom{m}{j}\binom{m-j}{k-j} .$$

As a consequence, we can write

$$(6.30)\qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = \binom{m}{j} \sum_{k=j}^{m} (-1)^{k+1}\binom{m-j}{k-j} ,$$

from which, by putting $h = k-j$ and taking into account the hypothesis $m - j \geq 1$, one finally gets

$$(6.31) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j} = (-1)^{j+1}\binom{m}{j}\sum_{h=0}^{m-j}(-1)^{h}\binom{m-j}{h} = 0.$$

Lemma 6.6

"Let j and m be any pair of positive integers such that $1 \le j \le m$. Then one has

$$(6.32) \qquad \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1} = (-1)^{j+1}"$$

Proof. By putting

$$(6.33) \qquad f(j) = \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k-1}{j-1},$$

for convenience, we can write

$$(6.34) \quad f(j)+f(j+1) = (-1)^{j+1}\binom{m}{j} + \sum_{k=j+1}^{m} (-1)^{k+1}\binom{m}{k}\left[\binom{k-1}{j-1}+\binom{k-1}{j}\right]$$

$$= \sum_{k=j}^{m} (-1)^{k+1}\binom{m}{k}\binom{k}{j},$$

where we have used the identity

$$(6.35) \qquad \binom{k-1}{j-1} + \binom{k-1}{j} = \binom{k}{j}.$$

On the basis of Lemma 6.5 from Eq. (6.34) we find

$$(6.36) \qquad f(j) + f(j+1) = 0,$$

for $1 \le j \le m-1$.

Since f(1)=1, Eq. (6.36) tells us that

$$f(j) = (-1)^{j+1} , \tag{6.37}$$

where $j = 1,2,\ldots, m-1$.

We complete the proof observing that the relation (6.37) also holds for j=m. In fact, putting j=m-1 we have from (6.36) and (6.37):

$$f(m) = -f(m-1) = (-1)^{m+1}.$$

Now let us go back to Eq. (6.28). Using the result (6.32), Eq. (6.28) becomes

$$m\int_0^\infty dt\ e^{-mt}\ \psi(n+1,0,-1;t) = -n!\ \sum_{j=1}^{m} \frac{1}{j^{n+1}} . \tag{6.38}$$

Then, making the substitution (6.38) into Eq. (6.23), we obtain

$$\psi^{(n)}(m+1)=(-1)^n\left[\psi(n+1,0,-1;0) + n!\sum_{j=1}^{m}\frac{1}{j^{n+1}}\right]. \tag{6.39}$$

Recalling that (see (6.10)):

$$\zeta(n+1) = -\frac{1}{n!}\ \psi(n+1,0,-1;0) ,$$

Eq. (6.39) finally produces the relation (6.18).

Remark 6.7

Using the identity

$$\binom{k}{j} = \frac{k}{j}\binom{k-1}{j-1} ,$$

Eq. (6.32) reads

$$(6.40) \qquad \sum_{k=j}^{m} \frac{(-1)^{k+1}}{k} \binom{m}{k}\binom{k}{j} = \frac{(-1)^{j+1}}{j} .$$

By putting h=k-j into (6.40), with the help of (6.29) one has

$$(6.41) \qquad (-1)^{j+1}\binom{m}{j} \sum_{h=0}^{m-j} (-1)^{h} \frac{1}{h+j} \binom{m-j}{h} = \frac{(-1)^{j+1}}{j} .$$

By putting in (6.41) n = m-j, we are led to the formula

$$(6.42) \qquad \sum_{h=0}^{n} (-1)^{h} \frac{1}{h+j} \binom{n}{h} = \frac{1}{j} \frac{1}{\binom{n+j}{j}} ,$$

which may be considered as a generalization of the well-known formula

$$\sum_{h=0}^{n} \frac{(-1)^{h}}{h+1} \binom{n}{h} = \frac{1}{n+1} ,$$

deducible from (6.42) when j = 1.

7. ASYMPTOTIC BEHAVIOUR OF $\Psi(\alpha,\beta,\gamma;x)$ for $x \to \infty$.

In order to derive the asymptotic behaviour of $\Psi(\alpha,\beta,\gamma;x)$ for large x, we recall for the sake of convenience the following generalization of Poincaré's definition of a asymptotic expansion [21] :

DEFINITION 7.1 " A sequence $\{\phi_s(x)\}$ of functions such that

$$(7.1) \qquad \frac{\phi_{s+1}(x)}{\phi_s(x)} \to 0$$

for $x \to +\infty$ and any $s = 0,1,2, \ldots,$ is called an <u>asymptotic sequence or scale</u> as $x \to +\infty$.

Consider now a scale $\{\phi_s(x)\}$ as $x \to +\infty$, and let $f(x)$, $f_n(x)$ $(n = 0,1,2,\ldots)$ be functions such that for every non-negative integer N, the quantity

$$(7.2) \qquad \left| \frac{f(x) - \sum_{s=0}^{N-1} f_s(x)}{\phi_N(x)} \right|$$

is bounded for $x \to +\infty$.

Then the series $\sum_{s=0}^{\infty} f_s(x)$ is said to be a <u>generalized asymptotic expansion</u> with respect to the scale $\{\phi_s(x)\}$, and one writes

$$(7.3) \qquad f(x) \sim \sum_{s=0}^{\infty} f_s(x); \quad \{\phi_s(x)\} \quad \text{as} \quad x \to +\infty \ . "$$

The following theorem holds:

THEOREM 7.2 "The function

$$(7.4) \qquad \psi(\alpha,\beta,\gamma;x) = \int_x^\infty dt\ t^{\alpha-1}\left[1-\left(1-\frac{e^{-t}}{t^\beta}\right)^\gamma\right], \quad (x>0)$$

admits the asymptotic behaviour

$$(7.5) \qquad \psi(\alpha,\beta,\gamma;x) \sim x^{\alpha-\beta-1}\ e^{-x}\sum_{s=0}^{\infty} A_s(x)\ \frac{1}{x^s},$$

for $x \to +\infty$, in the generalized sense of Poincaré (see Def. 7.1) with respect to the scale $\{\frac{1}{x^s}\}$, where

$$(7.6) \quad A_s(x) = (-1)^{s+1}\ \frac{1}{\Gamma(-\gamma)}\sum_{m=0}^{\infty}\frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1)e^{-mx}}{(m+1)!(m+1)^{s+1}\Gamma(-\alpha+(m+1)\beta+1)x^{m\beta}}$$

the series on the right of (7.6) being uniformly convergent for any $x > \bar{x}$, where $\bar{x}$ is such that the inequality $e^{-x} < x^\beta$ is verified".

The proof of this theorem will be obtained with the help of a few lemmas. More specifically;

Lemma 7.3. " The following inequality holds:

$$(7.7) \qquad \left|(1+y)^{\alpha-1} - \sum_{m=1}^{N-1}(-1)^{m-1}\ \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)}\ \frac{y^{m-1}}{(m-1)!}\right| \le$$

$$\le \frac{1}{(N-2)!}\left|\frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)}\right| y^{N-1}\ e^{|\alpha-N|y},$$

where $y \ge 0$ ".

Proof. Recall first that, as is known, if $f(y)$ is a function having continuous derivatives up to the $(N-1)$-th order enclosed, then

$$(7.8)\qquad f(y) - \sum_{k=0}^{N-2} \frac{f^{(k)}(0)}{k!} y^k = \frac{1}{(N-2)!} \int_0^y f^{(N-1)}(t)(y-t)^{N-2} dt.$$

If we deal with the case $f(y) = (1+y)^{\alpha-1}$ and put $k=m-1$, Eq. (7.8) gives rise to the inequality

$$(7.9)\qquad \left| (1+y)^{\alpha-1} - \sum_{m=1}^{N-1} (-1)^{m-1} \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{y^{m-1}}{(m-1)!} \right| \le$$

$$\le \frac{1}{(N-2)!} \int_0^y dt \left| f^{(N-1)}(t) \right| |y-t|^{N-2} ,$$

where

$$(7.10)\qquad f^{(N-1)}(t) = (-1)^{N-1} \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} (1+t)^{\alpha-N}.$$

Since $1+t \le e^t$ for any t such that $0 \le t \le y$, the assertion comes out immediately from (7.9).

Lemma 7.4. " Let N be a positive integer such that $N \geqslant 2$. Then

$$(7.11)\qquad \left| \Gamma(\alpha,x)x^{-\alpha}e^{x} - \sum_{m=1}^{N-1} (-1)^{m-1} \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{x^m} \right| \le$$

$$\le (N-1) \left| \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} \right| \frac{1}{\left[x-|\alpha-N|^N\right]} ,$$

for every $x > | \alpha - N |$".

Proof. Consider the Laplace transform

$$(7.12) \qquad \int_0^\infty dz \; e^{-zx} f(z)$$

where

$$(7.13) \quad f(z) = (1+z)^{\alpha-1} - \sum_{m=1}^{N-1} (-1) \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{(m-1)!} z^{m-1} .$$

In virtue of Lemma 7.3 one has

$$(7.14) \quad \left| \int_0^\infty dz \; e^{-zx} f(z) \right| \leq (N-1) \left| \frac{\Gamma(N-\alpha)}{\Gamma(1-\alpha)} \right| \frac{1}{\left[x - |\alpha - N| \right]^N} ,$$

for $x > | \alpha - N |$.

On the other hand, we can write

$$(7.15) \quad \int_0^\infty dz \; e^{-zx} f(z) = x^{-\alpha} e^{x} \Gamma(\alpha, x) - \sum_{m=1}^{N-1} (-1)^m \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \frac{1}{x^m} ,$$

where

$$(7.16) \qquad (\alpha, x) = x^{\alpha} e^{-x} \int_0^\infty dz \; e^{-zx} (1+z)^{\alpha-1} .$$

The Lemma follows then from (7.15) and (7.14).

Lemma 7.5. "Let N be a positive integer such that $N \geq 2$ and

$$(7.17) \quad A_s(x) = (-1)^{s+1} \frac{1}{\Gamma(-\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1) e^{-mx}}{(m+1)!(m+1)^{s+1} \Gamma(1-\alpha+(m+1)\beta) x^{m\beta}} ,$$

where α, β and γ are fixed parameters.

Then, if ε is any arbitrary positive number,

$$|x^{N-1}\{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha+1}e^{x} \sum_{s=0}^{N-2} A_s(x)\frac{1}{x^s}\}| \leq$$

(7.18)

$$\leq \frac{(N-1)(1+\varepsilon)^N}{|\Gamma(-\gamma)|} \sum_{m=0}^{\infty} \frac{|\Gamma(m+1-\gamma)||\Gamma(N-\alpha+(m+1)\beta)}{(m+1)!(m+1)^N|\Gamma(1-\alpha+(m+1)\beta)|} \frac{e^{-mx}}{x^{m\beta}} ,$$

for $x > [|\beta|+|\alpha-N|] \frac{1+\varepsilon}{\varepsilon}$, where $\psi(\alpha,\beta,\gamma;x)$ is defined by (1.6)".

Proof. Consider the integral representation of $\psi(\alpha,\beta,\gamma;x)$ as given by (1.9).

One has

$$(7.19) \qquad (1-Y)^{\gamma} = 1 + \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} Y^n ,$$

where

$$(7.20) \qquad Y \equiv \frac{e^{-x(1+y)}}{x^{\beta}(1+y)^{\beta}} < 1 ,$$

for each $y > 0$.

Since the series on the right of (7.19) converges uniformly for $|Y| \leq 1-\varepsilon$ (ε being such that $0 < \varepsilon < 1$), from (1.9) one gets integrating term by term

$$\psi(\alpha,\beta,\gamma;x) = -\frac{x^{\alpha}}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} \frac{e^{-nx}}{x^{n\beta}} \int_0^{\infty} dt(1+y)^{\alpha-1-n\beta}e^{-nxy} =$$

$$= - \frac{1}{\Gamma(-\gamma)} \sum_{n=1}^{\infty} \frac{\Gamma(n-\gamma)}{n!} n^{-\alpha+n\beta} \Gamma(\alpha-n\beta, nx) \quad ,$$

where the representation (7.16) has been used.

Putting $m = n-1$, Eq. (7.21) can be expressed as

$$\psi(\alpha,\beta,\gamma;x) = - \frac{1}{\Gamma(-\gamma)} x^{\alpha-\beta} e^{-x} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!} x^{-m\beta} e^{-mx} \quad \cdot$$

(7.22)

$$\cdot \quad \{ [(m+1)x]^{-\alpha+(m+1)\beta} e^{(m+1)x} \Gamma(\alpha-(m+1)\beta, (m+1)x) \} .$$

Using now (7.22) and recalling (7.17), we can write

$$\psi(\alpha,\beta,\gamma;x) x^{\beta-\alpha+1} e^{x} - \sum_{s=0}^{N-2} A_s(x) \frac{1}{x^s} =$$

$$= - \frac{1}{\Gamma(-\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!} x^{-m\beta+1} e^{-x} \quad \cdot$$

(7.23)

$$\cdot \quad \{ [(m+1)x]^{-\alpha+(m+1)\beta} e^{(m+1)x} \Gamma(\alpha-(m+1)\beta, (m+1)x) -$$

$$- \sum_{s=0}^{N-2} (-1)^s \frac{\Gamma(s-\alpha+(m+1)\beta+1)}{\Gamma(1-\alpha+(m+1)\beta) [(m+1)x]^{s+1}} \} .$$

Multiplying both sides of (7.23) by x^{N-1}, with the help of Lemma 7.4 one obtains the inequality (see Remark 7.6):

$$\left| x^{N-1}\{\psi(\alpha,\beta,\gamma;x)x^{\beta-\alpha-1}-\sum_{s=0}^{N-2}A_s(x)\ \frac{1}{x^s}\}\right| \leq$$

$$\leq \frac{(N-1)}{|\Gamma(-\gamma)|}\sum_{m=0}^{\infty}\frac{|\Gamma(m+1-\gamma)|}{(m+1)!}\ e^{-m\beta}\ e^{-mx} \quad .$$

(7.24)

$$\cdot\left|\frac{\Gamma(N-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)}\right| \frac{x^N}{[(m+1)x-|\alpha-(m+1)\beta-N|]^N} \quad ,$$

for $x > |\beta| + |\alpha-N|$.

Now we notice that

$$(7.25)\qquad \frac{x^N}{[(m+1)x-|\alpha-(m+1)\beta-N|]^N} \leq \frac{1}{(m+1)^N}\left[1+\frac{(m+1)|\beta|+|\alpha-N|}{(m+1)(x-|\beta|)-|\alpha-N|}\right]$$

Furthermore, for any $\varepsilon > 0$ there esists a value of x, say x_ε, such that for any $\underline{m}$:

$$(7.26)\qquad \frac{(m+1)|\beta|+|\alpha-N|}{(m+1)(x-|\beta|)-|\alpha-N|} < \varepsilon \quad ,$$

for each $x > x_\varepsilon$.

In fact, the validity of (7.26) is assured for any m whenever $x > x_\varepsilon$, where

$$(7.27)\qquad x_\varepsilon = [|\beta| + |\alpha-N|]\ \frac{1+\varepsilon}{\varepsilon} \quad .$$

Finally, Lemma (7.5) follows from (7.24) after taking into account (7.25) and (7.26).

Remark 7.6. The use of Lemma 7.4 in deriving the result (7.24) implies the evaluation of the Laplace transform

$$\int_0^\infty dt\, e^{-\lambda t}\, t^{N-1},$$

where $\lambda = (m+1)x - |\alpha-(m+1)\beta-N|$, which exists if and only if $(m+1)x > |\alpha-(m+1)\beta-N|$ for any m.

Since $|\alpha-(m+1)\beta-N| \le (m+1)|\beta| + |\alpha-N|$, we have

$$(m+1)x-|\alpha-(m+1)\beta-N| \ge (m+1)(x-|\beta|)-|\alpha-N|.$$

Thus we need to require that $(m+1)(x-|\beta|)-|\alpha-N| > 0$ for any m, the latter being satisfied when $x > |\beta|+|\alpha-N|$.

Lemma 7.6. "The series

$$(7.28)\qquad \sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)\Gamma(s-\alpha+(m+1)\beta+1)x^{-m\beta}e^{-mx}}{(m+1)!(m+1)^{s+1}\Gamma(1-\alpha+(m+1)\beta)},$$

which defines the function $(-1)^{s+1}\Gamma(-\gamma)A_s(x)$, converges absolutely and uniformly for any x greater than a certain $\bar{x}$ satisfying the inequality $e^{-x} < x^{\beta}$".

Proof. Since

$$(7.29)\qquad \frac{1}{(m+1)^{s+1}} \quad \frac{\Gamma(s+1-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \sim O(m^{-1})$$

as $m \to +\infty$, from a certain value of m, say m_0, onwards it turns out that

$$(7.30)\qquad \frac{1}{(m+1)^{s+1}} \left| \frac{\Gamma(s+1-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| < 1.$$

Hence

$$\frac{|\Gamma(m+1-\gamma)|\,|\Gamma(s+1-\alpha+(m+1)\beta)|\, x^{-m\beta} e^{-mx}}{(m+1)!\,(m+1)^{s+1}|\Gamma(1-\alpha+(m+1)\beta)|} \leq$$

(7.31)

$$\leq \quad \frac{|\Gamma(m+1-\gamma)|}{(m+1)!} \; e^{-mx} x^{-m\beta} \, ,$$

for $m \geq m_o$.

From (7.31) one deduces that the series (7.28) is majorised by

$$\sum_{m=m_o}^{\infty} \frac{\Gamma\,(m+1-\gamma)}{(m+1)!} \; e^{-mx} \; x^{-m\beta} . \tag{7.32}$$

Recall now that (see (3.1))

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+1-\gamma)e^{-mx}x^{-m\beta}}{(m+1)!} = \Gamma(-\gamma)\left[\left(1 - \frac{e^{-x}}{x^{\beta}}\right)^{\gamma} - 1\right] \; x^{\beta} e^{x} , \tag{7.33}$$

being the series on the left absolutely and uniformly convergent for any $x > \bar{x}$, where $\bar{x}$ is a certain value verifying the inequality $e^{-x} < x^{\beta}$. The assertion arises therefore from the fact that (7.32) is the m_o-th remainder of the series appearing in (7.33).

Lemma 7.7. "The series

$$(7.34)\quad \sum_{m=o}^{\infty} \frac{\Gamma(m+1-\gamma)}{(m+1)!(m+1)^N} \left| \frac{\Gamma(N-\alpha+(m+1)\beta)}{\Gamma(1-\alpha+(m+1)\beta)} \right| x^{-m\beta} e^{-mx},$$

which appears on the right of (7.18), converges uniformly for any x greater than a certain $\bar{x}$ verifying the inequality $e^{-x} < x$. Furthermore, one has

$$(7.35)\quad \left| x^{N-1} \{\psi(\alpha,\beta,\gamma;x) x^{\beta-\alpha+1} e^x - \sum_{s=o}^{N-2} A_s(x) \right| \frac{1}{x^s} \to \text{const},$$

as $x \to +\infty$, $A_s(x)$ being defined by (7.17) and

$$(7.36)\qquad \text{const} \le |\gamma|(N-1)(1+\varepsilon)^N \left| \frac{\Gamma(N-\alpha+\beta)}{\Gamma(1-\alpha+\beta)} \right| ,$$

where $N \ge 2$ and ε is any arbitrary positive number".

Proof. The first part of the lemma follows directly from Lemma (7.28).

As a consequence, the results (7.35) and (7.36) arise immediately from (7.18).

In virtue of the series of lemmas from (7.3) to (7.7), the basic Theorem 7.2 is thus completely proved.

8. SOME SPECIAL CASES.

a) "Asymptotic expansion of the incomplete Γ-function".

The expression (7.6) can be written as

$$A_s(x) = (-1)^{s+1} \{(-\gamma) \frac{\Gamma(s-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)} +$$

(8.1)

$$+ (1-\gamma)(-\gamma) \quad \frac{\Gamma(s-\alpha+\beta+1)\ e^{x}}{2!2^{s+1}\Gamma(-\alpha+2\beta+1)\ x^{\beta}} \quad + \ldots.\} \qquad ,$$

where the relation

$$(8.2) \qquad \frac{\Gamma(m+1-\gamma)}{\Gamma(-\gamma)} = (m-\gamma)(m-1-\gamma)\ \ldots\ (2-\gamma)(1-\gamma)(-\gamma)$$

has been employed.

Putting $\gamma=1$ into (8.1) and using the symbol

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} ,$$

we obtain

$$(8.3) \qquad A_s(x) = (-1)^s\ (1-\alpha+\beta)_s .$$

Then Eq. (7.5) becomes

$$(8.4) \quad \psi(\alpha,\beta,1;x) \equiv \Gamma(\alpha-\beta,x) \sim x^{\alpha-\beta-1} e^{-x} \sum_{s=0}^{\infty} (-1)^s \frac{(1-\alpha+\beta)_s}{x^s} ,$$

which gives the well-known asymptotic expansion for the incomplete Γ-function for fixed $(\alpha-\beta)$ and large x [22].

b) "Asymptotic expansion of the incomplete Debye function".

Let us remember that for $\beta=0$ and $\gamma=-1$, the function $-\psi(\alpha,\beta,\gamma;x)$ reduces to the incomplete Debye function $D(\alpha-1,x)$ as given by (6.8).

In this case, from (7.6) one gets

$$(8.5)\qquad A_s(x) = (-1)^{s+1} \frac{\Gamma(s-\alpha+1)}{\Gamma(1-\alpha)} \sum_{m=o}^{\infty} \frac{e^{-mx}}{(m+1)^{s+1}} .$$

Recalling now the function [23]

$$(8.6)\qquad \Phi(z,s,v) = \sum_{m=o}^{\infty} \frac{z^m}{(m+v)^s} ,$$

defined for $|z|<1$, $v\neq 0,-1,-2, \ldots$, the series on the right of (8.5) can be expressed by $\Phi(e^{-x}, s+1, 1)$.

Therefore (8.5) becomes

$$(8.7)\qquad A_s(x) = (-1)^{s+1}(1-\alpha)_s \Phi(e^{-x}, s+1, 1).$$

Taking account of (8.7), (7.5) specializes to

$$(8.8)\qquad D(\alpha-1,x) \sim x^{\alpha-1} e^{-x} \sum_{s=0}^{\infty} (-1)^s (1-\alpha)_s \frac{\Phi(e^{-x}, s+1,1)}{x^s}$$

for fixed α and large values of $x>0$.

Finally, let us point out that since [23] :

$$(8.9)\qquad \Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_o^{\infty} dt \frac{t^{s-1} e^{-vt}}{1-z e^{-t}} ,$$

for Re $v > 0$, the relation (8.8) can also be written as

$$(8.10) \quad D(\alpha-1,x) \sim x^{\alpha-1}e^{-x}\sum_{s=0}^{\infty}(-1)^s \frac{(1-\alpha)_s}{s!x^s}\int_0^{\infty}dt \; \frac{t^s e^{-t}}{1-e^{-(x+t)}} \quad .$$

9. SOME FUNCTIONAL RELATIONS.

It is easy to show that

$$(9.1) \quad \begin{aligned} \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma;x)\right] = -x^{-(\rho+1)}\Big\{&(\rho-\alpha+\gamma\beta)\Psi(\alpha,\beta,\gamma;x) + \\ &+\gamma\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right]\Big\}. \end{aligned}$$

In fact, we have

$$(9.2) \quad \frac{d}{dx}\left[x^{-\rho}\Psi(\alpha,\beta,\gamma,x)\right] = -x^{-(\rho+1)}\left\{\rho\Psi(\alpha,\beta,\gamma;x)+x^{\alpha}\left[1-\left(1-\frac{e^{-x}}{x\beta}\right)^{\gamma}\right]\right\}.$$

The result (9.1) is thus achieved with the help of the recurrence relation (4.5).

Putting $\rho = \alpha-\gamma\beta$, Eq. (9.2) takes the form

$$(9.3) \quad \begin{aligned} \frac{d}{dx}\left[x^{-(\alpha-\gamma\beta)}\Psi(\alpha,\beta,\gamma;x)\right] = -\gamma x^{-(\alpha-\gamma\beta+1)}\cdot \\ \cdot\left[\Psi(\alpha+1,\beta,\gamma;x)-\Psi(\alpha+1,\beta,\gamma-1;x)-\beta\Psi(\alpha,\beta,\gamma-1;x)\right], \end{aligned}$$

which for γ = 1 becomes the well-known functional relation for the incomplete Γ-function :

$$\frac{d}{dx}\left[x^{-(\alpha-\beta)}\Gamma(\alpha-\beta,x)\right] = -x^{-(\alpha-\beta+1)}\Gamma(\alpha-\beta+1,x),$$

where $\Gamma(\alpha-\beta, x) \equiv \Psi(\alpha, \beta, 1; x)$,

and $\Psi(\alpha, \beta, 0; x) = 0$.

Following the same procedure used in deriving Eq. (9.1) one can also demonstrate the more complicated relation

$$
\begin{aligned}
\frac{d^2}{dx^2}\left[x^{-(\alpha-\gamma\beta)}\Psi(\alpha, \beta, \gamma; x)\right] = \gamma x^{-(\alpha-\beta\gamma+1)}\big[\gamma\Psi(\alpha+2, \beta, \gamma; x) - (2\gamma-1)\Psi(\alpha+2, \beta, \gamma-1,\\
-2\beta(\gamma-1)\Psi(\alpha+1, \beta, \gamma-1; x) + (\gamma-1)(\beta+1)\Psi(\alpha+2, \beta, \gamma-2; x) + \qquad (9.4)\\
+\beta(\beta-1)\Psi(\alpha, \beta, \gamma-1; x) + 2\beta(\gamma-1)\Psi(\alpha+1, \beta, \gamma-2; x) + \beta^2(\gamma-1)\Psi(\alpha, \beta, \gamma-2; x)\big].
\end{aligned}
$$

One could at this point look for a general expression for the n-th derivative of the function $x^{-(\alpha-\beta)}\Psi(\alpha, \beta, \gamma; x)$ with respect to x. This task is quite cumbersome; here we limit ourselves to provide such a generalization for the case $\beta = 0$ (the parameters α and β being left free).

To this end, setting in (9.3) $\beta = 0$ one has

$$
\frac{d}{dx}\left[x^{-\alpha}K(\alpha, \gamma; x)\right] = -\gamma x^{-(\alpha+1)}\left[K(\alpha+1, \gamma; x) - K(\alpha+1, \gamma-1, x)\right], \qquad (9.5)
$$

where the symbol $K(\alpha, \gamma; x)$ stands for the function $\Psi(\alpha, 0, \gamma; x)$.

Now with the help of Eq. (9.5) we obtain

$$
\begin{aligned}
\frac{d^2}{dx^2}\left[x^{-\alpha}K(\alpha, \gamma; x)\right] = x^{-(\alpha+2)}\big[\gamma^2 K(\alpha+2, \gamma; x) + \qquad (9.6)\\
+\gamma(1-2\gamma)K(\alpha+2, \gamma-1; x) - \gamma(\gamma-1)K(\alpha+2, \gamma-2; x)\big].
\end{aligned}
$$

In the same manner we can write generally

$$(9.7)\quad \frac{d^n}{dx^n}[x^{-\alpha}K(\alpha,\gamma;x)] = x^{-(\alpha+n)}[a_n^{(n)}K(\alpha+n,\gamma;x)+a_{n-1}^{(n)}K(\alpha+n,\gamma-1,x)\cdot$$
$$\cdots + a_0^{(n)}K(\alpha+n,\gamma-n;x)],$$

where the coefficients $a_{n-i}^{(n)}$ $(i=0,1,2,\ldots,n)$ have to be determined.

In doing so, let us differentiate the expression (9.7) with respect to x. Using the result (9.5) one gets

$$\frac{d^{n+1}}{dx^{n+1}}[x^{-\alpha}K(\alpha,\gamma;x)] = x^{-(\alpha+n+1)}\{-\gamma a_n^{(n)}K(\alpha+n+1,\gamma;x)+$$

$$(9.8)\quad +[\gamma a_n^{(n)}-(\gamma-1)a_{n-1}^{(n)}]K(\alpha+n+1,\gamma-1;x)+$$
$$+[(\gamma-1)a_{n-1}^{(n)}-(\gamma-2)a_{n-2}^{(n)}]K(\alpha+n+1,\gamma-2;x)+\cdots$$
$$\cdots+(\gamma-n)a_0^{(n)}\cdot$$

On the other hand, comparing (9.8) with the expression which one obtains from (9.7) replacing n by n+1, we are led to the following relations

$$(9.9)\quad a_{n+1}^{(n+1)} = -\gamma a_n^{(n)}$$

$$(9.10)\quad a_0^{(n+1)} = (\gamma-n)\,a_0^{(n)},$$

and

$$(9.11)\quad a_{n-i}^{(n+1)} = (\gamma-i)\,a_{n-i}^{(n)} - (\gamma-i-1)\,a_{n-i-1}^{(n)},$$

where $i = 0,1,2,\ldots, n-1$.

The coefficients $a_n^{(n)}$ and $a_0^{(n)}$ are easily found. Indeed, iterating (9.9)

$$(9.12) \qquad a_{n+1}^{(n+1)} = (-1)^{n+1}\gamma^{n+1} ,$$

where the relation $a_1^{(1)} = -\gamma$ has been used. Eqs.(9.12) and (9.9) give

$$(9.13) \qquad a_n^{(n)} = (-1)^n \gamma^n .$$

Now from (9.10) we deduce that

$$(9.14) \qquad \begin{aligned} a_0^{(n+1)} &= (\gamma-n)a_0^{(n)} = (\gamma-n)(\gamma-n+1)\, a_0^{(n-1)} = \ldots = \\ &= (\gamma-n)(\gamma-n+1)\ldots(\gamma-1)\gamma , \end{aligned}$$

where we have substituted $a_0^{(1)} = \gamma$. Eqs. (9.14) and (9.10) yield

$$(9.15) \qquad a_0^{(n)} = (\gamma-n+1)(\gamma-n+2)\ldots(\gamma-1)\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-n+1)} .$$

Let us now calculate the coefficients $a_{n-1}^{(n)}$.

To this end, consider the relation (9.11) for i=0, i.e.

$$(9.16) \qquad a_n^{(n+1)} = \gamma a_n^{(n)} - (\gamma-1)\, a_{n-1}^{(n)} .$$

By iteration and with the help of (9.13), from (9.16) one has

$$(9.17) \qquad \begin{aligned} a_n^{(n+1)} &= \gamma a_n^{(n)} - (\gamma-1)\left[\gamma a_{n-1}^{(n-1)} - (\gamma-1)\, a_{n-2}^{(n-1)}\right] = \ldots \\ \ldots &= (-1)^n\left[\gamma^{n+1} + \gamma^n(\gamma-1) + \ldots + \gamma^2(\gamma-1)^{n-1} + \gamma(\gamma-1)^n\right] . \end{aligned}$$

Since the expression between the sauare brakets is a geometrical progression of the ratio $(\gamma-1)/\gamma$, Eq.(9.17) yields

$$(9.18) \qquad a_{n}^{(n+1)} = (-1)^{n}\gamma\left[\gamma^{n+1} - (\gamma-1)^{n+1}\right].$$

Finally from (9.18) and (9.16) we obtain

$$(9.19) \qquad a_{n-1}^{(n)} = (-1)^{n}\gamma\left[(\gamma-1)^{n} - \gamma^{n}\right].$$

Our purpose now is to calculate the coefficient $a_{n-i}^{(n)}$ for any $i=0,1,2,\dots, n-1$. To this end, we start from (9.11) and iterate $a_{n-i-1}^{(n)}$.

We have

$$a_{n-i}^{(n+1)} = (\gamma-i)\, a_{n-i}^{(n)} - (\gamma-i-1)\, a_{n-i-1}^{(n)} =$$

$$(9.20) \qquad = \sum_{k=0}^{n-i-1} a_{n-i-k}^{(n-k)} (\gamma-i)(\gamma-i-1)^{k}(-1)^{k} + (-1)^{n-i}(\gamma-i-1)^{n-i}\, a_{o}^{(i+1)}.$$

Finally, Eq. (9.20) gives

$$(9.21) \qquad a_{n-i}^{(n)} = (\gamma-i+1)\sum_{k_1=0}^{n-1-k_1} a_{n-i-k_1}^{(n-1-k_1)} (\gamma-i)^{k_1}(-1)^{k_1} + (-1)^{n-i}(\gamma-i)^{n-i}\, a_{o}^{(i)},$$

where

$$(9.22) \qquad a_o^{(i)} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}, \quad i = 0,1,2,\ldots,n-1.$$

To derive explicitly the coefficients $a_{n-i}^{(n)}$ in terms of n, i and γ, we use repeated iterations of (9.21). To facilitate our task, it is advisable to take into account that: the index i appearing in (9.21) can be interpreted as the difference between the upper and lower indices of the coefficients $a_{n-i}^{(n)}$ and $(n-i)$ can be regarded as the lower index of $a_{n-i}^{(n)}$. Hence, in virtue of these considerations from (9.21) we deduce that:

$$(9.23) \qquad a_{n-i-k_1}^{(n-1-k_1)} = (\gamma-i+2)\sum_{k_2=0}^{n-i-k_1-1} a_{n-i-k_1-k_2}^{(n-2-k_1-k_2)} (\gamma-i+1)^{k_2}(-1)^{k_2} + $$

$$+ (-1)^{n-i-k_1}(\gamma-i+1)^{n-i-k_1} a_o^{(i-1)},$$

for $i \geq 2$.

Inserting (9.23) into (9.21), we obtain:

$$(9.24) \qquad a_{n-i}^{(n)} = (\gamma-i+1)(\gamma-i+2)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{k_1}\sum_{k_2=0}^{n-i-k_1-1}(\gamma-i+1)^{k_2}\cdot$$

$$\cdot (-1)^{k_2} a_{n-i-k_1-k_2}^{(n-2-k_1-k_2)} +$$

$$+ (\gamma-i+1)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{n-i}(\gamma-i+1)^{n-i-k_1} a_o^{(i-1)} +$$

$$+ (-1)^{n-i}(\gamma-i)^{n-i} a_o^{(i)}.$$

After (i-1) iterations ($i \geqslant 2$), Eq. (9.21) yields

$$a_{n-i}^{(n)} = (\gamma-i+1)(\gamma-i+2)\cdots(\gamma-1)\gamma \cdot \sum_{k_1=0}^{n-i-1} (\gamma-i)^{k_1}(-1)^{k_1} \cdot$$

$$\cdot \sum_{k_2=0}^{n-i-k_1-1} (\gamma-i+1)^{k_2}(-1)^{k_2} \cdots \sum_{k_i=0}^{n-i-\sum_{m=1}^{i-1} k_m} (\gamma-1)^{k_i}(-1)^{k_1} a_{(n-i-\sum_{m=1}^{i} k_m)}^{(n-i-\sum_{m=1}^{i} k_m)}$$

(9.25)
$$+ (\gamma-i+1)\cdots(\gamma-1)a_0^{(1)}(-1)^{n-i}\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(-1)^{k_1}\cdot\cdots$$

$$\cdots \cdot \sum_{k_{i-1}=0}^{n-i-\sum_{m=1}^{i-2} k_m} (\gamma-2)^{k_{i-1}}(\gamma-1)^{n-i-\sum_{m=1}^{i-1} k_m} + \cdots$$

$$\cdots + (-1)^{n-i}(\gamma-i+1)\sum_{k_1=0}^{n-i-1}(\gamma-i)^{k_1}(\gamma-i+1)^{n-i-k_1} a_0^{(i-1)} +$$

$$+ (-1)^{n-1}(\gamma-i)^{n-i} a_0^{(i)} .$$

Taking account of

(9.26)
$$a_{n-i-\sum_{m=1}^{i} k_m}^{(n-i-\sum_{m=1}^{i} k_m)} = (-1)^{n-i-\sum_{m=i}^{i} k_m} \cdot \gamma^{n-i-\sum_{m=1}^{i} k_m} ,$$

(see (9.13), and of

$$(9.27)\qquad (\gamma-i+1)\ldots(\gamma-j)\, a_0^{(j)} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)} ,$$

where

$$(9.28)\qquad a_0^{(j)} = (\gamma-j+1)(\gamma-j+2)\ \ldots\ (\gamma-1)\gamma ,$$

(j = 1,2,..., i-1), Eq.(9.25) can also be written as

$$a_{n-i}^{(n)} = (-1)^{n-i}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}\Bigg\{\gamma^{n-i}\prod_{j=1}^{i}\sum_{k_j=0}^{n-1-\sum_{m=1}^{l-1}k_m}(\gamma-i+j-1)^{k_j}\gamma^{-}$$

$$+\sum_{j=1}^{i-1}(\gamma-j)^{n-i}\prod_{l=1}^{i-j}\sum_{k_l=0}^{n-1-\sum_{m=1}^{l-1}k_m}(\gamma-i+l-1)^{k_l}(\gamma-j)^{-k_l}+$$

$$(9.29)\qquad + (-1)^{n-i}(\gamma-i)^{n-i}\Bigg\} =$$

$$= (-1)^{n-i}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)}\Bigg\{\sum_{j=0}^{i-1}(\gamma-j)^{n-i}\prod_{l=1}^{i-j}\sum_{k_l=0}^{n-1-\sum_{m=1}^{l-1}k_m}(\gamma-i+l-1)^{k_l}(\gamma-j)^{-}$$

$$+ (\gamma-i)^{n-i}\Bigg\} .$$

Finally, using the notation

$$(9.30)\quad \alpha_{n\ell,ij} = (n-1)(1-\delta_{ij}) - \theta(\ell-2)\sum_{m=1}^{\ell-1} k_m ,$$

where δ_{ij} is Kronecker's symbol and

$$(9.31)\quad \theta(\ell-2) = \begin{cases} 1 & \text{for } \ell \geq 2, \\ 0 & \text{for } \ell < 2, \end{cases}$$

the coefficient $a^{(n)}_{n-i}$ as given by (9.29) can also be expressed in the more compact form

$$(9.32)\quad a^{(n)}_{n-i} = (-1)^{n-i} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-i+1)} \sum_{j=0}^{i} (\gamma-j)^{n-i} \prod_{\ell=1-\delta_{ij}}^{i-j} \sum_{k_\ell=0}^{\alpha_{n\ell,ij}} (\gamma-i+\ell-1)^{k_\ell} (\gamma-j),$$

where $i = 2,3,\ldots, n-1$.

In virtue of (9.13), (9.15), (9.19) and (9.32), we have determined explicitly the expression (9.7) for the n-th derivative of the function $x^{-\alpha}K(\alpha,\gamma;x)$.

<u>Remark</u> 9.1

For $\gamma = 1$, Eq.(9.7) reduces to the functional relation for the incomplete Γ -function :

$$(9.33)\quad \frac{d^n}{dx^n}\left[x^{-\alpha}\Gamma(\alpha,x)\right] = (-1)^n x^{-(\alpha+n)}\Gamma(\alpha+n,x).$$

In fact, when $\gamma = 1$ from (9.13) and (9.15) we have resepctively $a^{(n)}_{n-i} = (-1)^n$ and $a^{(n)}_{o} = 0$. Furthermore, Eq.

(9.21) yields $a_{n-i}^{(n)} = 0$ for $i = 1,2,\ldots, n-1$. The result (9.33) thus follows immediately from (9.7).

We point out that for $\gamma = -1$, Eqs.(9.13), (9.15) and (9.19) become respectively

$$a_n^{(n)} = 1, \tag{9.34}$$

$$a_o^{(n)} = (-1)^n n!\ , \tag{9.35}$$

and

$$a_{n-1}^{(n)} = 1 - 2^n. \tag{9.36}$$

On the other hand, from (9.32) we have that

$$a_{n-i}^{(n)} = i! \sum_{j=0}^{i} (-1)^i (1+j)^{n-i} \prod_{\ell=1-\delta_{ij}}^{\alpha_{n\ell,ij}} (2+i-\ell)^{k_\ell} (1+j)^{-k_\ell}, \tag{9.37}$$

for $i = 2,3,\ldots, n-1$.

Since (see (6.3))

$$K(\alpha_1 - 1;x) \equiv -D(\alpha - 1,x), \tag{9.38}$$

with the help of (9.34), (9.35), (9.36) and (9.37), Eq.(9.7) gives a relation for the n-th derivative of the function $x^{-\alpha} D(\alpha - 1,x)$, where $D(\alpha - 1,x)$ is the Debye function defined by (6.3). As far as we know, this formula is new.

We close this Section by noticing that, analogous to the manner in which we derived (9.7), one can obtain a formula for the n-derivative of the function $e^{-x}K(\alpha,\gamma;x)$.

10. ANOTHER RECURRENCE FORMULA.

The use of (5.1) and the relation (9.1) allows us to write down another recurrence formula besides (4.1).

In fact, by integrating term by term (9.1) with $-\rho=\mu$ and applying (5.1), we find the following relation:

$$(10.1)\quad \begin{aligned}&(\mu+\alpha-\gamma\beta)\psi(\alpha+\mu,\beta,\gamma;x)-\gamma\psi(\alpha+\mu+1,\beta,\gamma;x)+\gamma\psi(\alpha+\mu+1,\beta,\gamma-1;x)+\\&+\gamma\beta\psi(\alpha+\mu,\beta,\gamma-1;x)+x^{\mu}\Big[(-\alpha+\gamma\beta)\ \psi\ (\alpha,\beta,\gamma;x)+\\&+\gamma\psi(\alpha+1,\beta,\gamma;x)-\gamma\psi(\alpha+1,\beta,\gamma-1;x)-\gamma\beta\psi(\alpha,\beta,\gamma-1;x)\Big]=0.\end{aligned}$$

Remark 10.1

When $\gamma=1$, Eq. (10.1) gives the well-known recursive relation for the incomplete Γ-function:

$$(10.2)\quad \begin{aligned}&(\mu+a)\ \Gamma\ (a+\mu,x)-\Gamma(a+\mu+1,x)+x^{\mu}\Big[\Gamma(a+1,x)-\\&-a\Gamma(a,x)\Big]=0,\end{aligned}$$

where $a=\alpha-\beta$.

REFERENCES

[1] A.C.SCOTT, F.Y.F. CHU and D.W.Mc LAUGHLIN, "The Soliton: A New Concept in Applied Science", Proceeding IEEE 61 (1973), pp. 1443-1483.

[2] A.JEFFREY and T. KAKUTANI, "Weak Nonlinear Dispersive Waves: A Discussion Centered Around the Korteweg - de Vries Equation", SIAM Review 14 (1972), pp. 582-643.

[3] V.G.MAKHANKOV, "Dynamics of Classical Solitons (in Non-integrable Systems)", Physics Report 35 (1978), pp.1-128

[4] V.E. ZAKHAROV and L.D.FADDEEV, "The Korteweg - de Vries Equation: a Completely Integrable Hamiltonian System", Func. Anal. Appl. 5 (1971), pp. 280-287.

[5] M.D. KRUSKAL, R.M.MIURA, C.S. GARDNER and N.J.ZABUSKY, "Korteweg - de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion", J.Math. Phys. 9 (1968), pp. 1204-1209.

[6] R.M.MIURA, "Korteweg - de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation", J.Math. Phys. 9 (1968), pp. 1202-1204.

[7] G.B. WHITHAM, "Linear and Nonlinear Waves", John Wiley & Sons, New York, 1974, p.9.

[8] A.BARONE, F.ESPOSITO, C.J.MAGEE and A.C.SCOTT, "Theory and Applications of the Sine-Gordon Equation", Rivista del Nuovo Cimento 1 (1971), pp. 227-267.

[9] W.F.AMES, "Nonlinear Ordinary Differential Equations in Transport Processes", Academic Press, New York, 1968, p. 42 and p. 101.

[10] A.S. DAVYDOV, "Solitons in Molecular Systems", Preprint ITP - 78 - 62E, Kiev (1978).

[11] S.SAMUEL, "Grand Partition Function in Field Theory with Applications to Sine-Gordon Field Theory", Phys. Rev. D 18 (1978), pp. 1916-1932.

[12] F.W.J. OLVER, "Asymptotics and Special Functions",Academic Press, New York and London, 1974, p.45.

[13] N. Ja. VILENKIN, "Fonctions Speciales et Théorie de la Représentation des Groupes", Dunod, Paris, 1969.

[14] I.S. GRADSHTEYN and I.M. RYZHIK, "Table of Integrals, Series and Products",Academic Press, New York and London, 1965, p. 317.

[15] See [14], p. 942.

[16] F.G. TRICOMI, "Funzioni Ipergeometriche Confluenti", Edizioni Cremonese, Roma, 1954, p. 164.

[17] M. ABRAMOWITZ and I.A. STEGUN, "Handbook of Mathematical Functions", Dover, New York, 1965, p. 998.

[18] See [17], p. 257

[19] See [17], p. 260

[20] See [12], p. 25

[21] See [16], p. 174

[22] See [14], p.1075