

the decision time is bounded by a polynomial in $|x|$ and \bar{k} .

QED

Since no example is known of a problem which is strongly simple and not p -simple no application of theorem 4 can be provided which is different from the application given at the end of theorem 3.

As a conclusion of this paragraph we may observe that the results provided insofar have a twofold implication. On one side they can be used in order to characterize the computational complexity of one problem with respect to the given definitions, on the other side they establish conditions on the type of reductions that can be found among problems belonging to different classes, such as those discussed at the end of theorem 2 and theorem 3. As a further example we may observe that in the case of the reduction from PARTITION to MAX-CUT the existence of a much more succinct reduction than the one given by Karp is ensured by noting that the first problem is strongly simple while the second is weakly rigid.

4. STRONG NP-COMPLETENESS AND ITS RELATION TO RIGIDITY

In the preceding paragraph we have seen that in some cases the characterization of a problem B that is not fully approximable comes out of the fact that we can reduce an NP-complete combinatorial problem A^C into a subset of B^C in which the measure is bounded by a polynomial. Garey and Johnson give another way of considering subsets of the set INPUT of a problem to study the different characteristics of NPCO problems. Their paper (1978) is an attempt to understand the different roles that numbers play in NPCO problems. Let

us first consider, for example, the problem MAX-CUT that is a well-known NPCO problem. If we restrict to those graphs with unitary weights we obtain a seemingly easier problem SIMPLE-MAX-CUT, that, however, is still an NPCO problem. Different is the case of the problem JOB-SEQUENCING-WITH-DEADLINES: it has been shown to be NP-complete by Karp, but if we restrict to the case when all weights are unitary then the problem is solvable in $O(n \lg n)$. Moreover if the weights are at most k the problem is solvable by a classic dynamic approach in time bounded by a polynomial in k and in n (the number of jobs). Note that a polynomial algorithm must solve JOB-SEQUENCING-WITH-DEADLINES in time bounded by a polynomial in n and in $\lg k$.

In order to formalize these observations Garey and Johnson introduce another function of the input $\text{MAX:INPUT} \rightarrow \mathbb{Z}^+$ that captures the notion of the magnitude of the largest number occurring in the input. For example given a weighted graph G , $\text{MAX}(G)$ can be defined as the value of the maximum weighted edge. The following definitions formalize these concepts.

DEFINITION 8. A *pseudo-polynomial algorithm* is an algorithm that on input x runs in time bounded by a polynomial in $|x|$ and in $\text{MAX}(x)$.

DEFINITION 9. An NPCO problem is a *pseudopolynomial NPCO problem* if there is a pseudopolynomial algorithm that solves it.

DEFINITION 10. Given a problem P let P_q denote the problem obtained by restricting P to only those instances x in INPUT_q for which $\text{MAX}(x) \leq q(|x|)$

DEFINITION 11. An NPCO problem P is *NP-complete in the strong sense* if there exists a polynomial q such that P_q is NP-complete.

An example of pseudopolynomial NPCO problem is JOB-SEQUENCING-WITH-DEADLINES (Lawler and Moore (1969)) while MAX-CUT is NP-complete in the strong sense (it is sufficient to consider the constant polynomial $q(x) = 1$ to obtain SIMPLE-MAX-CUT).

The two classes of pseudopolynomial NPCO problems and of strong NP-complete problems are disjoint (obviously unless $P = NP$). The following proposition states the relationship between strong NP-completeness and full approximability.

PROPOSITION 2. If P is NP-complete in the strong sense then it is not fully approximable.

Garey and Johnson give another result that connects the two concepts of pseudopolynomial and fully approximable NPCO problem; for clarity sake, we will give it later as an immediate consequence of Theorem 6.

In many problems the optimal value of the measure and the MAX of the input have the same size or it is possible to establish a polynomial relation between them. This suggests the idea of comparing some of the different concepts introduced in the preceding paragraphs and in this one. First of all we can prove the following.

FACT 1. Let A be a pseudopolynomial optimization problem. If there exists a polynomial q such that for every $x \in \text{INPUT}_A$ $\text{MAX}(x) \leq q(m^*(x) - \tilde{m}(x), |x|)$, then given (x, k) , it is possible to decide in polynomial time if $m^*(x) \leq k$ or $m^*(x) > k$.

PROOF. The hypotheses imply that there exists a polynomial p such that, given x , $m^*(x)$ is computable within time $p(|x|, \text{MAX}(x))$ and, therefore, within time $p(|x|, q((m^*(x) - \tilde{m}(x)), |x|))$. We apply the pseudopolynomial algorithm to x for $p(|x|, q(k, |x|))$ steps. If the algorithm stops, it is decidable if $m^*(x) \leq k$ or $m^*(x) > k$; instead if the algorithm does not terminate in $p(|x|, q(k, |x|))$, then $m^*(x) > k$

QED

As all known pseudopolynomial algorithms make us of dynamic programming, it is possible, very often, to state fact 1 in a more interesting way.

FACT 2. Let A be a pseudopolynomial optimization problem. If there exists a polynomial q such that for, every $x \in \text{INPUT}_A$, $\text{MAX}(x) \leq q(m^*(x) - \tilde{m}(x), |x|)$, then A is p -simple.

THEOREM 5. Let A be a p -simple problem. If there exists a polynomial q such that, for every $x \in \text{INPUT}_A$, $(m^*(x) - \tilde{m}(x)) \leq q(\text{MAX}(x), |x|)$, then A is a pseudopolynomial NPCO problem.

PROOF. By the hypothesis for each k A_k^C is recognizable in time $Q(|x|, k)$. To obtain $m^*(x)$ we can use the following algorithm:

for $k := 0$ to $q(\text{MAX}(x), |x|)$

repeat the following step:

if $\langle x, k \rangle \in A_k^C$ then $m^*(x) = \tilde{m}(x) + k$

By hypothesis there are no more than $q(\text{Max}(x), |x|) + 1$ iterations of steps 2. As A^c is p-simple each iteration of step 2 takes no more than $Q(|x|, q(\text{MAX}(x), |x|) + 1)$. Therefore $m^*(x)$ is computable in at most $(q(\text{MAX}(x), |x|) + 1) \cdot Q(|x|, q(\text{MAX}(x), |x|))$. QED

COROLLARY. (Garey and Johnson (1978)). Let A be a fully approximable NPCO problem. If there exists a polynomial q such that for every $x \in \text{INPUT}_A$ $(m^*(x) - \tilde{m}(x)) \leq q(\text{MAX}(x), |x|)$ then A is a pseudopolynomial NPCO problem.

PROOF. Immediate from the previous theorem and the fact that a fully approximable problem is p-simple. QED

As the conditions of theorem 5 and Fact 1 are generally verified the two concepts of pseudopolynomial problem and p-simple problem coincide in many cases.

A natural question arises at this point: when the conditions of the theorems are not verified which of the two approaches gives a better information about the complexity of approximate algorithms?

Let us define

$$(P1) \quad \text{Max} \sum_{j=1}^n c_j y_j$$

$$\text{subject to} \quad \sum_{j=1}^n a_j y_j = b \quad y_j = 0, 1 \quad j = 1, 2, \dots, n$$

A natural definition of $\text{MAX}(\text{INPUT}_{P1})$ can be the following $\text{MAX}(x) = \max_j (c_j, a_j)$ and it is not hard to prove that P1 is pseudopolynomial (a classic dynamic approach solves it in $O(n^2 \text{MAX}(x))$); however even the problem to obtain any approximate solution is an NP-complete problem (Karp (1972)). Therefore P1 is a pseudopolynomial NPCO problem that is not

approximable.

Let us consider now:

$$(P2) \quad \prod_{j=1}^n c_j^{y_j}$$

$$\text{subject to } \prod_{j=1}^n a_j^{y_j} \leq b, \quad y_j = 0, 1 \quad j = 1, 2, \dots, n$$

This problem is fully approximable and we conjecture that it is not a pseudopolynomial problem because the classical method of deriving a pseudopolynomial algorithm from the dynamic programming approach does not work. Theorems 5 and 6 and the previous examples show that Paz and Moran's approach has a wider application for two different reasons. First their approach is straightforward and there is no need to introduce the function MAX whose definition can be ambiguous in some cases.

In addition we have proven that the two approaches are equivalent under restricted but reasonable hypotheses and we have shown that when $m^*(x)$ and $\text{MAX}(x)$ are not polynomially related the approach formulated by Paz and Moran remains adequate to study the complexity of approximation schemes for NPCO problems.

Before finishing this paragraph we want to observe that, when there is a polynomial relation between the value of the optimal solution and the value of MAX, there is a strong connection between the two concepts of strong NP-complete and weakly rigid.

THEOREM 6. Let A be a strong NP-complete optimization problem. If there exists a polynomial p such that for every $x \in \text{INPUT}_A$ $(m^*(x) - \tilde{m}(x)) \leq p(\text{MAX}(x), |x|)$ then A is weakly rigid.

PROOF. If A is NP-complete in the strong sense there must exist a polynomial q such that the following set

$Q = \{ \langle x, k \rangle \mid \langle x, k \rangle \in A^C, \text{MAX}(x) \leq q(|x|) \}$ is NP-complete.

Let us consider now the set

$Q' = \{ \langle x, k \rangle \mid \langle x, k \rangle \in A^C, \text{MAX}(x) \leq q(|x|), \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(\text{MAX}(x), |x|) \}$

As $Q \supseteq Q'$ in order to prove that $Q \equiv Q'$ it is sufficient to prove that

$Q - Q' \equiv \{ \langle x, k \rangle \mid \langle x, k \rangle \in A^C, \text{MAX}(x) \leq q(|x|), k > \tilde{m}(x) + p(\text{MAX}(x), |x|) \}$

is the empty set. In fact given $\langle x, k \rangle$, with $k > \tilde{m}(x) + p(\text{MAX}(x), |x|)$, we have by hypothesis $k > \tilde{m}(x) + p(\text{MAX}(x), |x|) \geq m^*(x)$ and therefore $\langle x, k \rangle \notin A^C$. Let us consider now

$Q'' = \{ \langle x, k \rangle \mid \langle x, k \rangle \in A^C, \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(q(|x|), |x|) \}$

Clearly Q'' is NP-complete and hence A is weakly rigid.

QED

5. CONCLUSIONS

In this paper we have shown that there exist close relations among different approaches to the classification of NP-complete optimization problems, giving also new results on the type of possible reductions among problems belonging to different classes. On the other side, it was proven that, violating some conditions, comparisons among different concepts do not hold any more.

Therefore we believe that, in the whole, our results are a useful contribution for a better understanding of properties of NPCO problems. We think that in order to provide meaningful characterizations of NOCO problems it is necessary to find the suitable level of abstraction because