compared, eventually exhibiting some examples which show that violating the conditions, the two approaches lead to different conclusions in the classification of NP-complete problems.

2. BASIC CONCEPTS AND TERMINOLOGY

In order to establish a formal ground for the study of the properties of optimization problems we first give an abstract notion of optimization problem which is broad enough to include most common problems of this kind. Following the literature (Johnson (1973), we consider an *NP-optimization problem* to be characterized by a polynomially decidable set INPUT of instances, a polynomially decidable set OUTPUT of possible solutions, a mapping SOL:INPUT $\rightarrow$ P(OUTPUT) which, given any instance x of the problem, in polynomial time nondeterministically provides the *approximate solutions* of x and a mapping m:OUTPUT $\rightarrow$ Z (where Z is the set of relative integers) which in polynomial time provides the *measure* of an approximate solution (if A is a maximization problem) or its opposite (if A is a minimization problem). Note that in this way we allow a uniform approach to both maximization and minimization problems.

Since we are interested in studying those optimization problems which are "associated" to NP-complete recognition problems we restrict ourselves to considering a particular class of NP-complete problems:

DEFINITION 1. Let A be an NP optimization problem. The *combinatorial problem associated* to A is the set

$$A^{c} = \left\{\langle x,k \rangle \mid x \in \mathrm{INPUT}_A \text{ and } k \in m(\mathrm{SOL}(x))\right\}.$$

On the base of this definition we exclude from our study those problems which are not directly related to optimization problems(*).

If A^{c} is NP-complete we say A is an NP-complete optimization (NPCO) problem.

We will denote $\tilde{m}(x)$ and $m^{*}(x)$ the worst and (respectively) the best solution of x with respect to the ordering of Z. In many cases the worst solution can be easily (in polynomial time) determined . In those cases we will refer to it as a trivial solution.

EXAMPLE. The problem MAX-CLIQUE is an NPCO problem. It is characterized by

INPUT = set of all finite graphs,
OUTPUT = set of all finite complete graphs,
SOL(x) = set of all complete subgraphs of a graph x
m(y) = no of nodes of y

The combinatorial problem $\{\langle x,k \rangle \mid x$ has a complete subgraph of k nodes$\}$ is a well known NP-complete recognition problem. In this case $\tilde{m}(x) = 1$ is clearly the trivial solution of the optimization problem.

For this particular class of NP-complete recognition problems the concept of reduction (Karp (1972)) can be spe-

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(*) In their paper Paz and Moran (1977) suggest that any NP recognition problem can be represented as an optimization problem but we prefer a more straightforward and explicit definition.

cialized and it can be extended to the associated optimization problems.

DEFINITION 2. Let A and B be two NP optimization problems. We say that *A is polynomially reducible to B* ($A \leq_p B$) if there exist two polynomially computable functions

$$f_1 : \mathrm{INPUT}_A \to \mathrm{INPUT}_B, \quad f_2 : \mathrm{INPUT}_A \times Z \to Z$$

such that

$$\langle f_1(x), f_2(x,k) \rangle \in B^c \text{ iff } \langle x,k \rangle \in A^c$$

Throughout this paper we will deal only with this kind of reductions. For simplicity we will say A reducible to B and we will drop the subscript p from $\leq_p$.

Since we are interested in discussing the approximability of NPCO problems and reductions between problems with a different behaviour with respect to this property, we first give some basic definitions that introduce the concept of approximate algorithm, of approximable problem and of fully approximable problem (Sahni (1975), Paz and Moran (1977)).

DEFINITION 3. Let A be an NPCO problem. We say that

i) A is an *approximate algorithm* for A if given any $x \in \mathrm{INPUT}_A$ A(x) is in $\mathrm{SOL}_A(x)$ and A is computable in polynomial time.

ii) A is an *ε-approximate algorithm* for A if it is an approximate algorithm for A for every $x \in \mathrm{INPUT}_A$

$$\left| \frac{m^*(x) - m(A(x))}{m^*(x)} \right| \leq \varepsilon$$

DEFINITION 4. Let A be an NPCO problem; we say that

i) A is *approximable* if given any $\varepsilon > 0$ there exists an ε-approximate algorithm;

ii) A is *fully approximable* if there exists a polynomial $\lambda x \lambda y[q(x,y)]$ such that for every ε there exists an ε-approximate algorithm A_ε that runs in time bounded by $q(|x|,1/\varepsilon)$

Many results in the recent literature are devoted to establishing whether a given problem is approximable or fully approximable or it cannot be approximated. For example it is known that the MAX-SUBSET-SUM problem is fully approximable while the MIN-CHROMATIC-NUMBER problem has been pròven not to be approximable for $\varepsilon < 1$ (if $P \neq NP$). A list of papers dealing with results in this area is provided by Garey and Johnson (1977). At present no result is known that shows that a problem is approximable but not fully approximable neither is known any precise characterization of the class of problems which are approximable or fully approximable. The results given by Paz and Moran (1977) and Garey and Johnson (1978) are nevertheless an important step forward in this direction. For this reason our aim has been to determine conditions for the comparison of these results and at the same time to develop this kind of research and to derive consequences which are useful for a better understanding of the properties of NP-complete optimization problems.

3. TRUNCATED COMBINATORIAL PROBLEMS AND THEIR PROPERTIES

The first approach (Paz and Moran (1977)) to the characterization of NP-complete optimization problems is based