1. INTRODUCTION

Various results on the properties of NP-complete optimization problems and on the characterization of these problems either with respect to their approximation properties or with respect to their combinatorial structure have been presented in the literature.

In particular we have considered the approaches given by Paz and Moran (1977) and by Garey and Johnson (1978) because of the interest of their results.

Paz and Moran introduce a classification of NP complete optimization problems based on the fact that considering only those inputs of the problem whose measure is bounded by an integer, it is possible to divide all the problems in different classes as regards their computational complexity (rigid, simple and p-simple problems). Furthermore these classes are then related to the approximability properties of the problem.

Under a different approach Garey and Johnson give another characterization which is based on the concept of strong NP-complete problem (limiting ourselves to those inputs,whose "value" is bounded by a polynomial in the length of the input, we still obtain an NP-complete problem) and of pseudopolynomial algorithm (an algorithm which is polynomial in the length of the input and in the magnitude of the greatest number occurring in the istance). Also in this case very interesting relations among these concepts and approximation properties are stated.

These papers, are, without any doubt, very interesting and new results of remarkable importance have been captured. Nevertheless, it seems to lack an attempt of organizing all these results in a unified framework as general as possible.

Furthermore any effort of comparison among different approaches has not been attempted.

The aim of our paper is therefore a first step in this direction. Starting from the observation that, intuitively, there is a similarity among some of the consequences of Paz and Moran, Garey and Johnson approaches, we have introduced a formal framework in which it is possible to establish clear connections among different concepts of the two approaches, at least under restricted but reasonable hypotheses. So, for istance, we have established under what conditions a pseudopolynomial problem is p-simple and viceversa. Beside this, our point of view allowed to derive some new consequences both concerning the classification of problems and the characterization of reductions that exist among different problems. We have stated what conditions must be verified to have a polynomial reduction from a rigid problem to a simple problem, from a p-simple problem to a p-simple problem and so on. Finally it seems interesting to us that some of these results can be interpreted as a formalization of facts that are used in practice when studying the solution of a particular problem, such as, for example, the fact that a problem with polynomially bounded objective function cannot be fully approximated.

In particular, in §.2 we give the basic terminology and notation. In §.3 we very briefly summarize the Paz and Moran approach with a slightly different formulation, giving new results such as those above stated concerning the characterization of reductions among problems belonging to different classes. In §.4 after recalling the main definitions and results of the Garey and Johnson approach, we establish under what conditions the results of these two approaches can be compared, eventually exhibiting some examples which show that violating the conditions, the two approaches lead to different conclusions in the classification of NP-complete problems.

2. BASIC CONCEPTS AND TERMINOLOGY

In order to establish a formal ground for the study of the properties of optimization problems we first give an abstract notion of optimization problem which is broad enough to include most common problems of this kind. Following the literature (Johnson (1973), we consider an *NP-optimization problem* to be characterized by a polynomially decidable set INPUT of instances, a polynomially decidable set OUTPUT of possible solutions, a mapping SOL:INPUT $\rightarrow$ P(OUTPUT) which, given any instance x of the problem, in polynomial time nondeterministically provides the *approximate solutions* of x and a mapping m:OUTPUT $\rightarrow$ Z (where Z is the set of relative integers) which in polynomial time provides the *measure* of an approximate solution (if A is a maximization problem) or its opposite (if A is a minimization problem). Note that in this way we allow a uniform approach to both maximization and minimization problems.

Since we are interested in studying those optimization problems which are "associated" to NP-complete recognition problems we restrict ourselves to considering a particular class of NP-complete problems:

DEFINITION 1. Let A be an NP optimization problem. The *combinatorial problem associated* to A is the set

$$A^c = \left\{ \langle x,k \rangle \mid x \in \text{INPUT}_A \text{ and } k \in m(\text{SOL}(x)) \right\}.$$

On the base of this definition we exclude from our study those problems which are not directly related to optimization problems(*).

If A^c is NP-complete we say A is an NP-complete optimization (NPCO) problem.

We will denote $\tilde{m}(x)$ and $m^*(x)$ the worst and (respectively) the best solution of x with respect to the ordering of Z. In many cases the worst solution can be easily (in polynomial time) determined . In those cases we will refer to it as a trivial solution.

EXAMPLE. The problem MAX-CLIQUE is an NPCO problem. It is characterized by

INPUT = set of all finite graphs,
OUTPUT = set of all finite complete graphs,
SOL(x) = set of all complete subgraphs of a graph x
m(y) = no of nodes of y

The combinatorial problem $\{\langle x,k \rangle \mid x$ has a complete subgraph of k nodes$\}$ is a well known NP-complete recognition problem. In this case $\tilde{m}(x) = 1$ is clearly the trivial solution of the optimization problem.

For this particular class of NP-complete recognition problems the concept of reduction (Karp (1972)) can be spe-

(*) In their paper Paz and Moran (1977) suggest that any NP recognition problem can be represented as an optimization problem but we prefer a more straightforward and explicit definition.

cialized and it can be extended to the associated optimization problems.

DEFINITION 2. Let A and B be two NP optimization problems. We say that A *is polynomially reducible to* B ($A \leq_p B$) if there exist two polynomially computable functions

$$f_1 : \mathrm{INPUT}_A \rightarrow \mathrm{INPUT}_B, \quad f_2 : \mathrm{INPUT}_A \times Z \rightarrow Z$$

such that

$$\langle f_1(x), f_2(x,k) \rangle \in B^c \text{ iff } \langle x,k \rangle \in A^c$$

Throughout this paper we will deal only with this kind of reductions. For simplicity we will say A reducible to B and we will drop the subscript p from $\leq_p$.

Since we are interested in discussing the approximability of NPCO problems and reductions between problems with a different behaviour with respect to this property, we first give some basic definitions that introduce the concept of approximate algorithm, of approximable problem and of fully approximable problem (Sahni (1975), Paz and Moran (1977)).

DEFINITION 3. Let A be an NPCO problem. We say that

i) A is an *approximate algorithm* for A if given any $x \in \mathrm{INPUT}_A$ A(x) is in $\mathrm{SOL}_A(x)$ and A is computable in polynomial time.

ii) A is an ε-*approximate algorithm* for A if it is an approximate algorithm for A for every $x \in \mathrm{INPUT}_A$

$$\left| \frac{m^*(x) - m(A(x))}{m^*(x)} \right| \leq \varepsilon$$

DEFINITION 4. Let A be an NPCO problem; we say that

i) A is *approximable* if given any $\varepsilon > 0$ there exists an ε-approximate algorithm;

ii) A is *fully approximable* if there exists a polynomial $\lambda x \lambda y[q(x,y)]$ such that for every ε there exists an ε-approximate algorithm A_ε that runs in time bounded by $q(|x|,1/\varepsilon)$

Many results in the recent literature are devoted to establishing whether a given problem is approximable or fully approximable or it cannot be approximated. For example it is known that the MAX-SUBSET-SUM problem is fully approximable while the MIN-CHROMATIC-NUMBER problem has been proven not to be approximable for $\varepsilon < 1$ (if $P \neq NP$). A list of papers dealing with results in this area is provided by Garey and Johnson (1977). At present no result is known that shows that a problem is approximable but not fully approximable neither is known any precise characterization of the class of problems which are approximable or fully approximable. The results given by Paz and Moran (1977) and Garey and Johnson (1978) are nevertheless an important step forward in this direction. For this reason our aim has been to determine conditions for the comparison of these results and at the same time to develop this kind of research and to derive consequences which are useful for a better understanding of the properties of NP-complete optimization problems.

3. TRUNCATED COMBINATORIAL PROBLEMS AND THEIR PROPERTIES

The first approach (Paz and Moran (1977)) to the characterization of NP-complete optimization problems is based on the complexity of the recognition of an infinite sequence of bounded subsets of the associated combinatorial problem. Informally, if we consider the search space that has to be explored in order to find approximate solutions to an optimization problem we may observe the following facts. Clearly, if the size of the search space is polynomial in the size of the input the problem itself is polynomially solvable. In the case of those problems which are in the class NP but which are not known to be polynomial an a priori evaluation of the size of the search space indicates that it grows exponentially. Nevertheless in many cases when we consider the search space that we have to explore in order to find approximate solutions whose measure does not exceed a certain bound, we may notice that it is polynomial. A typical example of this kind of problems is the problem MAX-CLIQUE in which the complete subgraphs of size k in a graph of size n are at most $\binom{n}{k}$, that is polynomial in n. Since this does not happen in all cases it suggests the following definition.

DEFINITION 5. Let A be an NPCO problem; let A^c be the associated combinatorial problem. *A truncated combinatorial problem* of A is a set

$$A^c_{\bar{k}} = \left\{ \langle x,k \rangle \in A^c \,|\, \tilde{m}(x) \leq k \leq \tilde{m}(x) + \bar{k} \right\}$$

where $\bar{k}$ is any nonnegative integer.

Note that the sequence $\{A^c_{\bar{k}}\}_{\bar{k}=0}^{\infty}$ approximates the set A^c in a sense which is analogous to the definition of limit recursion approximation (Gold (1965)).

DEFINITION 6. A is *simple* if, for every $\bar{k}$, $A^c_{\bar{k}}$ is polynomially decidable. A is *rigid* if it is not simple.

Note that if A is rigid there exists an integer $\bar{k}$ such that $A^c_{\bar{k}}$ is p-complete that is $A^c_{\bar{k}}$ is in P if and only if P = NP (see Sahni 75).

Examples of simple NPCO problems, besides MAX-CLIQUE, are MAX-SATISFIABILITY, MIN-CHROMATIC NUMBER etc.

Definitions 5 and 6 are slightly modified with respect to the corresponding definitions in Paz and Moran (1977). In fact we always start from the set A^c_0 in which all pairs $\langle x, \tilde{m}(x) \rangle$ are included and, as long as $\bar{k}$ increases,we go further and further from the worst solution to the optimal solution.

For example the problem MIN-CHROMATIC-NUMBER, which is rigid according to the original definitions, is simple in our case and this is because, given any h, the set of possible colourings of a graph of N nodes with N-h colours has polynomial size in N.

On the other side an example of rigid NPCO problem is provided by MAX-WEIGHTED-SATISFIABILITY because if we allow weights equal to zero even the set A^c_0 is in this case NP-complete because in order to decide whether a formula w has measure 0 we first need to prove that it is satisfiable (Ausiello et al. (1978)). Note that if we instead do not allow weights equal to zero the problem MAX-WEIGHTED-SATISFIABILITY can be proved to be simple.

Note that if a problem is simple, then its worst solution is actually a trivial solution, that is it can be always found in polynomial time.

The concept of simple problem can be strengthened in the following way:

DEFINITION 7. An NPCO problem A is p-*simple* if there is a polynomial Q such that, for every k, A^c_k is recognizable in deterministic time bounded by $Q(|x|,k)$.

Typical examples of p-simple problems are MAX-SUBSET-SUM, JOB-SEQUENCING-WITH-DEADLINES etc., while the above listed simple problems are not p-simple. We will discuss later on this claim.

Beside offering a first classification of NPCO problems, the concepts of simplicity and p-simplicity are relevant because it has been proven by Paz and Moran (1977) that a necessary condition for a problem A to be approximable (fully-approximable) is that A is a simple (p-simple) NPCO problem and clearly these properties still hold under our definitions.

Actually the fact that until now no problem has been shown to be approximable and not fully-approximable, determines a greater attention on the concept of p-simplicity; but in order to prove that a problem is not p-simple it is very hard to show that no algorithm which is polynomial in $|x|$ and k can exist. Much easier is to use the following definitions

DEFINITION 8. An NPCO problem A is *strongly simple* if, given any polynomial q, $A_q^c = \{\langle x,k\rangle \in A^c \,|\, \tilde{m}(x) \leq k \leq \tilde{m}(x)+q(|x|)\}$ is decidable in polynomial time. A is *weakly rigid* if there exists a polynomial p such that A_p^c is NP-complete.

Since a p-simple problem is strongly simple, to show that a problem is weakly rigid is a very easy method to prove that a problem is not p-simple and therefore not fully approximable. For example weakly rigid problems are MAX-CLIQUE, MAX-SATISFIABILITY, MIN-CHROMATIC-NUMBER and the proof is based on the fact that, for all these problems, for q(n) increasing more rapidly than n, $A_q^c = A^c$.

This fact suggests an even easier condition that is sufficient for a problem not to be fully approximable.

PROPOSITION 1. Let A Be an NPCO problem. If there exists a polynomial p such that for all $x \in \mathrm{INPUT}_A$, $m^*(x) - \tilde{m}(x) \leq p(|x|)$ then A is not fully approximable.

PROOF. In fact in order to be fully approximable, A should satisfy the property that A^c_p is recognizable in polynomial time but, by hypothesis, we have that $A^c_p = A^c$ and, hence, A^c_p is NP-complete.

QED

For some problems, like MAX-CLIQUE and MIN-CHROMATIC NUMBER, Proposition 1 can be immediately applied. In fact in these cases $p(|x|) = |x|$.
In some other case, in order to apply Proposition 1, we may prove a stronger result that is useful for showing that a problem is weakly rigid.

THEOREM 1. Let A and B be two NPCO problems; if there exists a reduction $f = \langle f_1, f_2 \rangle$ from A to B such that f satisfies the following property: $f_2(x,k) \leq p(|f_1(x)|)$ for some polynomial p and all $x \in \mathrm{INPUT}_A$, $k \in Q_A$, then B is not fully approximable.

PROOF. If B was fully approximable then for every polynomial q we should have B^c_q recognizable in polynomial time. If we now consider the set B^c_p if we could decide within polynomial time whether, given any pair $\langle y,h \rangle$ with $\tilde{m}(y) \leq h \leq \tilde{m}(y) + p(|y|)$, h is the measure of an approximate solution of y, then within polynomial time we could decide A^c. In fact in order to decide A^c in polynomial time, given a pair $\langle x,k \rangle$ we could compute in polynomial time $f_1(x)$ and $f_2(x,k)$ and since $f_2(x,k) \leq p(|f_1(x)|)$ we could use the decision procedure B_p^c to check whether $f_2(x,k)$ is the measure of an approximate solution of $f_1(x)$.

QED

Note that in theorem 1 the condition on f_2 may regard only a subset of B while in Proposition 1 all inputs must satisfy the hypothesis that $(m^*(x)-\tilde{m}(x)) \leq p(|x|)$.

Furthermore Theorem 1 partially characterizes the reductions between an arbitrary problem and a weakly rigid one. For example if we consider the trivial reduction (inclusion) from SIMPLE-MAX-CUT to MAX-CUT, we see that the image of SIMPLE-MAX-CUT is a subset of MAX-CUT where the measure is bounded by the number of nodes of the graph and this fact is sufficient to deduct that MAX-CUT is not fully approximable.

In the following we will continue the study of the characterization of reductions between problems belonging to different classes, and we will show how some of the considered properties can be inherited by polynomial reduction, under some natural hypothesis.

THEOREM 2. Let A and B be two NPCO problems such that $A \leq B$ via the reduction $f = \langle f_1, f_2 \rangle$; if A is rigid and if there exists a monotonous function g such that for every $x \in \text{INPUT}_A$, $k \in Q_A$ $f_2(x,k) \leq g(k)$ then B is rigid.

PROOF. If A is rigid there must be an integer $\bar{k}$ such that

$$A_{\bar{k}}^c = \left\{ \langle x,k \rangle \mid \langle x,k \rangle \in A^c \text{ and } \tilde{m}(x) \leq k \leq \tilde{m}(x)+\bar{k} \right\}$$

is P-complete. By hypothesis, if we take $\bar{\bar{k}} = g(\bar{k})$ then

$$B_{\bar{\bar{k}}}^c = \left\{ \langle y,h \rangle \mid \langle y,h \rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y)+g(\bar{k}) \right\}$$

contains $f(A^c_{\bar{k}})$ and, hence if there was a polynomial algorithm for $B^c_{\bar{\bar{k}}}$ it could be used to decide $A^c_{\bar{k}}$ in polynomial time. In fact in order to decide whether $\langle x,k \rangle$ belongs to $A^c_{\bar{k}}$ in the case $k \leq \bar{k}$ (otherwise we trivially know that $\langle x,k \rangle$ does not belong to $A^c_{\bar{k}}$), we may consider $\langle f_1(k), f_2(x,k) \rangle$ and decide whether it belongs to $B^c_{\bar{\bar{k}}}$. QED

REMARK. Note that under the same conditions if $A \leq B$ and B is simple A must be simple. This result shows that no polynomial reduction from a rigid problem to a simple problem is possible unless the function f_2 is such that for no computable function g it is true that for every x and every k $f_2(x,k) \leq g(k)$. In other words $f_2(x,k)$ cannot be dependent only on k but must eventually increase with respect to x.

Notice that theorem 2 strengthens another result given in Paz and Moran (1977) where g is not an arbitrary monotonous function but just a polynomial and the only considered case is when $f_2(x,k)$ is *equal* to $g(k)$.

When we pass from simple problems to strongly simple problems we obtain the following result.

THEOREM 3. Let A and B be two NPCO problems and $A \leq B$ via the reduction $f = \langle f_1, f_2 \rangle$. If there exists a polynomial t such that for all $x \in \mathrm{INPUT}_A$ and $k \in Q_A$ $f_2(x,k) - \tilde{m}(f_1(x)) \leq$ $\leq t(|x|, k - \tilde{m}(x))$ then B strongly simple implies A strongly simple.

PROOF. If B is strongly simple then for all polynomials p we know that the set B^c_p must be polynomially recognizable. Now, let us consider any polynomial r and the set

$$A^c_r = \left\{ \langle x,k \rangle \mid \langle x,k \rangle \in A^c \text{ and } \tilde{m}(x) \leq k \leq \tilde{m}(x) + r(|x|) \right\}$$

we shall show that A^c_r is polynomially decidable. In fact, given $\langle x,k\rangle$, if $k > \tilde{m}(x)+r(|x|)$ or if $k \leq \tilde{m}(x)$ we immediately know that $\langle x,k\rangle$ does not belong to A_r^c. On the other side, if $\tilde{m}(x) \leq k \leq \tilde{m}(x)+r(|x|)$ let us consider the following set:

$$f(A_r^c) = \Big\{\langle f_1(x), f_2(x,k'(x)+\tilde{m}(x))\rangle \,|\, \langle x,k'(x)+\tilde{m}(x)\rangle \in A^c \text{ and } 0 \leq k'(x) \leq r(|x|)\Big\} \;;$$

where $k'(x) = k - \tilde{m}(x)$

by hypothesis $f(A_r^c)$ is included in the set

$$S = \Big\{\langle y,h\rangle \,|\, \langle y,h\rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y)+t(|x|, r(|x|))\Big\}$$

Since we know that if A^c and B^c are NP-complete sets and $A^c \leq B^c$ via $\langle f_1, f_2\rangle$ then we must have $|x| \leq q(|f_1(x)|)$ for every x and a polynomial q, then there must exist a polynomial r' such that

$$B_{r'}^c = \Big\{\langle y,h\rangle \,|\, \langle y,h\rangle \in B^c \text{ and } \tilde{m}(y) \leq h \leq \tilde{m}(y) + r'(|y|)\Big\} \supseteq S.$$

So in order to decide whether $\langle x,k\rangle \in A_r^c$ we may use the reduction f and the polynomial algorithm that decides whether $\langle f_1(x), f_2(x,k)\rangle$ belongs to $B_{r'}^c$. Hence A_r^c is also polynomially decidable.

QED

An interesting consequence of this fact is that, given a problem *A* which is not strongly simple and a problem *B* which is strongly simple any reduction from *A* to *B* must violate the hypothesis.

This means that in a reduction between *A* and *B* the measure must increase exponentially. If we consider similar reductions given by Karp (1972) (e.g. EXACT-COVER$\leq$KNAPSACK) we notice that this is the case and by theorem 3 we may argue that no "easier" reduction may be found.

An analogous result holds in the case of p-simple problems. First of all we prove the following lemma:

LEMMA. Let A be an NPCO problem If A is p-simple, then, for every polynomial p, $A_p^c = \{\langle x,k\rangle \mid \langle x,k\rangle \in A^c \wedge \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(|x|)\}$ is recognizable in $Q(|x|, p(|x|))$ where Q is a polynomial.

PROOF. Let A be p-simple. Given a polynomial p, we can decide $\langle x,k\rangle \in Q_p^c$ in $Q(|x|, p(|x|)$.
In fact if $k > \tilde{m}(x) + p(|x|)$ or $k < \tilde{m}(x)$, it is obvious that $\langle x,k\rangle$ does not belong to A_p^c. Differently, we can use the following algorithmic procedure

1) compute $\bar{k} = p(|x|)$

2) decide if $\langle x,k\rangle \in Q_{\bar{k}}^c$ in $Q(|x|, \bar{k})$

QED

The following theorem holds:

THEOREM 4. Under the same hypotheses of Theorem 3, B p-simple implies A p-simple

PROOF. For every $\bar{k}$ we show that we can decide $A_{\bar{k}}^c$ in time polynomial in $|x|$ and $\bar{k}$. In fact, given $\langle x,k\rangle$, if $\tilde{m}(x) \leq k \leq \tilde{m}(x) + \bar{k}$ (the other cases are trivial), we consider f $(A_{\bar{k}}^c)$ which is included in the set $\bar{S} = \{\langle y,h\rangle \mid \langle y,h\rangle \in B^c \wedge \tilde{m}(y) \leq h \leq \tilde{m}(y) + t(|x|, \bar{k})\}$.
Furthermore if we consider the polynomial $r(u,\bar{k}) = t(q(u), \bar{k})$ where t and q are as in theorem 3, B_r^c contains $\bar{S}$ and, by the lemma, B_r^c is decidable in time $Q(|y|, r(|y|, \bar{k}))$. Using the reduction f and the property of B_r^c we may decide whether $\langle x,k\rangle \in A_{\bar{k}}^c$ within time

$$Q(|f_1(x)|, t(q(|f_1(x)|), \bar{k})) = Q(p(|x|), t(q(p(|x|)), \bar{k}))$$

(due to the polynomiality of the reduction f) what means that the decision time is bounded by a polynomial in $|x|$ and $\bar{k}$.

QED

Since no example is known of a problem which is strongly simple and not p-simple no application of theorem 4 can be provided which is different from the application given at the end of theorem 3.

As a conclusion of this paragraph we may observe that the results provided insofar have a twofold implication. On one side they can be used in order to characterize the computational complexity of one problem with respect to the given definitions, on the other side they establish conditions on the type of reductions that can be found among problems belonging to different classes, such as those discussed at the end of theorem 2 and theorem 3. As a further example we may observe that in the case of the reduction from PARTITION to MAX-CUT the existence of a much more succint reduction than the one given by Karp is ensured by noting that the first problem is strongly simple while the second is weakly rigid.

4. STRONG NP-COMPLETENESS AND ITS RELATION TO RIGIDITY

In the preceding paragraph we have seen that in some cases the characterization of a problem B that is not fully approximable comes out of the fact that we can reduce an NP-complete combinatorial problem A^c into a subset of B^c in which the measure is bounded by a polynomial.Garey and Johnson give another way of considering subsets of the set INPUT of a problem to study the different characteristics of NPCO problems. Their paper (1978) is an attempt to understand the different roles that numbers play in NPCO problems. Let us first consider, for example, the problem MAX-CUT that is a well-known NPCO problem. If we restrict to those graphs with unitary weights we obtain a seemingly easier problem SIMPLE-MAX-CUT, that, however, is still an NPCO problem. Different is the case of the problem JOB-SEQUENCING-WITH-DEADLINES: it has been shown to be NP-complete by Karp, but if we restrict to the case when all weights are unitary then the problem is solvable in $O(n \lg n)$. Moreover if the weights are at most k the problem is solvable by a classic dynamic approach in time bounded by a polynomial in k and in n (the number of jobs). Note that a polynomial algorithm must solve JOB-SEQUENCING-WITH-DEADLINES in time bounded by a polynomial in n and in $\lg k$.

In order to formalize these observations Garey and Johnson introduce another function of the input $\mathrm{MAX}: \mathrm{INPUT} \to Z^+$ that captures the notion of the magnitude of the largest number occurring in the input. For example given a weighted graph G, MAX(G) can be defined as the value of the maximum weighted edge. The following definitions formalize these concepts.

DEFINITION 8. *A pseudo-polynomial algorithm* is an algorithm that on input x runs in time bounded by a polynomial in $|x|$ and in MAX(x).

DEFINITION 9. An NPCO problem is a *pseudopolynomial NPCO problem* if there is a pseudopolynomial algorithm that solves it.

DEFINITION 10. Given a problem P let P_q denote the problem obtained by restricting P to only those instances x in INPUT_q for which $\mathrm{MAX}(x) \leq q(|x|)$

DEFINITION 11. An NPCO problem P is *NP-complete in the strong sense* if there exists a polynomial q such that P_q is NP-complete.

An example of pseudopolynomial NPCO problem is JOB-SEQUENCING-WITH-DEADLINES (Lawler and Moore (1969)) while MAX-CUT is NP-complete in the strong sense (it is sufficient to consider the costant polynomial $q(x) = 1$ to obtain SIMPLE-MAX-CUT).

The two classes of pseudopolynomial NPCO problems and of strong NP-complete problems are disjoint (obviously unless P = NP). The following proposition states the relationship between strong NP-completeness and full approximability.

PROPOSITION 2. If P is NP-complete in the strong sense then it is not fully approximable.
Garey and Johnson give another result that connects the two concepts of pseudopolynomial and fully approximable NPCO problem; for clarity sake, we will give it later as an immediate consequence of Theorem 6.
In many problems the optimal value of the measure and the MAX of the input have the same size or it is possible to establish a polynomial relation between them. This suggests the idea of comparing some of the different concepts introduced in the preceding paragraphs and in this one. First of all we can prove the following.

FACT 1. Let A be a pseudopolynomial optimization problem. If there exists a polynomial q such that for every $x \in \mathrm{INPUT}_A$ $\mathrm{MAX}(x) \leq q(m^*(x)-\tilde{m}(x), |x|)$, then given $\langle x,k \rangle$, it is possible to decide in polynomial time if $m^*(x) \leq k$ or $m^*(x) > k$.

PROOF. The hypotheses imply that there exists a polynomial p such that, given x, $m^*(x)$ is computable within time $p(|x|, MAX(x))$ and, therefore, within time $p(|x|, q((m^*(x)-\tilde{m}(x)), |x|))$. We apply the pseudopolynomial algorithm to x for $p(|x|, q(k,|x|))$ steps. If the algorithm stops, it is decidable if $m^*(x) \leq k$ or $m^*(x) > k$; instead if the algorithm does not terminate in $p(|x|, q(k,|x|))$, then $m^*(x) > k$

QED

As all known pseudopolynomial algorithms make us of dynamic programming, it is possible, very often, to state fact 1 in a more interesting way.

FACT 2. Let A be a pseudopolynomial optimization problem. If there exists a polynomial q such that for, every $x \in INPUT_A$, $MAX(x) \leq q(m^*(x)-\tilde{m}(x), |x|)$, then A is p-simple.

THEOREM 5. Let A be a p-simple problem. If there exists a polynomial q such that, for every $x \in INPUT_A$, $(m^*(x)-\tilde{m}(x)) \leq q(MAX(x), |x|)$, then A is a pseudopolynomial NPCO problem.

PROOF. By the hypothesis for each k A_k^c is recognizable in time $Q(|x|,k)$. To obtain $m^*(x)$ we can use the following algorithm:

for k : = 0 to $q(MAX(x), |x|)$

repeat the following step:

if $\langle x,k \rangle \in A_k^c$ then $m^*(x) = \tilde{m}(x) + k$

By hypothesis there are no more than $q(\mathrm{Max}(x),|x|)+1$ iterations of steps 2 . As A^c is p-simple each iteration of step 2 takes no more than $Q(|x|,q(\mathrm{MAX}(x),|x|)+1)$. Therefore $m^*(x)$ is computable in at most $(q(\mathrm{MAX}(x),|x|)+1)\cdot Q(|x|, q(\mathrm{MAX}(x),|x|))$. QED

COROLLARY. (Garey and Johnson (1978)). Let A be a fully approximable NPCO problem. If there exists a polynomial q such that for every $x \in \mathrm{INPUT}_A$ $(m^*(x)-\tilde{m}(x)) \leq q(\mathrm{MAX}(x),|x|)$ then A is a pseudopolynomial NPCO problem.

PROOF. Immediate from the previous theorem and the fact that a fully approximable problem is p-simple. QED

As the conditions of theorem 5 and Fact 1 are generally verified the two concepts of pseudopolynomial problem and p-simple problem coincide in many cases.

A natural question arises at this point: when the conditions of the theorems are not verified which of the two approaches gives a better information about the complexity of approximate algorithms?

Let us define

$$(\mathrm{P1}) \quad \mathrm{Max} \sum_{j=1}^{n} c_j y_j$$

$$\text{subject to} \sum_{j=1}^{n} a_j y_j = b \quad y_j = 0,1 \quad j = 1,2,\ldots n$$

A natural definition of $\mathrm{MAX}(\mathrm{INPUT}_{P1})$ can be the following $\mathrm{MAX}(x) = \max_j (c_j, a_j)$ and it is not hard to prove that P1 is pseudopolynomial (a classic dynamic approach solves it in $0(n^2\ \mathrm{MAX}(x)))$; however even the problem to obtain any approximate solution is an NP-complete problem (Karp (1972)). Therefore P1 is a pseudopolynomial NPCO problem that is not approximable.

Let us consider now:

(P2) $\prod_{j=1}^{n} c_j^{y_j}$

$$\text{subject to } \prod_{j=1}^{n} a_j^{y_j} \leq b \ , \quad y_j = 0,1 \qquad j = 1,2,\ldots n$$

This problem is fully approximable and we conjecture that it is not a pseudopolynomial problem because the classical method of deriving a pseudopolynomial algorithm from the dynamic programming approach does not work. Theorems 5 and 6 and the previous examples show that Paz and Moran's approach has a wider application for two different reasons. First their approach is straightforward and there is no need to introduce the function MAX whose definition can be ambiguous in some cases.

In addition we have proven that the two approaches are equivalent under restricted but reasonable hypotheses and we have shown that when $m^*(x)$ and MAX(x) are not polynomially related the approach formulated by Paz and Moran remains adequate to study the complexity of approximation schemes for NPCO problems.

Before finishing this paragraph we want to observe that, when there is a polynomial relation between the value of the optimal solution and the value of MAX , there is a strong connection between the two concepts of strong NP-complete and weakly rigid.

THEOREM 6. Let A be a strong NP-complete optimization problem. If there exists a polynomial p such that for every $x \in \text{INPUT}_A$ $(m^*(x)-\tilde{m}(x)) \leq p(\text{MAX}(x),|x|)$ then A is weakly rigid.

PROOF. If A is NP-complete in the strong sense there must exist a polynomial q such that the following set $Q = \{\langle x,k \rangle | \langle x,k \rangle \in A^c,\ MAX(x) \leq q(|x|)\}$ is NP-complete.
Let us consider now the set

$$Q' = \{\langle x,k \rangle | \langle x,k \rangle \in A^c, MAX(x) \leq q(|x|), \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(MAX(x), |x|)\}$$

As $Q \supseteq Q'$ in order to prove that $Q \equiv Q'$ it is sufficient to prove that

$$Q - Q' \equiv \{\langle x,k \rangle | \langle x,k \rangle \in A^c, MAX(x) \leq q(|x|),\ k \geq \tilde{m}(x) + p(MAX(x), |x|)\}$$

is the empty set. In fact given $\langle x,k \rangle$, with $k \geq \tilde{m}(x) + p(MAX(x), |x|)$, we have by hypothesis $k \geq \tilde{m}(x) + p(MAX(x), |x|) \geq m^*(x)$ and therefore $\langle x,k \rangle \notin A^c$. Let us consider now

$$Q'' = \{\langle x,k \rangle | \langle x,k \rangle \in A^c,\ \tilde{m}(x) \leq k \leq \tilde{m}(x) + p(q(|x|, |x|)\}$$

Clearly Q" is NP-complete and hence A is weakly rigid.

QED

5. CONCLUSIONS

In this paper we have shown that there exist close relations among different approaches to the classification of NP-complete optimization problems, giving also new results on the type of possible reductions among problems belonging to different classes. On the other side, it was proven that, violating some conditions, comparisons among different concepts do not hold any more.

Therefore we believe that, in the whole, our results are a useful contribution for a better understanding of properties of NPCO problems. We think that in order to provide meaningful characterizations of NOCO problems it is necessary to find the suitable level of abstraction because if a too general point of view is taken NPCO problems appear to be hardly distinguishable while if too many details are taken into consideration it is difficult to grasp similarities among different problems. The results stated in this paper are, as we feel,at the right level. For the same reason we would like to broaden our considerations and results to other approaches which stand at the same level of abstraction. In Ausiello, D'Atri, Protasi (1977) a distinction was introduced between convex and non convex problems (a problem is said to be convex if, for every integer k between the worst and the best solution, there is, at least, an approximate solution of measure k). It is interesting to observe that many examples show that the property of being non convex is related to the approximation properties of the problems.

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