Finally, by Propositions 10, 10^* and Theorem 9 we obtain:

THEOREM 11. - *Let S be a countably paracompact normal space and G a finite directed graph. Then there exists a natural bijection from the set of o-homotopy classes O(S,G) to the one of o^*-homotopy classes $O^*(S,G)$.*

Proof. In fact the assumption on *S* is equivalent to suppose that *S* and $S\times I$ are normal spaces. (See Introduction). ■

REMARK 1. - In general the previous result does not hold for any topological space. (See Example 13.5).

REMARK 2. - In the foregoing conditions it follows that the sets *Q(S,G)*, $Q(S,G^*)$, $Q^*(S,G)$, $Q^*(S,G^*)$ can be identified.

PART TWO. DUALITY THEOREM FOR REGULAR FUNCTIONS BETWEEN PAIRS.

5) Balanced functions.

We can characterize the regular functions between pairs, similarly to Propositions 2, 3, by the following:

PROPOSITION 12. - *Let* f: S,S' → G,G' *be a function from a pair of topological spaces* S,S' *to a pair of finite directed graphs* G,G' *and* f': S' → G' *the restriction of* f: S → G *to* S'. *Then* f *is an o-regular function, iff* f(x) *is a head of* <f(x)> *in* G, *for all* x ∈ S; *while* f'(x) *is a head of* <f'(x)> *in* G', *for all* x ∈ S'. *Moreover,* f *is c.o-regular, iff also the subsets* <f(x)> *are totally headed in* G *and all the subsets* <f'(x)> *are totally headed in* G'. ■

REMARK. - Consequently, *if* G *is an undirected graph, a function* f: S,S' → G,G' *is strongly regular, iff* $<f(x)>$ *is totally headed in* G *for all* $x \in S$*, and so is* $<f'(x)>$ *in* G' *for all* $x \in S'$.

Unfortunately the considerations developed in Part one in order to obtain the Duality Theorem for regular functions can not be directly generalized to regular functions between pairs, since there does not exist an o^*-pattern of any c.o-regular function $f: S,S' \to G,G'$ in general. Hence we must add the following new condition:

$T_G(<f(x')>) \cap T_{G'}(<f'(x')>) \neq \phi, \ \forall\, x' \in S'$,

and consequently we put:

DEFINITION 6. - *Let* $f: S,S' \to G,G'$ *be a function from a pair of topological spaces* S,S' *to a pair of finite directed graph* G,G' *and let* $f': S' \to G'$ *be the restriction to* S' *of* $f: S \to G$. *The function* f *is said to be* balanced *in* (S,S') *or simply a* b.function *if, for all* $x' \in S'$ *and for all* $v \in G$*, it is* $x' \in \overline{V^f} \Rightarrow x' \in \overline{V^{f'}}$ *: i.e., for all* $x' \in S'$, $<f(x')> = <f'(x')>$.

REMARK 1. - If the restriction f' of a b.function f is c.o-regular, by R_a and Proposition 3 it results $T_G(<f(x')>) \cap T_{G'}(<f'(x')>) \neq \phi$, since now we have $T_{G'}(<f'(x')>) \subseteq T_G(<f(x')>)$; while for a c.o-regular function it can happen that $T_G(<f(x')>) \cap T_G(<f'(x')>) = \phi$. (See Example 13.1).

REMARK 2. - We can also write *b.o-regular function,...,b.homotopy ,...* instead of *balanced o-regular function,..., balanced homotopy,...* .

PROPOSITION 13. - *Under the assumptions of Definition 6, if* S' *is an open set of* S*, all the functions* $f: S,S' \to G,G'$ *are balanced in* (S,S').

Proof. - By Proposition 1 we have $<f(x')> = \cap\{f(U_{x'}) \ / \ U_{x'}$ is a neighbourhood of x' in $S\}$, while for S' it results $<f'(x')> = \cap\ \{f'(U_{x'} \cap S')\}$. Now, since S' is open in S, it follows $\cap\{f'(U_{x'} \cap S')\} = \cap\{f(U_{x'})\}$. ■

6) *Patterns of a function between pairs.*

As in Definitions 2, 3 we have:

DEFINITION 7. - *Let* f:S,S' → G,G' *be a function from a pair of topological spaces* S,S' *to a pair of finite directed graphs* G,G'. *A function* g: S,S' → G,G' *is called an* o-pattern *(resp.* o^*-pattern*) of* f, *if* g: S → G *is an o-pattern (resp.* o^**-pattern) of* f: S → G *and its restriction* g': S' → G' *is an o-pattern (resp.* o^**-pattern) of* f': S' → G'.

REMARK. - For an o-pattern g of f, we have the following relations:

i) $\forall\, x \in S-S'$, $\quad g(x) \in H_G(<f(x)>)$

ii) $\forall\, x' \in S'$, $\quad g(x') \in H_G(<f(x')>) \;\cap H_{G'}(<f'(x')>)$.

DEFINITION 8. - *Under the assumptions of Definition 7, the function* f: S,S' → G,G' *is called* q.o-regular *(resp.* q.o^*-regular, c.q.regular), *if such are the function* f: S → G *and its restriction* f': S' → G'.

REMARK. - Also for pairs, we can get results similar to those of Remarks 1, 2 to Definition 3 and of Proposition 4.

Instead of Proposition 5, we have only:

PROPOSITION 14. - *If a q.o-regular function* f: S,S' → G,G' *is balanced in* (S,S'), *there exists an o-pattern of* f.

Proof. - For all $x \in S-S'$, we proceed as in ii) of the proof of Proposition 5. While, for all $x' \in S'$, we choose as $g(x')$ the vertex with the lowest index among the vertices of $H_{G'}(<f'(x')>) \subseteq H_G(<f'(x')>) = H_G(<f(x')>)$. ■

REMARK 1. - In general there are no patterns of functions that are only balanced or q.o-regular. The condition of q.o-regularity, indeed, is necessary by Proposition 5, while the condition of b.q.o-regularity is only sufficient. (See Example 13.2).

REMARK 2. - In general an o-pattern of a b.function $f: S,S' \to G,G'$ is not balanced. (See Example 13.2).

DEFINITION 9. - *Two functions* f,g: S,S' → G,G' *from a pair of topological spaces* S,S' *to a pair of finite directed graphs* G,G' *are called* c.o-homotopic *(resp.* c.o*-homotopic*) if there exists a homotopy* F *between* f *and* g, *which is a c.o-regular (resp. c.o*-regular) function.*

By following the proofs of Propositions 6, 7 and by using Definitions 7, 8, we can obtain properties similar to Propositions 6,7, since both the functions from S to G and the ones from S' to G' satisfy the conditions. But, on account of Remark 2 to Proposition 14, in general, the constructed o-patterns are not balanced. Nevertheless, by Proposition 13, we have:

PROPOSITION 15. - *Let* S *be a topological space,* S' *an open subset of* S, G *a finite directed graph,* G' *a subgraph of* G *and* f:S,S' → G,G' *a b.c.q.regular function from* S,S' *to* G,G'. *Then:*

i) all its o-patterns are b.c.o-regular functions,

ii) two o-patterns of f *are b.c.o-homotopic to each other.* ∎

7) Duality Theorem for complete homotopy classes when S' *is open.*

Now we just state the Duality Theorem for the c.homotopy, when S' is an open subspace of S.

Similarly to Proposition 8, we can prove that the c.o-homotopy is an equivalence relation in the set of c.o-regular functions from S,S' to G,G'. Then it follows:

DEFINITION 10. - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite directed graph and* G' *a subgraph of* G. *We denote by* $Q_C(S,S';G,G')$ *(resp.* $Q^*_C(S,S';G,G')$*) the set of c.o-homotopy (resp. c.o*-homotopy) classes.*

REMARK. - $Q^*_C(S,S';G,G')$ coincides with $Q_C(S,S';G^*,G'^*)$ and $Q_C(S,S';G,G')$ with $Q^*_C(S,S';G^*,G'^*)$.

THEOREM 16. - *Let* S *be a topological space,* S' *an open subspace of* S, G *a finite directed graph and* G' *a subgraph of* G. *Then there exists a natural bijection* Φ *from the set of complete o-homotopy classes* $Q_C(S,S';G,G')$ *to the one of complete o*-homotopy classes* $Q^*_C(S,S';G,G')$.

Proof. - It is similar to that one of Theorem 9, by using Propositions 15, 15*. We just observe that, since *S'* is open, the functions are balanced, hence the sought patterns can be constructed. ■

REMARK. - The proof of Theorem 9 can not be generalized for any subspace *S'* of *S*. In step iii), indeed, in order to construct an o-pattern *z* of *h*, it is necessary that *h* is a balanced o*-pattern of *f*. (See Example 13.1).

8) *Inductive limits.*

Let *S'* be any subspace of *S* and *U* any open neighbourhood of *S'*. We have:

DEFINITION 11. - *We denote by* $F_C(S,S';G,G')$ *the set of c.o-regular functions from* S,S' *to* G,G', *by* $F_U = F_C(S,U;G,G')$ *the set of c.o-regular functions from* S,U *to* G,G' *and by* $Q_U = Q_C(S,U;G,G')$ *the set of c.o-homotopy classes of functions from* S,U *to* G,G'. *Dually, we can consider* $F^*_C(S,S';G,G')$, $F^*_U = F^*_C(S,U;G,G')$ *and* $Q^*_U = Q^*_C(S,U;G,G')$.

Now we consider the collection of sets $\mathcal{U}_{S'}$ = { *U* / *U* is an open neighbourhood of S'} and, since $U_{S'}$ is decreasingly filtrated, it follows:

PROPOSITION 17. - *The family of sets* $\{ F_U \,/\, U \in U_{S'}\}$ *with associated maps* $\{ \lambda_V^U \,/\, U,V \in U_{S'},\ V \subseteq U\}$ *is an inductive family if* $\lambda_V^U : F_U \to F_V$ *is the identical embedding.* ■

PROPOSITION 18. - *The associated map* λ_V^U $(U,V \in U_{S'},\ V \subseteq U)$ *defined in Proposition 17, is compatible with the c.o-homotopy in* F_U *and* F_V.

Proof. - If $f,g\colon S,U \to G,G'$ are c.o-homotopic, such are also the functions $f,g\colon S,V \to G,G'$. ■

PROPOSITION 19. - *Let* $\Lambda_V^U\colon Q_U \to Q_V$ *be the function induced by the identical embedding* $\lambda_V^U\colon F_U \to F_V$, *then the family of sets* $\{Q_U \,/\, U \in U_{S'}\}$ *with associated maps* $\{\Lambda_V^U \,/\, U,V \in U_{S'},\ V \subseteq U\}$ *is an inductive family.*

Proof. - The family $\{Q_U\}$ is inductive since, given $U,V,W \in U_{S'} \,/\, U \subseteq V \subseteq W$, from $\lambda_W^U = \lambda_W^V \lambda_V^U$ it results $\Lambda_W^U = \Lambda_W^V \Lambda_V^U$. ■

Now, if we consider the family of bijections $\{\Phi_U\colon Q_U \to Q_U^* \,/\, U \in U_{S'}\}$ (see Theorem 16), we obtain:

THEOREM 20. - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite directed graph and* G' *a subgraph of* G. *Then there exists a natural bijection* Φ *from the inductive limit* $\varinjlim Q_U$ *to* $\varinjlim Q_U^*$.

Proof. - Let $U,V \in U_{S'}$ be and $V \subseteq U$. We see that the diagram:

$$\begin{array}{ccc} Q_U & \xrightarrow{\Phi_U} & Q_U^* \\ {\scriptstyle \Lambda_V^U}\downarrow & & \downarrow{\scriptstyle \Lambda^{*U}_V} \\ Q_V & \xrightarrow[\Phi_V]{} & Q_V^* \end{array}$$

is commutative. Following, indeed, the proof of Theorem 9, we must just observe that the identical embedding of a pattern of $f \in F_U$ is a pattern of $f \in F_V$. Consequently, there exists a natural bijection Φ from $\varinjlim Q_U$ to $\varinjlim Q^*_U$, since $\forall\, U \in \mathcal{U}_{S'}$, Φ_U is a natural bijection by Theorem 16. (See [8], 40.1). ■

9) Neighbourhood completely regular functions and homotopies.

The inductive limits of § 8 can be regarded also as sets of regular functions and homotopy classes.

DEFINITION 12. - *Let* f: S,S' → G,G' *be a c.o-regular (resp. c.o*-regular) function from a pair of topological spaces* S,S' *to a pair of finite directed graphs* G,G'. *The function* f *is called* neighbourhood completely o-regular *(resp.* neighbourhood completely o*-regular*) in* (S,S')*, or simply* n.c.o-regular *(resp.* n.c.o*-regular*) if there exists an open neighbourhood* U *of* S'*, such that the function* f: S,U → G,G' *is c.o-regular (resp. c.o*-regular). The open neighbourhood* U *is called a* balancer *of* f: S,S' → G,G'

REMARK 1. - We call U a balancer since by Proposition 13 the function $f\colon S,U \to G,G'$ is balanced.

REMARK 2. - A function $f\colon S,S' \to G,G'$ can be b.c.o-regular without being n.c. o-regular. (See Example 13.3).

PROPOSITION 21. - *The inductive limit* $\varinjlim F_U$ *coincides with the set* $F_{nc}(S,S;G,G')$ *of the n.c.o-regular functions from* S,S' *to* G,G'.

Proof. - In fact, two c.o-regular functions $f\colon S,U \to G,G'$ and $g\colon S,V \to G,G'$ $(U,V \in \mathcal{U}_{S'})$ are equivalent iff $f\colon S \to G$ coincides with $g\colon S \to G$, since $\lambda^U_{U\cap V}(f) = \lambda^V_{U\cap V}(g)$. ■

DEFINITION 13. - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite* *directed graph and* G' *a subgraph of* G. *Two n.c.o-regular (resp. n.c.o***-regular) functions* f,g: $S,S' \to G,G'$ *are called* n.c.o-homotopic *(resp.* n.c.o*-homotopic*), if there exist an open neighbourhood* W *of* $S'\times I$ *and a homotopy* F: $S\times I, S'\times I \to G,G'$ *between* f *and* g *such that* F: $S\times I, W \to G,G'$ *is a c.o-regular (resp. c.o***-regular) function.* F *is called a* n.c.o-homotopy *(resp.* n.c.o*-homotopy).

REMARK. - $W \cap (S\times\{0\})$ and $W \cap (S\times\{1\})$ can be considered respectively balancers of f and g.

LEMMA 22. - *Let* S *be a topological space and* S' *a subspace of* S. *Then, for every neighbourhood* W *of* $S'\times I$ *in* $S\times I$, *there exists a neighbourhood* U *of* S', *such that* $S'\times I \subseteq U\times I \subseteq W$.

Proof. - If x is a point of S', then, for all $t \in I$, there is a neighbourhood of (x,t) of the form $U_x^{(t)}\times U_t \subseteq W$. Since I is compact, there exists a finite set, namely $U_{t_1},\dots,U_{t_n}$, of neighbourhoods which covers I. Thus, if we put $U_x = U^{(t_1)} \cap \dots \cap U^{(t_n)}$, we have $U_x\times I$ is a neighbourhood of $\{x\}\times I$ included in W. By choosing $U = \cup\, U_x$, $\forall x \in S'$, the assertion immediately follows. ■

Directly, for open neighbourhoods we have:

PROPOSITION 23. - *Under the assumptions of Definition 13, let* F *be a n.c.o-homotopy. Then there exists a balancer of* F *of the form* $U\times I$, *where* $U \in \mathcal{U}_{S'}$. ■

PROPOSITION 24. - *The n.c.o-homotopy relation is an equivalence relation in the set* $F_{nc}(S,S';G,G')$ *of n.c.o-regular functions from* S,S' *to* G,G'.

Proof. - The relation obviously satisfies the reflexive and symmetric properties. Also the transitive property is true: in fact, by using the same notations of the proof of Proposition 8, the homotopy K is c.o-regular by the same proposition. Moreover, f is n.c.o-regular, because if we construct by Proposition 23 a balancer $U\times I$

of F and a balancer $V\times I$ of J, $(U\cap V)\times I$ is a balancer of K. ■

DEFINITION 14. - *Under the assumptions of Definition 13, we call* $Q_{nc}(S,S';G,G')$ *(resp.* $Q^*_{nc}(S,S';G,G')$*) the set of n.c.o-homotopy (resp. n.c.o*-homotopy) classes.*

REMARK. - We note that $Q^*_{nc}(S,S';G,G')$ coincides with $Q_{nc}(S,S';G^*,G'^*)$ and $Q^*_{nc}(S,S';G^*,G'^*)$ with $Q_{nc}(S,S';G,G')$.

PROPOSITION 25. - *The inductive limit* $\varinjlim Q_U$ *coincides with the set* $Q_{nc}(S,S';G,G')$ *of the n.c.o-homotopy classes.*

Proof. - $\forall\ U \in \mathcal{U}_{S'}$, let $\phi_U\colon F_U \to F_{nc}(S,S';G,G')$ be the identical embedding. Since ϕ_U is compatible with the respective homotopy relations, we denote by $\Phi_U\colon Q_U \to Q_{nc}(S,S';G,G')$ the induced function. Now the diagram:

$$\begin{array}{ccc} Q_U & & \\ {\scriptstyle \Lambda^U_V}\big\downarrow & \overset{\Phi_U}{\searrow} & \\ & & Q_{nc}(S,S';G,G') \\ & \underset{\Phi_V}{\nearrow} & \\ Q_V & & \end{array} \qquad (\forall\ U,V \in \mathcal{U}_{S'}\ /\ V \subseteq U)$$

is commutative, then we can define a function $\Phi\colon \varinjlim Q_U \to Q_{nc}(S,S';G,G')$. Moreover Φ is onto by definition. Finally, we see that Φ is one to one. Let, indeed, $\alpha,\beta \in Q_U\ /\ \Phi(\alpha) = \Phi(\beta)$ be, then, if $f \in \alpha$ and $g \in \beta$, there exists a balancer V such that f and g are c.o-homotopic. Consequently, we have $\Lambda^U_{U\cap V}\,\alpha = \Lambda^U_{U\cap V}\,\beta$. ■

Then, Theorem 20 becames:

THEOREM 26 . - *Let* S *be a topological space,* S' *a subspace of* S, G *a finite directed graph,* G' *a subgraph of* G. *Then there exists a natural bijection from the set of neighbourhood complete o-homotopy classes* $Q_{nc}(S,S';G,G')$ *to the one of neighbouhood complete o*-homotopy classes* $Q^*_{nc}(S,S';G,G')$. ■

10) Duality Theorem for homotopy classes.

In addition to the Extension Theorem R_c, we need the following:

PROPOSITION 27. - *Let* S *be a normal topological space,* S' *a closed subspace of* S, X *a closed subset of* S', G *a finite directed graph,* G' *a subgraph of* G *and* f: S,S' → G,G' *an o-regular function. Then there exist a closed neighbourhood* W *of* X *and an o-regular function* g: S,S' → G,G'*, which is o-homotopic to* f *and such that* g: S,S'∪W → G,G' *is o-regular.*

Proof. - It is similar to that one of Theorem 20 in [2], by putting $X^* = X$, rather than $X^* = S'$. ■

Moreover, if we recall the definition of singularity (see Background), by R_d the Extension Theorem can be completed by the following:

PROPOSITION 28. - *Let* S *be a normal topological space,* S' *a closed subspace of* S, G *a finite directed graph,* G' *a subgraph of* G *and* f: S,S' → G,G' *a c.o-regular function. Then there exists a closed neighbourhood* W *of* S' *and a function* g: S,S' → G,G'*, which is o-homotopic to* f *and such that the function* g: S,W → G,G' *is c.o-regular.* (See [2], Corollary 22). ■

Similarly, we have (see also [2], Corollaries 12, 19) :

PROPOSITION 29. - *Under the assumptions of Proposition 27, if* f: S,S' → G,G' *is c.o-regular, so is also the function* g: S,S'∪W → G,G'. ■

PROPOSITION 30. - *Let* S×I *a normal topological space,* S' *a closed subspace of* S, G *a finite directed graph and* G' *a subgraph of* G*. Then two c.o-homotopic n.c.o-regular functions* f,g: S,S' → G,G' *are also n.c.o-homotopic.*

Proof. - Let the open neighbourhood U be a balancer of f and g, and let $F\colon S\times I, S'\times I \to G,G'$ be a c.o-homotopy between f and g. We define the c.o-homotopy $J : S\times I,$

$S'\times I \to G,G'$, given by:

$$J(x,t) = \begin{cases} f(x) & \forall x \in S, \quad \forall t \in [0,\frac{1}{3}] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in [\frac{1}{3},\frac{2}{3}] \\ g(x) & \forall x \in S, \quad \forall t \in [\frac{2}{3},1]. \end{cases}$$

(See [3],Theorem 16).

Since S is normal, there exists a closed neighbourhood W of S' included in U. We put $Z = W\times[0,\frac{1}{4}] \cup S'\times[\frac{1}{4},\frac{3}{4}] \cup W\times[\frac{3}{4},1]$ and we note that the function $J\colon S\times I,Z \to G,G'$ is c.o-regular, since $W\times[0,\frac{1}{4}] \subset U\times[0,\frac{1}{3}[$, $S'\times[\frac{1}{4},\frac{3}{4}] \subset S'\times I$ and $W\times[\frac{3}{4},1] \subset U\times]\frac{2}{3},1]$. Moreover, we can apply Propositions 27, 29, since Z is closed, $S'\times[\frac{1}{4},\frac{3}{4}]$ is a closed subset of Z and $S\times I$ is normal. Then we can construct a closed neighbourhood T of $S'\times[\frac{1}{4},\frac{3}{4}]$ and a c.o-regular function $K\colon S\times I,Z\cup T \to G,G'$ which is also a homotopy between f and g, by choosing the closed neighbourhoods $L^{(i,j,k)}$, which we employ, disjoined from $S\times\{0\}$ and $S\times\{1\}$. Finally, since $Z\cup T$ is a closed neighbourhood of $S'\times I$, it follows immediately that f and g are n.c.o-homotopic. ■

THEOREM 31. - *Let* S×I *be a normal topological space,* S' *a closed subspace of* S, G *a finite directed graph and* G' *a subgraph of* G. *Then there exists a natural bijection from the set of n.c.o-homotopy classes* Q_{nc}(S,S';G,G') *to the one of o-homotopy classes* Q(S,S';G,G').

Proof. - Let $j\colon F_{nc}(S,S';G,G') \to F(S,S';G,G')$ be the identical embedding. Since two n.c.o-homotopic functions are also o-homotopic, j induces a function J from $Q_{nc}(S,S';G,G')$ to $Q(S,S';G,G')$. Moreover, J is onto by R_h and Proposition 28 and it is one to one by R_e and Proposition 30. ■

Finally, by Theorems 31, 31^* and 26 we obtain (see Theorem 11):

THEOREM 32. - *Let* S *be a countably paracompact normal space,* S' *a closed subspace of* S, G *a finite directed graph and* G' *a subgraph of* G. *Then there exists a natural bijection from the set of o-homotopy classes* Q(S,S',G,G') *to the one of* o^**-homotopy classes* Q^*(S,S';G,G'). ■

REMARK 1. - In general the previous result does not hold for any topological space. (See Example 13.4).

REMARK 2. - In the foregoing conditions it follows that the sets $Q(S,S';G,G')$, $Q(S,S';G^*,G'^*)$, $Q^*(S,S',G,G')$, $Q^*(S,S';G^*,G'^*)$ can be identified.

11) Case of n *subspaces and of* n *subgraphs.*

The previous results between pairs can be easily generalized to the case between $(n+1)$-tuples. (See [2], § 8 b).

Let S be a topological space, G a finite directed graph, $S_1,\dots,S_n$ subspaces of S and $G_1,\dots,G_n$ subgraphs of G such that S_j is a subspace of S_i and G_j is a subgraph of G_i, $\forall i,j = 1,\dots,n$, $j>i$.

In this case we have to consider functions $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ between $(n+1)$-tuples and their restrictions $f_1\colon S_1 \to G_1,\dots, f_n\colon S_n \to G_n$.

We only remark that:

1) A function $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ is said to be *balanced* in $(S,S_1,\dots,S_n)$ if:

   i) $\forall x_1 \in S_1$, $\quad <f(x_1)> = <f_1(x_1)>$,

   ii) $\forall x_2 \in S_2$, $\quad <f(x_2)> = <f_1(x_2)> = <f_2(x_2)>$,

   ...

   n) $\forall x_n \in S_n$, $\quad <f(x_n)> = <f_1(x_n)> = \dots = <f_n(x_n)>$. (See Definition 6).

2) If the subspaces $S_1,\dots S_n$ are open in S, all the functions $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ are balanced in $(S,S_1,\dots,S_n)$. (See Proposition 13). Hence the Duality Theorem for complete homotopy classes holds when all the subspaces are open. (See Theorem 16).

3) If we denote by $\mathcal{U}_{S_1},\dots,\mathcal{U}_{S_n}$ the collections of open neighbourhoods respectively of the subspaces $S_1,\dots,S_n$, in $\mathcal{U}_{S_1}\times\dots\times\mathcal{U}_{S_n}$ we can consider the subset $\mathcal{U}$ of all the n-tuples $U = (U_1,\dots,U_n)$ such that $U_1 \supseteq \dots \supseteq U_n$. By putting $U \leqslant V \Leftrightarrow U_1 \subseteq V_1,\dots, U_n \subseteq V_n$, it follows that $\mathcal{U}$ is decreasingly filtrated, then the families of sets $\{F_U \,/\, U \in \mathcal{U}\}$ and $\{Q_U \,/\, U \in \mathcal{U}\}$ are inductive and there exists a natural bijection between $\varinjlim Q_U$ and $\varinjlim Q^*_U$. (See Theorem 20).

4) A function $f\colon S, S_1,\dots,S_n \to G,G_1,\dots,G_n$ is called *n.c.o-regular* in $(S,S_1,\dots,S_n)$ if in $\mathcal{U}$ there exists a n-tuple $U = (U_1,\dots,U_n)$ such that the function $f\colon S,U_1,\dots,U_n \to G,G_1,\dots,G_n$ is c.o-regular. (See Definition 12).

Then two n.c.o-regular functions $f,g\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ are called *n.c.o-homotopic* if there exists a homotopy $F\colon S{\times}I,S_1{\times}I,\dots,S_n{\times}I \to G,G_1,\dots,G_n$ which is a n.c.o-regular function (See Definition 13).

Hence by 3) we obtain a natural bijection between the sets of n.c.homotopy classes $Q_{nc}(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$ and $Q^*_{nc}(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$. (See Theorem 26).

Moreover, if $S{\times}I$ is a normal space and $S_1,\dots,S_n$ are closed subspaces of S, we also observe that:

5) We can generalize the Normalization Theorems (R_b, R_e) following the construction used in [3], Final remark i).

6) For the generalization of the Extension Theorems (see R_c, Proposition 27), let $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ be an o-regular function. By following what we said in [2], §8 b, it results:

i) We can construct a closed neighbourhood U_n of S_n and an o-regular function $g^{(1)}\colon S,S_1 \cup U_n,\dots,S_{n-1} \cup U_n,U_n \to G,G_1,\dots,G_n$ o-homotopic to f.

ii) Let V_n be a closed neighbourhood of S_n such that $S_n \subseteq V_n \subseteq A_n \subseteq U_n$, where A_n is an open set. We construct a closed neighbourhood U_{n-1} of $S_{n-1} \cup U_n$ and an o-regular function $g^{(2)}\colon S,S_1 \cup U_{n-1},\dots,S_{n-2} \cup U_{n-1},U_{n-1} \to G,G_1,\dots,G_{n-1}$ o-homotopic to $g^{(1)}$ by choosing the closed neighbourhoods, which we employ in the construction of $g^{(2)}$, disjoined from V_n. Consequently, also the function $g^{(2)}\colon S, S_1 \cup U_{n-1},\dots,S_{n-2} \cup U_{n-1},U_{n-1},V_n \to G,G_1,\dots,G_n$ is o-regular and o-homotopic to f.

iii) Let V_{n-1} be a closed neighbourhood of $S_{n-1} \cup V_n$ such that $S_{n-1} \cup V_n \subseteq V_{n-1} \subseteq A_{n-1} \subseteq U_{n-1}$, where A_{n-1} is an open set. Then we go on as in step ii).

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n) Let $g^{(n-1)}\colon S,S_1 \cup U_2,U_2,V_3,\dots,V_n \to G,G_1,\dots,G_n$ be the o-regular function, o-homotopic to f, which follows from the previous process. Then, let V_2 be a closed neighbourhood of $S_2 \cup V_3$ such that $S_2 \cup V_3 \subseteq V_2 \subseteq A_2 \subseteq U_2$, where A_2 is an open set. We construct a closed neighbourhood U_1 of $S_1 \cup U_2$ and an o-regular func tion $g^{(n)}: S,U_1 \to G,G_1$ o-homotopic to $g^{(n-1)}$ by choosing the closed neighbourhoods, which we employ in the construction of $g^{(n)}$ disjoined from V_2. Consequently, also the function $g^{(n)}: S,U_1,V_2,\dots,V_n \to G,G_1,\dots,G_n$ is o-regular and o-homotopic to f. Since $U_1,V_2,\dots, V_n$ are respectively closed neighbourhoods of $S_1,\dots,S_n$, the function $g^{(n)}$ is the sought extension.

7) Similarly to Theorem 31, from 6) it follows that there exists a natural bijection between the sets of o-homotopy classes $Q_{nc}(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$ and $Q(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$, when $S\times I$ is a normal space and the subspaces S_i are closed.

8) Finally, by 4) we obtain the *conclusive theorem* (see Theorem 32):

THEOREM 33. - *Let* S *be a countably paracompact normal space,* G *a finite directed graph,* $S_1,\dots,S_n$ *closed subspaces of* S *and* $G_1,\dots,G_n$ *subgraphs of* G, *such that* S_j *is a subspace of* S_i *and* G_j *is a subgraph of* G_i, $\forall$ i,j = 1,...,n, j > i. *Then there exists a natural bijection from the set of o-homotopy classes* $Q(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$ *to the one of* o^**-homotopy classes* $Q^*(S,S_1,\dots,S_n;G,G_1,\dots,G_n)$. ∎

12) Duality Theorems for homotopy groups.

If we apply the previous results to the particular case of homotopy groups (see [8]), we obtain:

THEOREM 34. - *Let* G *be a finite directed graph and* v *a vertex of* G. *Then there exists a natural isomorphism between the* m*-th o-homotopy group* $Q_m(G,v)$ *and the* m*-th* o^**-homotopy group* $Q^*_m(G,v)$.

Proof. - At first, let I^m be the unite m-cube and $\dot{I}^m$ its boundary. We note that now $Q_m(G,v)$ and $Q^*_m(G,v)$ coincide with $Q(I^m,\dot{I}^m;G,v)$ and $Q^*(I^m,\dot{I}^m;G,v)$ respectively, and every function $f: I^m,\dot{I}^m \to G,v$ is a loop. Since I^m is a compact normal space and $\dot{I}^m$ a closed subspace, by Theorem 32 there exists a natural bijection between $Q_m(G,v)$ and $Q^*_m(G,v)$, for all $m \geq 0$. Moreover, if $f,h: I^m,\dot{I}^m \to G,v$ are two loops with balancers U, V respectively, we can also call sum of loops f,h the function f_*h

given by:

$$(f*h)(x_1,\dots,x_m) = \begin{cases} f(3x_1,x_2,\dots,x_m) & \forall\, x_1 \in [0,\frac{1}{3}], \\ f(1,x_2,\dots,x_m) = h(0,x_2,\dots,x_m) & \forall\, x_1 \in [\frac{1}{3},\frac{2}{3}], \\ h(3x_1-2,x_2,\dots,x_m) & \forall\, x_1 \in [\frac{2}{3},1]. \end{cases}$$

It follows that $f*h$ is a n.c.o-regular function, since there exists a balancer W of $f*h$ of the form $W = U' \cup [\frac{1}{3},\frac{2}{3}] \times I^{m-1} \cup V'$, where U', V' are the correspondents of U, V which result from concentrating f, h on $[0,\frac{1}{3}] \times I^{m-1}$, $[\frac{2}{3},1] \times I^{m-1}$ respectively. Moreover the operation $*$ is compatible with the n.c.o-homotopy, since the previous sum of n.c.o-homotopies is a n.c.o-regular function. Hence $*$ induces an operation in $Q_{nc}(I^m,\dot{I}^m,G,v)$.

Now if $g\colon I^m,U \to G,v$ and $k\colon I^m,V \to G,v$ are two o*-patterns of $f\colon I^m,U \to G,v$ and $h\colon I^m, V \to G,v$ respectively, it follows that $g*k\colon I^m,W \to G,v$ is an o*-pattern of $f*h\colon I^m,W \to G,v$. Then the natural bijection in Theorem 26 between $Q_{nc}(I^m,\dot{I}^m,G,v)$ and $Q^*_{nc}(I^m,\dot{I}^m, G,v)$ is an isomorphism.

Finally, $Q_{nc}(I^m,\dot{I}^m,G,v)$ is isomorphic to $Q(I^m,\dot{I}^m,G,v) = Q_m(G,v)$. There exists, indeed, a natural bijection by Theorem 31 and the loop $f*h$ is o-homotopic to the loop $f+h$, given by:

$$(f+h)(x_1,\dots,x_m) = \begin{cases} f(2x_1,x_2,\dots,x_m) & \forall\, x_1 \in [0,\frac{1}{2}] \\ h(2x_1-1,x_2,\dots,x_m) & \forall\, x_1 \in [\frac{1}{2},1]. \end{cases}$$

(See [7], Properties 3.3, 3.7). Thus the theorem follows. ∎

THEOREM 35. - *Let* G *be a finite directed graph,* G' *a subgraph of* G *and* v *a vertex of* G'. *Then there exists a natural isomorphism between the relative o-homotopy group* $Q_m(G,G',v)$ *and the relative o*-homotopy group* $Q^*_m(G,G',v)$.

Proof. - Let J^{m-1} be the union of the $(m-1)$-faces of I^m, different from the face $x_m = 0$. We note that $Q_m(G,G',v)$ and $Q^*_m(G,G',v)$ coincide with $Q(I^m,\dot{I}^m,J^{m-1};G,G',v)$ and $Q^*(I^m,\dot{I}^m,J^{m-1};G,G',v)$ respectively, and every function $f\colon I^m,\dot{I}^m,J^{m-1} \to G,G',v$ is a relative loop. By Theorem 33 there exists a natural bijection between $Q_m(G, G',v)$ and $Q^*_m(G,G',v)$ for $m \geqslant 1$. Proceeding as before we obtain a natural isomorphsm between $Q_{nc}(I^m,\dot{I}^m,J^{m-1};G,G',v)$ and $Q_{nc}(I^m,\dot{I}^m,J^{m-1};G,G',v)$ for $m > 1$. Then we have also a natural isomorphism between $Q_m(G,G',v)$ and $Q^*_m(G,G',v)$. ■

REMARK. - We define as sum of loops f, h the function $f*h$ instead of $f+h$, since we always must obtain a n.c.o-regular function. (See Remark to Proposition 8). Nevertheless in the proof of Theorem 34 we can also choose as sum of loops the function $f+h$, since G' is a singleton.

13) *Examples.*

13.1) There exists a c.o-regular function without o^*-patterns.

Let $S = [0,1]$ be the unit interval, $S' = \{0\}$ the subspace of S, $G = \{a,b; b \to a\}$ the directed graph and $G' = \{a\}$ the subgraph of G. Then the function $f: S,S' \to G,G'$ given by:

$$\begin{cases} f(0) = a \\ f(]0,1]) = \{b\} \end{cases}$$

is not balanced since $\{a,b\} = \langle f(0)\rangle \supset \langle f'(0)\rangle = \{a\}$. Moreover, there is no pattern of f, in fact it is $T_G(\langle f(0)\rangle) = \{b\}$ and $T_{G'}(\langle f'(0)\rangle) = \{a\}$, hence it follows $T_G(\langle f(0)\rangle) \cap T_{G'}(\langle f'(0)\rangle) = \phi$.

13.2) There exists a non-balanced o^*-pattern of a b.c.o-regular function.

Let $S = I\times I$ be the topological space, $S' = I\times\{0\}$ the subspace of S, $G = \{a,b;\ a \to b,\ b \to a\}$ the directed graph and $G' = \{a,b; a \to b\}$ the subgraph of G. Then the function given by:

$$\begin{cases} f(\{0\}\times I) = \{a\} \\ f(]0,1]\times I) = \{b\} \end{cases}$$

is c.o-regular and balanced since:

$$\langle f(0,t)\rangle = \langle f'(0,t)\rangle = \begin{cases} \{a,b\} & \text{for } t = 0, \\ \{b\} & \forall\, t \in]0,1]. \end{cases}$$

For it is $T_{G'}(\{a,b\}) = \{b\}$, it follows that the function $g: S,S' \to G,G'$, given by:

$$\begin{cases} g(0,0) = b \\ g(\{0\} \times]0,1]) = \{a\} \\ g(]0,1] \times I) = \{b\} \end{cases}$$

is an o^*-pattern of f. But the function g is not balanced since $<g(0,0)> = \{a,b\} \supset \{b\} = <g'(0,0)>$. Nevertheless g is also an o-pattern of itself. In fact we have $H_G(<g(0,0)>) \cap H_{G'}(<g'(0,0)>) = \{b\}$.

13.3) There exists a b.c.o-regular function which is not n.c.o-regular.

Let $S = I\times I$ be the topological space, $S' = I\times\{0\}$ the subspace of S, $G = \{a,b; a \to b, b \to a\}$ the directed graph and $G' = \{a,b; a \to b\}$ the subgraph of G. Then the function $f\colon S,S' \to G,G'$ given by:

$$\begin{cases} f([0,\tfrac{1}{2}] \times \{0\}) = \{a\} \\ f(]\tfrac{1}{2},1] \times \{0\}) = \{b\} \\ f([0,\tfrac{1}{2}[\times]0,1]) = \{a\} \\ f([\tfrac{1}{2},1] \times]0,1]) = \{b\} \end{cases}$$

is b.c.o-regular since:

$$<f(t,0)> = <f'(t,0)> = \begin{cases} \{a\} & \forall\, t \in [0,\tfrac{1}{2}[\\ \{a,b\} & \text{for } t = \tfrac{1}{2} \\ \{b\} & \forall t \in]\tfrac{1}{2},1]. \end{cases}$$

But f is not n.c.o-regular. For every open neighbourhood U of S', indeed, the function $\hat{f}\colon U \to G'$ is not o-regular since it is $b \not\to a$ and $B^{\hat{f}} \cap A^{\hat{f}} \neq \phi$

13.4) There exist a pair of topological spaces S,S' *and a pair of directed graphs* G,G' *such that* Q(S,S':G,G') *and* O^*(S,S';G,G') *are not equipotent* (see [9]).

Let $S = \{x,x',y,y'\}$ be the topological space with the collection of open sets given by $\phi,\{x\},\{x'\},\{x,x'\},\{x,x',y\},\{x,x',y'\},S$ and let $G = \{a,a',b,b'; a \to b, a \to b', a' \to b, a' \to b'\}$ be the directed graph. We obtain that:

i) All the non-bijective o-regular (resp. o^*-regular) functions are o-homotopic (resp. o^*-homotopic) among themselves and particularly they are o-homotopic (resp. o^*-homotopic) to the constant function $f_0\colon (x,x',y,y') \to (a,a,a,a)$.

ii) There exist only the following four o-regular bijective functions:

$f_1: (x,x',y,y') \to (b,b',a,a')$, $f_2: (x,x',y,y') \to (b',b,a,a')$, $f_3: (x,x',y,y') \to (b,b',a',a)$, $f_4:(x,x',y,y') \to (b',b,a',a)$ and the following four o*-regular bijective functions: $f_1^*: (x,x',y,y') \to (a,a',b,b')$, $f_2^*: (x,x',y,y') \to (a',a,b,b')$, $f_3^*(x,x',y,y') \to (a,a',b',b)$, $f_4^*: (x,x',y,y') \to (a',a,b',b)$. We note that f_1,f_2,f_3,f_4 (resp. f_1^*,f_2^*,f_3^*,f_4^*) are not c.o-regular (resp. c.o*-regular) functions.

iii) The functions f_1,f_2,f_3,f_4 (resp. f_1^*,f_2^*,f_3^*,f_4^*) are not o-homotopic (resp. o*-homotopic) either among themselves or to f_0. Thus both $Q(S,G)$ and $Q^*(S,G)$ consist of five classes.

iv) Let $S' = \{y\}$ and $G' = \{a\}$ be. It follows that $Q(S,S':G,G')$ consists of the three classes $\{f_0\},\{f_1\},\{f_2\}$, while $Q^*(S,S':G,G')$ consists only of the class $\{f_0\}$.

v) Every c.o-regular (resp. c.o*-regular) function is c.o-homotopic (resp. c.o*-homotopic) to the constant function. Then Theorem 26 holds since $Q_{nc}(S,S';G,G')$ (resp. $Q^*_{nc}(S,S';G,G')$ consists of the class $\{f_0\}$.

13.5) There exist a topological space S *and a directed graph* G *such that* Q(S,G) *and* Q*(S,G) *are not equipotent* (see [9]).

Let $S = \{x,x',y,y',y''\}$ be the topological space with the collection of open sets given by $\phi,\{x\},\{x'\},\{x,x'\},\{x,x',y\},\{x,x',y'\},\{x,x,y''\},\{x,x',y,y'\},\{x,x',y,y''\},\{x,x',y',y''\}$, S and let $G = \{a,a',b,b',b''; a \to b, a \to b', a \to b'', a' \to b, a' \to b', a' \to b''\}$ be the directed graph. By the results of 13.4, in order to obtain regular functions which do not belong to the class of constant functions, it is necessary that the range of S consists of the vertices a, a' and of two vertices at least among b,b',b''. Thus we consider functions which are not c.o-regular and such that:

i) The image of $\{x,x'\}$ is given by two of the three elements b,b',b''.

ii) The image of $\{y,y',y''\}$ is given by the two elements a,a'.

Then there exist $6 \cdot 6 = 36$ possibities, and, consequently, $Q(S,G)$ consists of 37 classes.

On the contrary for the o*-regularity condition, we consider functions which are not c.o*-regular and such that:

i*) The image of $\{x,x'\}$ is given by the two elements a,a'.

ii*) The image of $\{y,y',y''\}$ is given by at least two of the three elements b,b',b''.

Then there exist 2·24 = 48 possibilities, and, consequently, $Q^*(S,G)$ consists of 49 classes.

We remark that Theorem 9 holds since $Q_c(S,G)$ (resp. $Q_c^*(S,G)$) consists of the class $\{f_0\}$.

REMARK. - The topological space considered in Examples 4, 5 are quasi-compact, T_0, non-T_1 spaces. For other similar examples which concern quasi-compact T_1, non-T_2 spaces, see [9].

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