b) We get
\[ J'(\mathbb{R},\mathbb{R}) \cong \mathbb{R} \times T^*\mathbb{R}. \]

This isomorphism is the unique map \( J'(\mathbb{R},\mathbb{R}) \rightarrow \mathbb{R} \times T^*\mathbb{R} \) that makes commutative the following diagram, for each function \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[
\begin{array}{ccc}
J'(\mathbb{R},\mathbb{R}) & \longrightarrow & \mathbb{R} \times T^*\mathbb{R} \\
\downarrow \quad j'f & & \downarrow (f,df) \\
\mathbb{R} & \longrightarrow & \mathbb{R} \times T^*\mathbb{R}
\end{array}
\]

c) There is a unique map (which is an isomorphism)
\[ J^2(\mathbb{R},\mathbb{R}) \rightarrow \mathbb{R} \times sT^2\mathbb{R} \]
such that the following diagram is commutative, for each curve \( c : \mathbb{R} \rightarrow \mathbb{R} \),

\[
\begin{array}{ccc}
J^2(\mathbb{R},\mathbb{R}) & \longrightarrow & \mathbb{R} \times sT^2\mathbb{R} \\
\downarrow \quad j^2c & & \downarrow (id_R,\alpha^2c) \\
\mathbb{R} & \longrightarrow & \mathbb{R} \times sT^2\mathbb{R}
\end{array}
\]

2 - JETS OF SECTIONS.

Let \( n = (E,p,\mathbb{R}) \) be a bundle.

1 DEFINITION.

The JET SPACE, of order \( i \), OF SECTIONS \( \mathbb{R} \rightarrow E \), is the set

\[ J^iE \equiv \bigcup_{p \in \mathbb{R}} f^p_{\rho_p^i}, \]

where

- \( f^p_{\rho_p^i} \) is the set of \( C^\infty \) sections \( \mathbb{R} \rightarrow E \) defined in a neighbourhood of \( p \);
- \( \rho_p^i \) is the restriction of the equivalence relation defined in (1,1).

Let us remark that we get
\[ J^i(M, N) = J^i(M \times N), \]

considering \( M \times N \) as a trivial bundle on \( M \).

2 PROPOSITION.

a) \( J^i E \) is a submanifold of \( J^i(M, E) \).

b) The following diagram is commutative, for each \( i > j \)

\[
\begin{array}{ccc}
J^i E & \xrightarrow{\sigma_{ji}} & J^i(M, E) \\
\downarrow & & \downarrow \\
J^j E & \xleftarrow{\sigma_{ij}} & J^j(M, E)
\end{array}
\]

\[ \sigma_j \]

\[ \Pi' \circ \sigma^j \]

\[ M \]

c) The triple \( n^j_i = (J^i E, \sigma^j_i, J^j E) \) is a bundle.

d) The triple \( n^i = (J^i E, \sigma^i, M) \) is a bundle.

e) If \( f : M \to E \) is a section,

then \( j^i f : M \to J^i E \) is a section.

3 PROPOSITION.

We get a natural isomorphism

\[ J^0 E \cong E \]

4 THEOREM.

\[ n^{01} = (J^i E, \sigma^{01}, J^0 E) \]

is an affine subbundle of \( (J^i(M, E), \sigma^{01}, J^0(M, E)) \)
over the inclusion \( J^0 E \to J^0(M, E) \),
whose vector bundle is \( n^{01} = (T^* M \otimes_E v T_E \otimes_E, E) \).

PROOF.

We have

\[ J^i E = \bigsqcup_{e \in E} \{ e \} \]
where \( \phi_e : T_{p(e)}M \rightarrow T_{e}E \)

is any linear map such that

a) \( T_{p} \circ \phi_e = id_{T_{p(e)}M} \)

5 THEOREM.

\( n^{12} \equiv (J^2E, \sigma^{12}, J'(E)) \) is an affine subbundle of \( (J^2(M,E), \sigma^{12}, J'(M,E)) \)

over the inclusion \( J'E \rightarrow J'(M,E) \), whose vector bundle is

\( n^{12} \equiv (J'E \times_{E} (T^*M \times T^*M \otimes E) \otimes TE), (\sigma^{12}, J'(M,E)) \)

PROOF.

We have

\[ J^2E = \bigoplus_{e \in J'E} (\phi_{e}^{-1}) \]

where

\[ \phi_{e} : T_{p(e)}M \rightarrow T_{e}E \]

is any linear map as in (1.7), that satisfies the further condition

a) \( T_{p}^{2} \circ \phi_{e} = id_{T_{p(e)}M} \)

This theorem can be generalized to higher orders.

6 PROPOSITION.

There is a unique map

\[ c : J'E \times_{M} T^*M \rightarrow T^*E \]

such that, for each section \( u : M \rightarrow TM \), \( v : M \rightarrow E \), the following diagram is commutative

\[ \begin{array}{ccc}
M & \xrightarrow{(jv,u)} & J'E \times_{M} TM \\
\downarrow T_{v}u & & \downarrow c \\
TE & \rightarrow &
\end{array} \]

Such a map is given by
c : (φ_e, u(p(e))) → φ_e(u(p(e))) ∈ T E.

c is an affine morphism on hTE and a linear morphism on J'E → E. Hence the following diagram is commutative

\[
\begin{array}{cccc}
J'E & \xrightarrow{\pi^0_1} & E \\
\downarrow{\pi'} & & \downarrow{\pi_E} \\
J'E \times_M TM & \xrightarrow{c} & TE \\
\downarrow{\pi^0_1 \times id_{TM}} & & \downarrow{h} \\
hTE & \xrightarrow{h} & hTE
\end{array}
\]

7 PROPOSITION.

There is a unique map

\[c^* : T^* M \times_M J' TM \rightarrow T^* TM\]

such that the following diagram is commutative

\[
\begin{array}{cccc}
T^* M \times_M J' TM \times_M TM & \xrightarrow{(soc^*, c)} & T^* TM \times_{TM} TM \\
\downarrow & & \downarrow <,> \\
0 & \xrightarrow{=} & R
\end{array}
\]

8 PROPOSITION.

Let \( \eta = (E, p, M) \) be an affine (vector) bundle, whose vector bundle is \( \eta = (\bar{E}, \bar{p}, M) \).

Then \( n^i = (J^i E, \sigma^i, M) \) is an affine (vector) bundle and \( J^i E = J^i \bar{E} \).

PROOF.

The affine (vector) operations on \( E \) are compatible with respect to the equivalence relations \( \rho^i_p \).
9 COROLLARY.

Let $k > h > 0$.

Let $n \equiv (E, p, M)$ be a vector bundle.

Then $n^h k = (J^k E, o^h k E)$ is an affine bundle, whose vector bundle $-h k\ n$ is the pull back bundle of $(\text{Ker } o^h k, o^h k, M)$ with respect to the map $o^h$.

Namely the following diagram is commutative

\[
\begin{array}{ccc}
J^h k E & \rightarrow & \text{Ker } o^h k \\
\downarrow & & \downarrow o^h k \\
J^h & \rightarrow & M
\end{array}
\]

Moreover, if $h \equiv k - 1$, we get

\[J^h k E = J^h E \times^k_M (V T^* M \otimes E),\]

where $V T^* M$ is the $k$-symmetrized tensor product of $T^* M$ over $M$.

Let us remark that, if $E \equiv M \times F$ (i.e. $n$ is a trivial bundle), then we get

\[J^k E = E \otimes_M \text{Ker } o^k = F \times \text{Ker } o^k .\]

In such a case, we put

\[j^k u \equiv \Pi^2 \circ j^k E .\]

In particular, for $F \equiv R$, we get

\[j^1 f = df .\]

10 PROPOSITION.

Let $n' \equiv (E', p', M)$ and $n'' = (E'', p'', M)$ be vector bundles.

There is a unique linear map $t: J^k E' \otimes_M J^k E'' \rightarrow J^k (E' \otimes_M E'')$

such that the following diagram is commutative

\[
\begin{array}{ccc}
J^k E' \otimes_M J^k E'' & \rightarrow & J^k (E' \otimes_M E'') \\
(J^k u', J^k u'') & \rightarrow & J^k (u' \otimes u'')
\end{array}
\]

for each section $u' : M \rightarrow E'$, $u'' : M \rightarrow E''$. 

11 COROLLARY.

Let \( n = (E, p, M) \) be a vector bundle.

There is a unique linear map on \( M \)
\[
\begin{align*}
\ker & \quad J^{k-1}_E \quad J^k_E, \\
\text{where } & \quad \ker \text{ is the kernel of the linear morphism } J^k(M \times \mathbb{R}) \to J^0(M \times \mathbb{R}) \text{ on } M
\end{align*}
\]
such that
\[
\begin{align*}
\ker & \quad J^{k-1}_E \quad J^k_E, \\
\text{where } & \quad \ker \text{ is the kernel of the linear morphism } J^k(M \times \mathbb{R}) \to J^0(M \times \mathbb{R}) \text{ on } M
\end{align*}
\]
where \( K \) is the Kernel of the linear morphism \( J^k(M \times \mathbb{R}) \to J^0(M \times \mathbb{R}) \) on \( M \),

As a particular case, we get
\[
j^1(fu) = f \cdot j^1u + df \circ u,
\]
being
\[
t'(df \circ u) = df \circ u
\]

12 PROPOSITION.

Let \( n = (E, p, M) \) be an affine bundle, whose vector bundle is \( \tilde{n} = (\tilde{E}, \tilde{p}, \tilde{M}) \).

There is a unique map
\[
\begin{align*}
J^1 \times \tilde{E} & \quad J^1 \tilde{E} \\
\text{such that, for each vertical curve } & \quad c, c' : R \to E, \text{ the following diagram is commutative}
\end{align*}
\]

13 PROPOSITION.

Let \( n = (E, p, M) \) be a vector bundle and let \( n^* = (E^*, p^* M) \) be the dual one.

There is a unique map
such that, for each section \( u : M \to E \) and \( v : M \to E^* \), the following diagram is commutative:

\[
\begin{array}{ccc}
J'^*E \times_M J'^*E & \xrightarrow{\pi} & T^*M \\
\downarrow{(j^! u, j^! v)} & & \downarrow{\pi^2(j^! <u,v>)} \\
M & & \\
\end{array}
\]

Such a map is bilinear.

14 PROPOSITION.

Let \( n = (E, p, M) \) be a vector bundle and let \( g : E \times_M E \to \mathbb{R} \) be a pseudo-Riemannian structure.

There is a unique map \( g : J'^*E \times_M J'^*E \to T^*M \) such that the following diagram is commutative, for each section \( u, v : M \to E \):

\[
\begin{array}{ccc}
J'^*E \times_M J'^*E & \xrightarrow{g} & T^*M \\
\downarrow{(j^! u, j^! v)} & & \downarrow{\pi^2 \circ j^!(g(u,v))} \\
M & & \\
\end{array}
\]

3 LIE DERIVATIVES.

1 DEFINITION.

A \( k \)-LIE-DERIVABLE bundle is a 4-plet

\( n = (E, p, M; B) \),

where \( (E, p, M) \) is a bundle and

\[
B : E \times_M J^kTM \to TE
\]

is a bundle morphism on \( hT E \) and a linear morphism on \( E \).