b) Reversing all the terms of b), we get the inverse isomorphism

\[ s^{-1} : T^*TM \rightarrow T^*TM. \]

c) \((T^*TM, h, T^*M \times_M T^*M)\) is the pull-back bundle of
\((T^*TM, \nu, T^*M \times_M T^*M)\) with respect to the map

\[ i := (id_{T^*M} \times (- id_{TM}) : T^*M \times_M T^*M \rightarrow T^*M \times_M T^*M. \]

The induced map

\[ \omega = \nu : T^*TM \rightarrow T^*TM \]

is an isomorphism (the SYMPLECTIC ISOMORPHISM) such that the following diagram is commutative

\[ \begin{array}{ccc}
T^*TM & \xrightarrow{\omega} & T^*TM \\
\downarrow{(\pi_{T^*M}, T^*_PM)} & & \downarrow{\nu} \\
T^*M \times_M T^*M & \xrightarrow{i} & T^*M \times_M T^*M
\end{array} \]

c) In an analogous way we get the isomorphism

\[ \omega \circ s^{-1} : T^*TM \rightarrow T^*TM \]

5 DEFINITION.

The SYMMETRIC SUBMANIFOLD of \( T^*TM \) is

\[ s \in T^*TM \equiv \{ \alpha \in T^*TM \mid s(\alpha) = \alpha \} \]

4 - Lie derivative of tensors.

1 Let \( \mathcal{M} \) be the category, whose objects are manifolds and whose morphisms are diffeomorphisms.

Let

\[ T(r,s) : \mathcal{M} \rightarrow \mathcal{M}. \]
be the covariant functor defined as follows:

\[ T_{(r,s)} M \equiv \mathcal{P}_{s} T M \otimes T^{*} M \]

b) if \( f : M \rightarrow N \) is a diffeomorphism, then

\[ T_{(r,s)} f \equiv \mathcal{P}_{s} f \otimes T^{*} f^{-1} : T_{(r,s)} M \rightarrow T_{(r,s)} N. \]

2 Let \( M \) and \( N \) be manifolds.

Let \( \phi : R \times M \rightarrow N \)
be a map (defined at least locally). Then

\[ \alpha \phi : M \rightarrow T N \]

is the map given by

\[ \alpha \phi(x) = T\phi(x)(0,1). \]

3 Let \( M \) be a manifold.

Let \( u : M \rightarrow T M \)
be a vector field and

let \( C : R \times M \rightarrow M \)
be the (locally defined) group of local diffeomorphisms generated by \( u \).

Namely we have \( u = \circ C \).

Let \( v : M \rightarrow T_{(r,s)} M \)
be a tensor field.

Let \( C v : R \times M \rightarrow T_{(r,s)} M \)
be the (locally defined) map, given, \( \forall \lambda \in R \), by the tensor field

\[ (C v)_{\lambda} \equiv T_{(r,s)} C_{\lambda}^{-1} \circ v \circ C_{\lambda} : M \rightarrow T_{(r,s)} M. \]

Let us remark that \( \alpha(Cv) \) takes its values in the subspace

\( \nu T_{(r,s)} M \rightarrow T T_{(r,s)} M \), since \( (C v)_{\lambda} \) is a section of \( T_{(r,s)} M \).

Then we can give the following definition.

4 DEFINITION.

The LIE DERIVATIVE of \( v : M \rightarrow T_{(r,s)} M \) with respect to \( u : M \rightarrow TM \) is

the tensor field
Let $u : M \to TM$ be a section. There is a unique map

$$\alpha(Tu) : T^*M \to T^*TM$$

such that the following diagram is commutative and exact in $TM$

$$\begin{array}{ccccccccc}
\text{id} & \to & T^*M & \xrightarrow{\alpha(Tu)} & T^*TM & \xrightarrow{u(M)} & T^*M & \to & M \times O \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M & \xrightarrow{u} & TM & & & & & & \\
\end{array}$$

**Proposition.**

We have

$$L_u v = \prod_{T(r,s)M} \circ \beta(Cv) : M \to T(r,s)M$$

**Proof.**

It suffices to give the proof for $v : M \to T(1,0)M$ and $v : M \to T(0,1)M$.

In the first case, we get

$$Cv = (T_2C^{-1}) \circ (\Pi',\nu) \circ (\Pi^1,c) : R \times M \to TM,$$

hence

$$\beta(Cv) = (T(T_2C^{-1}) \circ (\Pi',Tv) \circ (\Pi^1,T_1C)(0,1) =$$

$$= ((\beta T_2C^{-1}) \circ \Pi_{TM} + T_2T_2C^{-1}) \circ (Tv) \circ \beta C =$$

$$= (s \circ T\alpha C^{-1}) \circ \Pi_{TM} + \text{id}_{TM}) \circ (Tv) \circ u =$$

$$= s \circ Tu \circ v + Tv \circ u.$$

In the second case, we get

$$Cv = (T_2^*C) \circ (\Pi^1,\nu) \circ (\Pi^1,c) : R \times M \to T^*M$$
hence \[ a(Cv) = (TT^*_2 C) \circ (\Pi^1 T) v \circ (TT^*_1 C)(0,1) = \]
\[ =((\alpha T^*_2 C) \circ \Pi_{TM} + T^*_2 C_0) \circ (TV) \circ a C = \]
\[ =(-s \circ \alpha(T_2 \circ C) \circ \Pi_{TM} + id_{TT^*_M}) \circ (TV) \circ u = \]
\[ = -s \circ \alpha(Tu) \circ v + T_1 \circ v \circ u \]

Let us remark that both tensors in (\#) are on the same affine fiber on \( h \mathbb{T}_{(r,s),M} \).

5 Connection on a bundle.

Let \( \mathbb{n} = (E,p,M) \) be a bundle.

1 DEFINITION.

A **PSEUDO-CONNECTION** on \( \mathbb{n} \) is an affine bundle morphism on \( h \mathbb{T} E \)

\[ \Gamma : \mathbb{T} E \rightarrow \mathcal{V} \mathbb{T} E \]

whose fiber derivatives are \( l \).

A **PSEUDO-HORIZONTAL SECTION** is a section

\[ H : h \mathbb{T} E \rightarrow \mathbb{T} E \]

Hence the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{T} E & \xrightarrow{\Gamma} & \mathcal{V} \mathbb{T} E \\
\downarrow \scriptstyle{h} & & \downarrow \scriptstyle{h} \\
\mathbb{T} E & \xrightarrow{H} & \mathbb{T} E \\
\end{array}
\]

Let us remark that \( \Gamma : \mathbb{T} E \rightarrow \mathcal{V} \mathbb{T} E \) is characterized by the map \( \Gamma' : \mathbb{T} E \rightarrow \mathcal{V} \mathbb{T} E \) given by \( \mathbb{T} E \xrightarrow{\Gamma} \mathcal{V} \mathbb{T} E \xrightarrow{\Pi^2} \mathcal{V} \mathbb{T} E \).