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DUALITY THEOREMS FOR REGULAR HOMOTOPY
OF FINITE DIRECTED GRAPHS. (*)

RIASSUNTO. - *Dati uno spazio topologico normale e numerabilmente paracompatto S ed un grafo finito ed orientato G si prova che tra gli insiemi $Q(S,G)$ e $Q^*(S,G)$ delle classi di o -omotopia e di o^* -omotopia esiste una biiezione naturale. Nelle stesse condizioni, se S' è un sottospazio chiuso di S e G' un sottografo di G , esiste ancora una biiezione naturale tra gli insiemi $Q(S,S';G,G')$ e $Q^*(S,S';G,G')$ delle classi di omotopia. Si mostra infine che in condizioni meno restrittive per lo spazio S le precedenti biiezioni possono non sussistere.*

INTRODUCTION

In the extension from the undirected graphs to the directed ones, we have two possible definitions of regular function. In fact, given a topological space S and a finite directed graph G , a function $f: S \rightarrow G$ is called *o -regular* (resp. *o^* -regular*) if for all $v, w \in G$ such that $v \neq w$ and $v \nrightarrow w$, it is $\overline{f^{-1}(v)} \cap f^{-1}(w) = \phi$ (resp. $f^{-1}(v) \cap \overline{f^{-1}(w)} = \phi$). Therefore we can deal with two different homotopies, the o -homotopy and the o^* -homotopy. Hence we examine the problem

(*) Work performed under the auspices of the *Consiglio Nazionale delle Ricerche* (CNR, GNSAGA), Italy.

set of o -homotopy (resp. o^* -homotopy) classes. We note that $Q^*(S, S'; G, G')$ coincides with $Q(S, S', G^*, G'^*)$ and $Q(S, S'; G, G')$ with $Q^*(S, S'; G^*, G'^*)$.

A function $f: S, S' \rightarrow G, G'$ is called *c.o-regular* (resp. *c.o*-regular*) if both $f: S \rightarrow G$ and $f': S' \rightarrow G'$ are *c.o-regular* (resp. *c.o*-regular*) functions.

As before, the *Duality Principle* holds for functions between pairs.

Main results of [2], [3].

R_a : $X \subseteq G$ is totally headed, iff it is totally tailed. (See [3], Proposition 4).

If S is a normal topological space and S' is a closed subspace of S , we have:

R_b : (The first Normalization Theorem). Let $f: S \rightarrow G$ (resp. $f: S, S' \rightarrow G, G'$) be an o -regular function. Then there exists a *c.o-regular* function, o -homotopic to f . (See [3], Theorems 12, 15).

R_c : (Extension Theorem between pairs). Let $f: S, S' \rightarrow G, G'$ be an o -regular function. Then there exist a closed neighbourhood U of S' and an o -regular function $g: S, S' \rightarrow G, G'$, which is o -homotopic to f and such that the function $g: S, U \rightarrow G, G'$ is o -regular, i.e. $g(U) \subseteq G'$ and the restriction $\hat{g}: U \rightarrow G'$ of g to U is o -regular. (See [2], Theorem 20).

R_d : In the construction of R_c , if there exist n vertices $p_1, \dots, p_n \in G$ and m vertices $q_1, \dots, q_m \in G'$, such that $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap \overline{Q_1^{f'}}$ $\cap \dots \cap \overline{Q_m^{f'}} = \emptyset$, then also it follows $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap \overline{Q_1^{\hat{g}}} \cap \dots \cap \overline{Q_m^{\hat{g}}} = \emptyset$. Similarly, from $\overline{P_1^f} \cap \dots \cap \overline{P_n^f} \cap X = \emptyset$ it results $\overline{P_1^g} \cap \dots \cap \overline{P_n^g} \cap U = \emptyset$. (See [2], Corollary 21).

Moreover, if $S \times I$ is normal, then it results:

R_e : (The first Normalization Theorem for homotopies). Let $f, g: S \rightarrow G$ (resp. $f, g: S, S' \rightarrow G, G'$) be two o -homotopic *c.o-regular* functions. Then, between the functions f and g , there also exists an o -homotopy, which is a *c.o-regular* function. (See [3], Theorem 16).

By Duality Principle, the results dual to the previous ones are also true.