REMARK ON A BROWDER'S FIXED POINT THEOREM(*)

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ABSTRACT - *In a recent paper, F.E. Browder discussed continuous self-mappings of contractive type in a complete metric space. Browder showed that such mappings have a fixed point and the sequence of iterates of any point, in an invariant subset, converges to the fixed point. In the present paper, the result of Browder is obtained for mappings which are not necessarily continuous.*

1. Introduction. Let (M,d) be a complete metric space, $f : M \to M$ a mapping and let ψ be a contractive gauge function (i.e. ψ is a function from R_+, the non-negative reals, to R_+ non-decreasing and continuous from the right, $\psi(0) = 0$, and $\psi(r) < r$ for all $r > 0$).

In a recent paper [1] , in order to get a fixed point theorem of great generality, F.E. Browder proved the following result.

THEOREM 1 - Let M_o be a subset of M such that f carries M_o into M_o. For each x in M_o, suppose that there exists a positive integers $n(x)$ and for each $n \geq n(x)$ and for each y in M_o, three subsets $J_1(x,y,n)$, $J_2(x,y,n)$, $J_3(x,y,n)$ of $Z_+ \times Z_+$ (Z_+ is the set of non-negative integers) such that for each $n \geq n(x)$, y in M_o,

$$d(f^n x, f^n y) \leq \psi(\max(\sup_{(i,j)\in J_1} d(f^i x, f^j y), \sup_{(r,s)\in J_2} d(f^r x, f^s x), \sup_{(\ell,t)\in J_3} d(f^\ell y, f^t y))).$$

Then f has a fixed point x_o in M such that for each x in M_o, $(f^n x)$ converges to x_o as $n \to \infty$.

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In the present paper we shall develop this point of view a little further. Therefore we investigate mappings which are not necessarily continuous but satisfy the following weaker condition.

(A) the function $D(x) = d(x,fx)$ has the following property: if $x_n \to x_\circ$ then a subsequence (x_{n_p}) exists such that $D(x_\circ) \leq D(x_{n_p})$.

As a special case of our result we obtain a fixed point theorem of L.Ciric (Theorem 1, [2]) and a fixed point theorem of the first author (Corollary, [3])

2. Let $O(f,x)$ denote the orbit of x under f, i.e. $O(f,x) = \bigcup_{0 \leq n} \{f^n x\}$. We begin this section with the following definition.

DEFINITION 1 - We shall say that x has bounded orbit under f if $\operatorname{diam}(O(f,x)) < +\infty$.

LEMMA 1 - Let ψ be a contractive gauge function, $t_\circ > 0$. Then the sequence $\psi^n(t_\circ) \to 0$ as $n \to \infty$ (where ψ^n is the n - th iterate of ψ).

THEOREM 2 - Suppose that:

(1) for some $x_\circ$ in M, $\operatorname{diam}(O(f,x_\circ)) < +\infty$,

(2) for each x in M there exists a positive integer $n(x)$ and for each $n \geq n(x)$ and for each y in M, three subsets $J_1(x,y,n)$, $J_2(x,y,n)$, $J_3(x,y,n)$ of $Z_+ \times Z_+$ such that for each $n \geq n(x)$, y in M

$$d(f^n x, f^n y) \leq \psi(\max(\sup_{(i,j)\in J_1} d(f^i x, f^j y), \sup_{(r,s)\in J_2} d(f^r x, f^s x), \sup_{(\ell,t)\in J_3} d(f^\ell y, f^t y))).$$

Then f has a unique fixed point u in M and $(f^n x)$ converges to u in M as $n \to \infty$ for each x in M which has bounded orbit under f.

Proof. - For the x_o in (1), let $m_o = n(x_o)$ and inductively $x_n = f^{m_{n-1}} x_{n-1}$, $m_n = n(x_n)$. We show that (x_n) is a Cauchy sequence. It suffices to show that for a given $\varepsilon > 0$, $d(x_{n+1}, x_{n+k+1}) < \varepsilon$ for all k positive integer, when n is large enough. For this purpose, let n be fixed and denote $d_i = \operatorname{diam}(O(f, x_{n-i}))$, $m(k) = m_{n+1} + m_{n+2} + \ldots + m_{n+k}$. Then

$$d(x_{n+1}, x_{n+k+1}) = d(f^{m_n} x_n, f^{m_n}(f^{m(k)} x_n)) \leq \psi(\max(\sup_{(i,j)\in J_1} d(f^i x_n, f^j(f^{m(k)} x_n)),$$

$$\sup_{(r,s)\in J_2} d(f^r x_n, f^s x_n), \sup_{(\ell,t)\in J_3} d(f^\ell(f^{m(k)} x_n), f^t(f^{m(k)} x_n))))$$

$$\leq \psi(d_o),$$

where $J_1(x_n, f^{m(k)} x_n, m_n)$, $J_2(x_n, f^{m(k)} x_n, m_n)$, $J_3(x_n, f^{m(k)} x_n, m_n)$ are as in (2), since ψ is nondecreasing and all terms under the max and sup operations are bounded by d_o. Let u and v two points of $O(f, x_n)$; u and v may be put in the form $u = f^p x_n$, $v = f^{p+q} x_n$, $q \geq 0$. Hence

$$d(u,v) = d(f^p x_n, f^{p+q} x_n) = d(f^{m_{n-1}+p} x_{n-1}, f^{m_{n-1}+p}(f^q x_{n-1})) \leq \psi(\max(\sup_{(i,j)\in J_1}$$

$$d(f^i x_{n-1}, f^j(f^q x_{n-1})), \sup_{(r,s)\in J_2} d(f^r x_{n-1}, f^s x_{n-1}), \sup_{(\ell,t)\in J_3}$$

$$d(f^\ell(f^q x_{n-1}), f^t(f^q x_{n-1})))) \leq \psi(d_1).$$

It follows that $d_\circ \leq \psi(d_1)$. By routine calculation one can easily show that the following inequality holds

$$d(x_{n+1},x_{n+k+1}) \leq \psi^{n+1}\,(\mathrm{diam}\,(O(f,x_\circ))).$$

It follows that (x_n) is a Cauchy sequence as it derives from the LEMMA 1. Say u in M such that $u = \lim_n x_n$. Now, by an argument similar to that used above, one can easily show that

$$d(x_n,f(x_n)) \leq \psi^n(\mathrm{diam}\,(O(f,x_\circ)))$$

so that $\lim_n D(x_n) = \lim_n d(x_n,f\,x_n) = 0$. From (A), for a subsequence (x_{n_p}), we obtain $0 \leq D(u) \leq D(x_{n_p})$ and hence $D(u) = 0$, i.e., u is a fixed point of f. The uniqueness of the fixed point may be established by use of (2). It now remains to be shown the last assertion of the theorem. For this end, let x in M be such that has bounded orbit under f. As above, let $x_\circ = x$, $m_\circ = n(x_\circ)$ and inductively $x_n = f^{m_{n-1}}\,x_{n-1}$, $m_n = n(x_n)$. Since, already, we have showed that

$$\mathrm{diam}(O(f,x_n)) \leq \psi(\mathrm{diam}(O(f,x_{n-1})))$$

for all n, it follows that

$$\mathrm{diam}(O(f,x_n)) \leq \psi^n\,(\mathrm{diam}(O(f,x))).$$

Being the sequence $(\mathrm{diam}(O(f,f^n x)))$ monotone and containing a convergent subsequence $(\mathrm{diam}(O(f,x_n)))$, it follows that it is convergent and

$$\lim_n\ \mathrm{diam}(O(f,f^n x)) = 0\,,$$

i.e.,that $(f^n\,x)$ is a Cauchy sequence. Therefore, $(f^n\,x) \to u$ as $n \to \infty$.

This complete the proof.

REMARK 1. - If we set $\psi(r) = q\cdot r$ for some $q < 1$, ψ is a contractive gauge function. It follows that the L.Ciric's Theorem 1 ([2]) is a special case of Theorem 2.

REMARK 2. - We shall recall that a version of Theorem 2 is given in [3] by the first author. In [3] one assume conditions which ensure that (1) is true for every x_0 in M and (2) is true for a $n = n(x)$ and $J_1 = \{(0,0)\}$, $J_2 = J_3 = \emptyset$.

B I B L I O G R A P H Y

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