Summary. - On concluding $|S|$, J. Szép suggested the study of a special algebra $S(\cdot, \times)$, where "·" is a group operation (instead of $a \cdot b$ we shall write $ab$) and "×" is a semigroup operation with an idempotent element $e$; moreover

$$\forall a, b, c \in S : (a \times b)c = ac \times bc, \quad c(a \times b) = ca \times c^{-1} a$$

where $c^{-1}$ is the inverse of $c$ in $S(\cdot)$. The aim of this work is to analyze such an algebra.
ON A SPECIAL ALGEBRA WITH TWO OPERATIONS

by

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In [S] J. Szép suggested the study of a special algebra; the aim of this work is to analyze such an algebra.

Let $S(\cdot, x)$ be an algebra where $\cdot$ is a group operation (instead of $a \cdot b$ we shall write $ab$) and $\times$ is a semigroup operation with an idempotent element $e$; moreover

$$(a) \quad \forall a,b,c \in S : (a \times b)c = ac \times bc, \quad c(a \times b) = caxc^{-1}a$$

where $c^{-1}$ is the inverse of $c$ in $S(\cdot)$.

The following results hold:

1) $S(\times)$ is an idempotent semigroup. In fact, let $1$ denote the unit element of $S(\cdot)$; since $e \times e = e$, we have $a = 1 \cdot a = ee^{-1} a = (e \times e)c^{-1}a = (1 \times 1)a = axa$ for every $a \in S$.

2) For every $a \in S : a^{2} = a \cdot a = a(axa) = a^{2}axa = a^{2} \times 1$.

3) Let $G_{1} = \{x \in S | x \times 1 = x \}$, $G_{2} = \{x \in S | 1 \times x = x \}$; then the following result holds: $u \in G_{1} \implies 1 \times u = 1$, $u \in G_{2} \implies u \times 1 = u^{2}$.

In fact $u \in G_{1} \implies u \times 1 = u \implies u^{-1}(u \times 1) = u^{-1}u \implies 1 \times u = 1$;

$u \in G_{2} \implies 1 \times u = u \implies u(1 \times u) = u \implies u \times 1 = u^{2}$.

4) For every $u \in S : u \in G_{2} \implies u^{2} = 1$. In fact (see 3),

$u \in G_{2} \implies u^{2} = (u \times 1)u = u \times (1 \times u) = uxu = u$

and hence

$u \in G_{2} \implies u^{2} = u \implies (u \times 1)u = u \implies u^{2} \cdot u = u \implies u^{2} = 1$.

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5) \( G_1 \cap G_2 = \{1\} \) (see 3))

6) The sets \( G_1, G_2 \) are normal subgroups of \( S(\cdot) \).

The following steps prove the assertion:

(i) If \( u,v \in G_1 \) then \( u1 = u, \ v1 = v \) so \( uv = (u1)v = uv \cdot v = uv \cdot (v1) = (u1)v = uv1 \) and hence \( u \cdot v \) is in \( G_1 \). (Similarly \( u, v \in G_2 \Rightarrow u \cdot v \) is in \( G_2 \))

(ii) If \( u \in G_1 \Rightarrow 1 \cdot u = 1 \Rightarrow (1 \cdot u)^{-1} = u^{-1} \Rightarrow u^{-1} \cdot 1 = u^{-1} \Rightarrow u^{-1} \in G_1 \); 
    \( u \in G_2 \Rightarrow u^{-1} \in G_2 \) (see 4)).

(iii) For every \( a \in G_1, \ b \in S \) \( b^{-1}ab \cdot 1 = b^{-1}(a \cdot 1) \cdot b \cdot 1 = (b^{-1}ab \cdot b) \cdot 1 = b^{-1}ab \cdot b = b^{-1}ab \) and hence \( b^{-1}ab \in G_1 \). 
For every \( a \in G_1, \ b \in S \) \( b^{-1}ab = b^{-1}(1 \cdot a) \cdot b = (b^{-1} \cdot b) \cdot b = b^{-1}ab \) and hence \( b^{-1}ab \in G_2 \).

7) \( S = G_1 \cdot G_2 \). In fact for \( x \in S \) we consider the element \( a = x \cdot 1 \in G_1 \).

So it follows that \( a = (1 \cdot x^{-1}) \cdot x \), i.e. \( x \cdot x^{-1} = ax^{-1} \) and hence, since \( 1 \cdot x^{-1} \in G_2 \), \( x = (1 \cdot x^{-1})^{-1} \cdot a = a \cdot (1 \cdot x^{-1})^{-1} \in G_1 \cdot G_2 \).

From 5), 6), 7) it follows that

8) \( G = G_1 \& G_2 \) (where \( \& \) denotes the direct product)

9) \( 1 \cdot g_1g_2 = g_2, \ g_1g_2 \cdot 1 = g_1 \) where \( g_1 \in G_1, \ g_2 \in G_2 \). In fact, since \( (g_1^{-1}g_2^{-1} \cdot 1)g_1g_2 = (1 \cdot g_1g_2) \in G_2 \), it follows \( g_1^{-1}g_2^{-1} \cdot 1 \in G_1 ^{-1}G_2 \).

But \( g_1^{-1}g_2^{-1} \cdot 1 \in G_1 \) and so (by 5) \( g_1^{-1}g_2^{-1} \cdot 1 = g_1^{-1} \). This means that \( 1 \cdot g_1g_2 = (g_1^{-1}g_2^{-1} \cdot 1)g_1g_2 = g_1^{-1}g_1g_2 = g_2 \) (Likewise for \( g_1g_2 \cdot 1 = g_1 \)).

10) If \( a = g_1g_2, \ b = h_1h_2 \) where \( g_1, h_1 \in G_1, g_2, h_2 \in G_2 \) then \( a \cdot b = g_1g_2 \cdot h_1h_2 = (1 \cdot h_1h_2^{-1}g_1 \cdot g_1g_2 = h_2^{-1}g_1g_2 \cdot h_2g_1 = g_1h_2 \).
So we have the following

THEOREM 1. For the algebra $S(\cdot, x)$ the following properties hold

1) $S = G_1 \times G_2$ and $g^2 = 1$ \quad $\forall g \in G_2$.

2) $g_1 g_2 x h_1 h_2 = g_1 h_2 \quad (g_1, h_1 \in G_1, g_2, h_2 \in G_2)$.

Conversely we can prove

THEOREM 2. Let the group $S(\cdot)$ be the direct product of two subgroup $G_1, G_2$ such that $g^2 = 1$ for every $g \in G_2$. A semigroup operation "$\times$" with an idempotent element exists in $S$ such that

$$\forall a, b, c \in S: (a \times b)c = ac \times bc, \quad c(a \times b) = ca \times c^{-1} b.$$ 

Proof. A few calculations show that the required operation is defined as follows:

$g_1 g_2 x h_1 h_2 = g_1 h_2 \quad$ for every $g_1, h_1 \in G_1, g_2, h_2 \in G_2$.

REMARK. Finally we observe that theorems analogous to theorems 1 and 2 can be proved if in place of (a) one has

$$(b) \quad \forall a, b, c \in S: (a \times b)c = ac \times bc, \quad c(a \times b) = c^{-1} a \times cb$$

REFERENCES


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