Summary. - *On concluding* |S|, *J. Szép suggested the study of a special algebra* $S(\cdot,\times)$, *where* "$\cdot$" *is a group operation (instead of* $a\cdot b$ *we shall write* ab*) and* "$\times$" *is a semigroup operation with an idempotent element* e*; moreover*

$$(\alpha) \qquad \forall a,b,c \in S : (a\times b)c = ac\times bc, \; c(a\times b) = ca\times c^{-1}a$$

where c^{-1} *is the inverse of* c *in* $S(\cdot)$.

The aim of this work is to analyze such an algebra.

ON A SPECIAL ALGEBRA WITH TWO OPERATIONS

by

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In [S] J. Szép suggested the study of a special algebra; the aim of this work is to analyze such an algebra.

Let $S(\cdot,\times)$ be an algebra where "$\cdot$" is a group operation (instead of $a\cdot b$ we shall write ab) and "$\times$" is a semigroup operation with an idempotent element e; moreover

(α) $\forall a,b,c \in S : (a\times b)c = ac\times bc \ , \ c(a\times b) = ca\times c^{-1}a$

where c^{-1} is the inverse of c in $S(\cdot)$.

The following results hold:

1) $S(\times)$ is an idempotent semigroup. In fact, let 1 denote the unit element of $S(\cdot)$; since $e\times e = e$, we have $a=1\cdot a= ee^{-1}a=(e\times e)e^{-1}a=(1\times 1)a=a\times a$ for every $a \in S$.

2) For every $a \in S : a^2 = a\cdot a = a(a\times a) = a^2\times a^{-1}a= a^2\times 1$.

3) Let $G_1 = \{x\in S | x\times 1 = x\}$, $G_2 = \{x\in S | 1\times x = x\}$; then the following result holds : $u \in G_1 \Longrightarrow 1\times u = 1$, $u \in G_2 \Longrightarrow u\times 1 = u^2$.

In fact $u \in G_1 \Longrightarrow u\times 1 = u \Longrightarrow u^{-1}(u\times 1) = u^{-1}u \Longrightarrow 1\times u = 1$;

$u \in G_2 \Longrightarrow 1\times u = u \Longrightarrow u(1\times u) = u^2 \Longrightarrow u\times 1 = u^2$.

4) For every $u\in S : u \in G_2 \Longrightarrow u^2 = 1$. In fact (see 3),1))

$$u \in G_2 \Longrightarrow u^2\times u = (u\times 1)\times u = u\times(1\times u) = u\times u = u$$

and hence

$$u \in G_2 \Longrightarrow u^2\times u = u \Longrightarrow (u\times 1)u = u \Longrightarrow u^2\cdot u = u \Longrightarrow u^2 = 1.$$

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5) $G_1 \cap G_2 = \{1\}$ (see 3))

6) The sets G_1, G_2 are normal subgroup of $S(\cdot)$.

The following steps prove the assertion:

(i) if $u,v \in G_1$ then $u\times 1 = u$, $v \times 1 = v$ so $uv = (u\times 1)v = uv\times v = uv\times(v\times 1) =$ $= (u\times 1)v\times 1 = uv\times 1$ and hence $uv \in G_1$. (Similarly $u,v\in G_2 \Longrightarrow uv \in G_2$)

(ii) $u\in G_1 \Longrightarrow 1\times u = 1 \Longrightarrow (1\times u)u^{-1} = u^{-1} \Longrightarrow u^{-1}\times 1 = u^{-1} \Longrightarrow u^{-1} \in G_1$;

$u \in G_2 \Longrightarrow u^{-1} \in G_2$ (see 4)).

(iii) For every $a \in G_1$, $b\in S$: $b^{-1}ab\times 1 = b^{-1}(a\times 1)\, b\times 1 = (b^{-1}a\times b)b\times 1 =$

$= (b^{-1}ab\times b^2)\times 1 = b^{-1}ab\times(b^2\times 1) = b^{-1}ab\times b^2 = b^{-1}ab$ and hence $b^{-1}ab \in G_1$.

For every $a\in G_2$, $b\in S$: $b^{-1}ab = b^{-1}(1\times a)b = (b^{-1}\times ba)b = 1\times bab \in G_2$.

7) $S = G_1 \cdot G_2$. In fact for $x\in S$ we consider the element $a = x\times 1 \in G_1$.

So it follows that $a = (1\times x^{-1})x$, i.e. $1\times x^{-1} = ax^{-1}$ and hence, since $1\times x^{-1} \in G_2$, $x = (1\times x^{-1})^{-1}\cdot a = a\cdot(1\times x^{-1})^{-1} \in G_1\cdot G_2$.

From 5),6),7) it follows that

8) $G = G_1 \otimes G_2$ (where $\otimes$ denotes the direct product)

9) $1\times g_1g_2 = g_2$, $g_1g_2\times 1 = g_1$ where $g_1\in G_1$, $g_2 \in G_2$. In fact, since

$(g_1^{-1}g_2^{-1}\times 1)g_1g_2 = (1\times g_1g_2) \in G_2$,it follows $g_1^{-1}g_2^{-1}\times 1 \in g_1^{-1}G_2$.

But $g_1^{-1}g_2^{-1}\times 1 \in G_1$ and so (by 5) $g_1^{-1}g_2^{-1}\times 1 = g_1^{-1}$. This means that

$1\times g_1g_2 = (g_1^{-1}g_2^{-1}\times 1)g_1g_2 = g_1^{-1}g_1g_2 = g_2$ (Likewise for $g_1g_2\times 1 = g_1$).

10) If $a = g_1g_2$, $b = h_1h_2$, where $g_1,h_1\in G_1, g_2,h_2 \in G_2$ then $a\times b = g_1g_2\times h_1h_2 =$

$= (1\times h_1h_2g_2^{-1}g_1^{-1})g_1g_2 = h_2g_2^{-1}g_1g_2 = h_2g_1 = g_1h_2$.

So we have the following

THEOREM 1. For the algebra $S(\cdot,\times)$ the following properties hold

1) $S = G_1 \otimes G_2$ and $g^2 = 1$ $\forall g \in G_2$.

2) $g_1 g_2 \times h_1 h_2 = g_1 h_2$ $(g_1, h_1 \in G_1 \ ,\ g_2, h_2 \in G_2\ )$.

Conversely we can prove

THEOREM 2. Let the group $S(\cdot)$ be the direct product of two subgroup G_1, G_2 such that $g^2 = 1$ for every $g \in G_2$. A semigroup operation "$\times$" with an idempotent element exists in S such that

$$\forall\ a,b,c \in S : (a\times b)c = ac\times bc \ , \qquad c(a\times b) = ca\times c^{-1}b \ .$$

Proof. A few calculations show that the required operation is defined as follows:

$$g_1 g_2 \times h_1 h_2 = g_1 h_2 \quad \text{for every} \quad g_1, h_1 \in G_1,\ g_2, h_2 \in G_2 \ .$$

REMARK. Finally we observe that theorems analogous to theorems 1 and 2 can be proved if in place of (α) one has

(β) $\forall\ a,b,c \in S : (a\times b)c = ac\times bc \qquad c(a\times b) = c^{-1}a\times cb$

R E F E R E N C E S

[S] J. Szép : *On a finite algebra with two operations.* Act.Math.Academiae Scientiarum Hungaricae Tomus 26 (3-4),(1975),347-348.

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