REMARKS ON SZÉPS'S DECOMPOSITION OF SEMIGROUPS.

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J. Szēp has given in [3] a disjoint decomposition for an arbitrary semigroup. Let S be a semigroup without non-zero annihilators (every semigroup can be easily reduced to this case): then

$$(1) \qquad S = \bigcup_{i=0}^{5} S_i$$

holds, where the semigroups $S_i (i=0,1,\dots,5)$ are mutually disjoint and

$$S_0 = \{a \in S : a S \subsetneq S \text{ and } \exists\, x \in S,\ x \neq 0, \text{ such that } a x = 0\}$$

$$S_1 = \{a \in S : a S = S \text{ and } \exists\, y \in S,\ y \neq 0, \text{ such that } ay = 0\},$$

$$S_2 = \{a \in S : a \notin S_0 \cup S_1,\ a S \subset S \text{ and } \exists\, x_1, x_2 \in S,\ x_1 \neq x_2, \text{ such that } ax_1 = a x_2\},$$

$$S_3 = \{a \in S : a \notin S_0 \cup S_1,\ a S = S \text{ and } \exists\, y_1, y_2 \in S,\ y_1 \neq y_2, \text{ such that } ay_1 = ay_2\},$$

$$S_4 = \{a \in S : a \notin \bigcup_{i=0}^{3} S_i \text{ and } a S \subset S\},$$

$$S_5 = \{a \in S : a \notin \bigcup_{i=0}^{3} S_i \text{ and } a S = S\},$$

It follows that for a <u>finite</u> semigroup S one has

(2) $$S = S_0 \cup S_2 \cup S_5 \ .$$

The finiteness of S is not a necessary condition for the validity of (2). F. Migliorini and J. Szép [1] proved that the same decomposition holds if S is a regular semigroup without (left) magnifying elements. The next Theorem 1 gives another sufficient condition.

Let S be a completely regular semigroup, i.e. for every $a \in S$ there exists x in S such that $a = axa$ (that is, S is regular), and $ax = xa$. It is well known that S is completely regular if and only if it is a disjoint union of groups,

(3) $$S = \bigcup_{\alpha \in I} G_{e_\alpha} , \qquad G_{e_\alpha} \cap G_{e_\beta} = \emptyset \qquad (\alpha \neq \beta) ,$$

where G_{e_α} is a maximal subgroup of S, with identity e_α .

Theorem 1 - *Let S be a completely regular semigroup. Then $S_1 = S_3 = S_4 = \emptyset$.*

Proof. We prove that $S_4 = \emptyset$. Assume the contrary: then, given $a \in S_4$, we have $a \in G_{e_\alpha}$ for a suitable $\alpha \in I$, and $G_{e_\alpha} \neq S$. By the definition of S_4, the elements of the set aS are all distinct; hence an analogous conclusion holds for the set $e_\alpha S$ (otherwise $e_\alpha s = e_\alpha s'$ would imply $as = as'$). It follows that $e_\alpha s = s$ for any $s \in S$ (otherwise $e_\alpha s_0 \neq s_0$ would imply $e_\alpha e_\alpha s_0 \neq e_\alpha s_0$, which is impossible), i.e. e_α is a left identity of S; then e_α is the identity of G_{e_β}, with $\beta \neq \alpha$ (contradiction).

Now, $S_4 = \emptyset$ implies $S_1 = S_3 = \emptyset$, by Corollary 1.5 of [1].

Theorem 2 - *Given a completely regular semigroup S and its decomposition $S = S_0 \cup S_2 \cup S_5$, the latter three semigroups are completely regular.*

Proof. : a) For any $a \in S_0$ there is an $\alpha \in I$ such that $a \in G_{e_\alpha}$: we show that $G_{e_\alpha} \subseteq S_0$ (it will easily follow that S_0 is a disjoint union of groups). Let $b \neq 0$ such that $ab = 0$ (recall the definition of S_0): then $G_{e_\alpha} ab = G_{e_\alpha} b = 0$, i.e. $G_{e_\alpha} \subseteq S_0$.

b) Let $a \in S_2$: then $a \in G_{e_\alpha}$ for a suitable $\alpha \in I$. The definition of S_2 gives $ab_1 = ab_2$, with $b_1 \neq 0$, $b_2 \neq 0$, $b_1 \neq b_2$, and it follows that $g\, b_1 = g\, b_2$ for any $g \in G_{e_\alpha}$, and so $g \in S_2$.

c) Let $a \in S_5$ and $a \in G_{e_\alpha}$: then $a\, S = S$ and $e_\alpha S = S$. It follows $g\, S = S$ for any $g \in G_{e_\alpha}$, and, since $S_1 = S_3 = \emptyset$, we have $G_{e_\alpha} \subseteq S_5$.

Corollary 1 - *If S is a completely regular semigroup, then the conclusions of Theorem 2.1 and Corollary 2.2 of [1] hold without the assumptions concerning the magnifying elements.*

Let us now apply Szép's decomposition to the case of a nearly right simple (n.r.s.) semigroup: for its definition, cfr. [2]. It can be characterized as a semigroup which is the disjoint union of its principal right ideals. But we may also consider decomposition (1) in this case.

Theorem 3 - *Let* S *be a n.r.s. semigroup (without non-zero annihilators), which is* not *right simple.*

Then

(4) $$S = S_2 \cup S_4 .$$

Proof.: Since in S there are no nonzero annihilators, it follows from the definition of n.r.s. semigroup that also $0 \notin S$. Therefore $S_0 = S_1 = \emptyset$, and it is not difficult to see that $S_3 = S_5 = \emptyset$.

Corollary 2 - *If* S *is n.r.s. and periodic (and not right simple), then* $S = S_2$. *Moreover,* S *is completely regular.*

Remarks: (i) Although a right simple semigroup is n.r.s., its decomposition is not a particular case of(4). In fact, if S is right simple, one has $S = S_3 \cup S_5$.

(ii) In general, a n.r.s. semigroup, is not completely regular, and conversely. On the other hand, it is shown in [2] that a n.r.s. semigroup is right regular.

REFERENCES

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