4 SPECIAL FRAMES.

A classification of the most important types of frames can be performed taking into account the vanishing of quantities occurring, in $D\bar{P}$.
So we get a chain of four types, characterized by a more and more rich structure of the fosition space $\mathbb{P}$.

<u>Affine frames</u>.

1 DEFINITION

The frame $\mathcal{P}$ is AFFINE if

$$\check{D}^2 \bar{P} = 0 \ .$$

2 We have interesting characterizations of affine frames.

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is affine.

b) $\check{D}\bar{P}$ depends only on time, i.e. $\check{D}\bar{P}$ is factorizable as follows

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\check{D}\bar{P}} & \\ \downarrow t & & \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \\ \mathbb{T} & \nearrow & \end{array}$$

c) We have $\quad \check{\check{P}} = 0$

d) $\check{P}$ depends only on time, i.e. $\check{P}$ is factorizable as follows

$$\begin{array}{ccc} \mathbb{T} \times \mathbb{E} & \xrightarrow{\check{P}} & \\ \downarrow id_T x t & & \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \\ \mathbb{T} \times \mathbb{T} & \nearrow & \end{array}$$

e) Let $\sigma \in \mathbb{T}$; then , $\forall\ \tau \in T$, the map

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S}_\sigma \to \mathbb{S}_\tau$$

is affine, i.e.

$$\tilde{P}_{e'}(\tau) = \breve{P}_e(\tau) + \check{P}_{(\tau,\sigma)} \quad (e' - e) \ .$$

f) $\forall\ \tau \in \mathbb{T}$, the map

$$\bar{P}_{|\$_\tau} : \$_\tau \to \mathbb{U}$$

is affine, i.e.

$$\bar{P}(e') = \bar{P}(e) + \frac{1}{2}\,\epsilon_{\mathbf{P}}(\tau)(e'-e) + \Omega_{\mathbf{P}}(\tau) \times (e'-e) \ \underline{\cdot}$$

PROOF.

It sufficies to prove f) $\Longrightarrow$ e), the other implications being immediate. f) $\Longrightarrow$ e).

Let $$D_1\tilde{P}(\tau,e') = D_1\tilde{P}(\tau,e) + \check{D}_2 D_1\tilde{P}(\tau) \ (e'-e),$$

with $$t(e) \equiv \sigma \equiv t(e') \ .$$

Then, by integration, we get

$$\tilde{P}(\tau,e') = \tilde{P}(\tau,e) + A(\tau)(e'-e) + B(\tau,e-e')$$

where

$$A(\tau) : \bar{\$} \to \bar{\$}$$

is a linear map.

Moreover, for (II,1.10 a) and (II.1.10 b) also B is linear with respect to $(e-e')$.

Then $$\tilde{P}(\tau,e') = \tilde{P}(\tau,e) + \check{D}_2\tilde{P}(\tau)(e'-e) \qquad \underline{\cdot}$$

Here by abuse of notation we have written

$$\check{D}\,\bar{P} : \mathbb{T} \to \bar{\$}^* \otimes \bar{\$} \quad ; \ \check{P} : \mathbb{T} \times \mathbb{T} \to \bar{\$}^* \otimes \$ \ , \ \ldots$$

as $\check{D}\,\bar{P}$, $\check{\check{P}}$, ... depend only on time.

Hence the motion of an affine frame $\mathcal{P}$ is characterized by the motion of one of its particles

$$P_q : \mathbb{T} \to \mathbb{E} \qquad \text{and by} \qquad \epsilon_{\mathcal{P}} : \mathbb{T} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \,, \; \bar{\Omega} : \mathbb{T} \to \bar{\mathbb{S}} \,.$$

3 Let $\mathcal{P}$ be affine. since $\check{P}$ depends only on time, we can get a reduction of the representation of $T\,\mathbb{P}$ by $\check{T}\mathbb{E}_{/\mathcal{P}}$, writing

$$(\mathbb{E} \times \bar{\mathbb{S}})_{/\mathcal{P}} \cong (\mathbb{P} \times \mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} = \mathbb{P} \times (\mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} \,.$$

THEOREM.

a) Let $\bar{\mathbb{P}}$ be the quotient space

$$\bar{\mathbb{P}} \equiv (\mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} \,,$$

given by

$$[\tau,u] = [\tau',u'] \iff u' = \check{P}_{(\tau',\tau)}(u) \,.$$

Then $\bar{\mathbb{P}}$ results into a vector space, putting

$$\lambda\,[\tau,u] \equiv [\tau,\lambda u]$$

$$[\tau,u] + [\tau',u'] \equiv [\tau, u + \check{P}_{(\tau,\tau')}(u)]$$

For each $\tau \in \mathbf{T}$, the map

$$\begin{aligned} \bar{\mathbb{P}} &\longrightarrow \bar{\mathbb{S}} \\ [\tau',u] &\longrightarrow \check{P}_{(\tau,\tau')}(u) \,, \end{aligned}$$

is an isomorphism.

b) Let $\sigma_{\mathcal{P}}$ be the map

$$\sigma_{\mathcal{P}} : \mathbb{P} \times \bar{\mathbb{P}} \to \mathbb{P},$$

given by $(q,[\tau,u]) \mapsto p(\mathcal{P}(\tau,q) + u)$.

Then the triple $(\mathbb{P},\bar{\mathbb{P}},\sigma_{\mathcal{P}})$ is a three dimensional affine space.

c) For each $\tau \in \mathbb{T}$, the maps

$$p_{\tau} : \mathbb{S}_{\tau} \to \mathbb{P} \qquad \text{and} \qquad \mathcal{P}_{\tau} : \mathbb{P} \to \mathbb{S}_{\tau}$$

are affine isomorphisms.

d) We get the splittings $T\,\mathbb{P} = \mathbb{P} \times \bar{\mathbb{P}}$ and $T^2\mathbb{P} = \mathbb{P} \times \bar{\mathbb{P}} \times \bar{\mathbb{P}} \times \bar{\mathbb{P}}$,

writing

$$[e,u] = (p(e),[t(e),u]) \quad \text{and} \quad [e,u,v,w] = (p(e),[t(e)u],[t(e),v][t(e)w]])$$

$\Gamma_{\mathcal{P}}$ results to be time independent and it is the affine connection of $\mathbb{P}$

$$\check{\Gamma}_{\mathcal{P}} : s\,T^2\mathbb{P} \to {}_{\nu}T^2\,\mathbb{P}$$

$$(q,[\tau,u],[\tau,u],[\tau,w]) \mapsto (q,[\tau,u],0,[\tau,w]) \; .$$

PROOF.

It follows from the fact that, $\forall\ \tau',\tau \in \mathbb{T}$, the map

$$\tilde{P}_{(\tau',\tau)} : \mathbb{S}_{\tau} \to \mathbb{S}_{\tau'}$$

is affine and from the properties

$$\tilde{\mathcal{P}}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)} = \tilde{P}_{(\tau'',\tau)} \quad , \quad \tilde{P}_{(\tau,\tau)} = id_{\mathbb{S}_{\tau}}$$

4 We get simplified formulas for $T\,p$, T^2p, $T\,P$, T^2P and $\check{\Gamma}_{\mathcal{P}}$.

COROLLARY.

We have

a) $T\,p(e,u) = (p(e),\ [t(e),\ \hat{P}(e)(u)])$.

b) $T\,P(\tau,\lambda;q,[\tau',u]) = (P(\tau,q),\ \lambda\bar{P}(P(\tau,q) + P_{(\tau,\tau')}(u))$

c) $\dot{\Gamma}_{\mathcal{P}}(\tau;q,[\tau,u],[\tau,v],[\tau,w]) =$

$$= (q,[\tau,u],0,[\tau,w + \epsilon_P(\tau)\ (u) + 2\Omega_P(\tau) \times u + \bar{\bar{P}}(P(\tau,q))]).$$

Rigid frames.

5 DEFINITION.

The frame $\mathcal{P}$ is RIGID if it is affine and

$$\epsilon_{\mathcal{P}} = 0 \quad \dot{-}$$

6 We have interesting characterizations of rigid frames.

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is rigid.

b) Let $\sigma \in \mathbb{T}$; then, $\forall\ \tau \in \mathbb{T}$, the map

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S} \to \mathbb{S}_\tau$$

preserves the distances, i.e.

$$||\tilde{P}_{(\tau,\sigma)}(e) - \tilde{P}_{(\tau,\sigma)}(e')|| = ||e-e'||$$

c) $\forall\ \sigma \in \mathbb{T}$, the map

$$\bar{P}_{|\mathbb{S}_\sigma} : \mathbb{S}_\sigma \to \mathbb{U}$$

is affine and

$$\bar{P}(e') = \bar{P}(e) + \Omega_P(\sigma) \times (e'-e) \quad .$$

d) We have $\check{\check{P}} = 0$ and $\check{P} : \mathbb{T} \times \mathbb{E} \to S\,U(\bar{\mathbb{S}})$.

PROOF.

a)<===>c) trivial.

a) ===> b) $\epsilon_{\mathcal{P}}$ is the Lie derivative of g with respect $\bar{P}$, i.e. the derivative with respect to time of the deformations tensor $\underset{\sim}{g} \circ(\check{P},\check{P}) - \underset{\sim}{g}$. Then $\epsilon_{\mathcal{P}} = 0$, by integration with respect to time, gives the result.

b) ===>d) It is known (the proof is a purely algebric computation, making use of an orthogonal basis) that, if A is an affine euclidean space and $f : A \to A$ is a map which preserves the norm, then f is an affine map with unitary derivative. Then we see that $P(\tau,\sigma)$ is affine adn $\tilde{D}P_{(\tau,\sigma)} \in U(\bar{\mathbb{S}})$.

d) ===>a) $\check{P}_{(\tau',\tau)} \in U(\mathbb{S})$ gives

$$\check{P}_{(\tau,\tau')} = \check{P}^{t}_{(\tau',\tau)}$$

hence, deriving respect to τ'

$$\check{P}_{(\tau',\tau)} \circ \check{P}^{t}_{(\tau',\tau)} = id_{\bar{\mathbb{S}}} ,$$

we get

$$D_1\check{P}_{(\tau',\tau)} \circ \check{P}^{t}_{(\tau',\tau)} + \check{P}_{(\tau',\tau)} \circ D_1 \check{P}^{t}_{(\tau',\tau)} = 0$$

and, for $\tau' \equiv \tau$,

$$\epsilon_{\mathcal{P}}(\tau) = S\,D_1 \check{P}_{(\tau,\tau)} = D_1 \check{P}_{(\tau,\tau)} + D_1\check{P}^{t}_{(\tau,\tau)} = 0 \;\dot{-}$$

Hence the motion of a rigid frame $\mathcal{P}$ is characterized by the motion of one of its particles $\mathcal{P}_q : \mathbb{T} \to \mathbb{E}$ and by $\Omega_{\mathcal{P}} : \mathbb{T} \to \bar{\mathbb{S}}$.

7 Let $\mathcal{P}$ be rigid.

THEOREM.

$\mathcal{P}$ results into an affine euclidean space. In fact $g_{\mathcal{P}}$ results to be time independent and we can define the map

$$g_{\mathcal{P}} : \bar{\mathbb{P}} \to \mathbb{R}$$

which is given by $\quad [\tau, u] \mapsto \frac{1}{2} u^2 \quad .$

The affine connection $\check{\Gamma}_{\mathcal{P}}$ results into the Riemannian connection of $\mathcal{P}$.

Translating frames.

8 DEFINITION

A frame $\mathcal{P}$ is TRANSLATING if it is rigid and

$$\Omega_{\mathcal{P}} = 0 \quad .$$

9 We have interesting characterizations of translating frames .

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is translating

b) Let $\sigma \in \mathbb{T}$; then $\forall \; \tau \in T$, the map,

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S}_{\sigma} \to \mathbb{S}_{\tau}$$

is affine, with derivative $\quad D\tilde{P}_{(\tau,\sigma)} = id_{\bar{\mathbb{S}}}$, i.e.

$$\tilde{P}_{e'}(\tau) = \tilde{P}_{e}(\tau) + (e'-e) \quad .$$

c) $\forall \; \tau \in \mathbb{T}$, the map

$$\bar{P}_{|\mathbb{S}_{\tau}} : \mathbb{S}_{\tau} \to \mathbb{U}$$

is constant, i.e. $\quad \bar{P}(e') = \bar{P}(e) \quad .$

Hence the motion of a translating frame is characterized by the motion of one of its particles $P_q : \mathbb{T} \to E$.

We will write $\bar{P} : \mathbb{T} \to \mathbb{U}$, $\bar{\bar{P}} = D\bar{P} : \mathbb{T} \to \bar{\$}$. $\hat{P} = d_{\mathbb{E}} - \underline{t} \otimes \bar{P} : \mathbb{T} \to \bar{\mathbb{E}}^* \otimes \bar{\$}$.

10 Let $\mathcal{P}$ be translating. Since $\check{P} = id_{\bar{\$}}$, we can get a further reduction of the representation of $T\mathbb{P}$ by $\check{T}\mathbb{E}_{/\mathcal{P}}$, writing

$$(\mathbb{E} \times \bar{\$})_{/\mathcal{P}} \cong (\mathbb{P} \times \mathbb{T} \times \bar{\$})_{/\mathcal{P}} = \mathbb{P} \times \bar{\$} \quad .$$

THEOREM.

Let $\mathcal{P}$ be translating.

a) The map

$$\bar{\mathbb{P}} \to \bar{\$} ,$$

given by

$$[\tau, u] \to u ,$$

is well defined and it is an isomorfism.

Then the map

$$\sigma_P : \mathbb{P} \times \bar{\$} \to P,$$

given by

$$(q,u) \mapsto p(P(\tau,q)+u),$$

does not depend on the choice of $\tau \in T$.

b) The triple $(P, \$, \epsilon_{\mathcal{P}})$ is an affine euclidean space,

11 We get simplified formulas for $T\,p$, $T^2 p$, $T\,P$, $T^2 P$, $\Gamma_{\mathcal{P}}$.

PROPOSITION

Let P translating

a) $T\,p(e,u) = (p(e), u - u^\circ\, \bar{P}(t(e)))$

$$T^2 p\,(e,u,v,w) = (p(e), u-u^\circ\,\bar{P}(t(e))\ v-v^\circ\,\bar{P}(t(e)), w-w^\circ\,\bar{P}(t(e)) + \bar{\bar{P}}(t(e))).$$

b) $TP(\tau,\lambda;q,u) = (P(\tau,q),\lambda\bar{P}(\tau) + u$,

$$T^2P(\tau,\lambda,\mu,\nu;q,u,v,w) = (P(\tau,q),\lambda\bar{P}(\tau)+u,\mu\bar{P}(\tau)+\nu,\lambda\bar{\bar{P}}(\tau)+\nu\bar{P}(\tau)+w)$$

c) $\Gamma'_{\mathcal{P}}(\tau;q,u,u,w) = (q,u,0,w + \bar{\bar{P}}(\tau))$.

<u>Inertial frames.</u>

12 DEFINITION.

A frame P is inertial if it is translating and

$$\bar{\bar{P}} = 0.$$

13 PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is inertial,

b) $\mathcal{P}$ is translating and $D\,\bar{P} = 0$.

c) $\tilde{P}$ is are affine map, i.e. (taking into account the properties (II.1.10)

$$\tilde{P}(\tau,e) = e + \bar{P}(\tau-t(e)), \qquad \text{with } \bar{P} \in U .$$

d) $\bar{P} : \mathbb{E} \to \mathbb{U}$ is a constant map .

Hence an inertial frame is characterized by its constant velocity.

14 PROPOSITION.

We have

a) $T\,p(e,u) = (p(e),u-u^\circ\bar{P})$

$$T^2p(e,u,v,w) = (p(e),u-u^\circ\bar{P},v-v^\circ\bar{P}, w-w^\circ\bar{P})$$

b) $T\,P(\tau,\lambda;q,u) = (P(\tau,q),\lambda\bar{P}+u)$

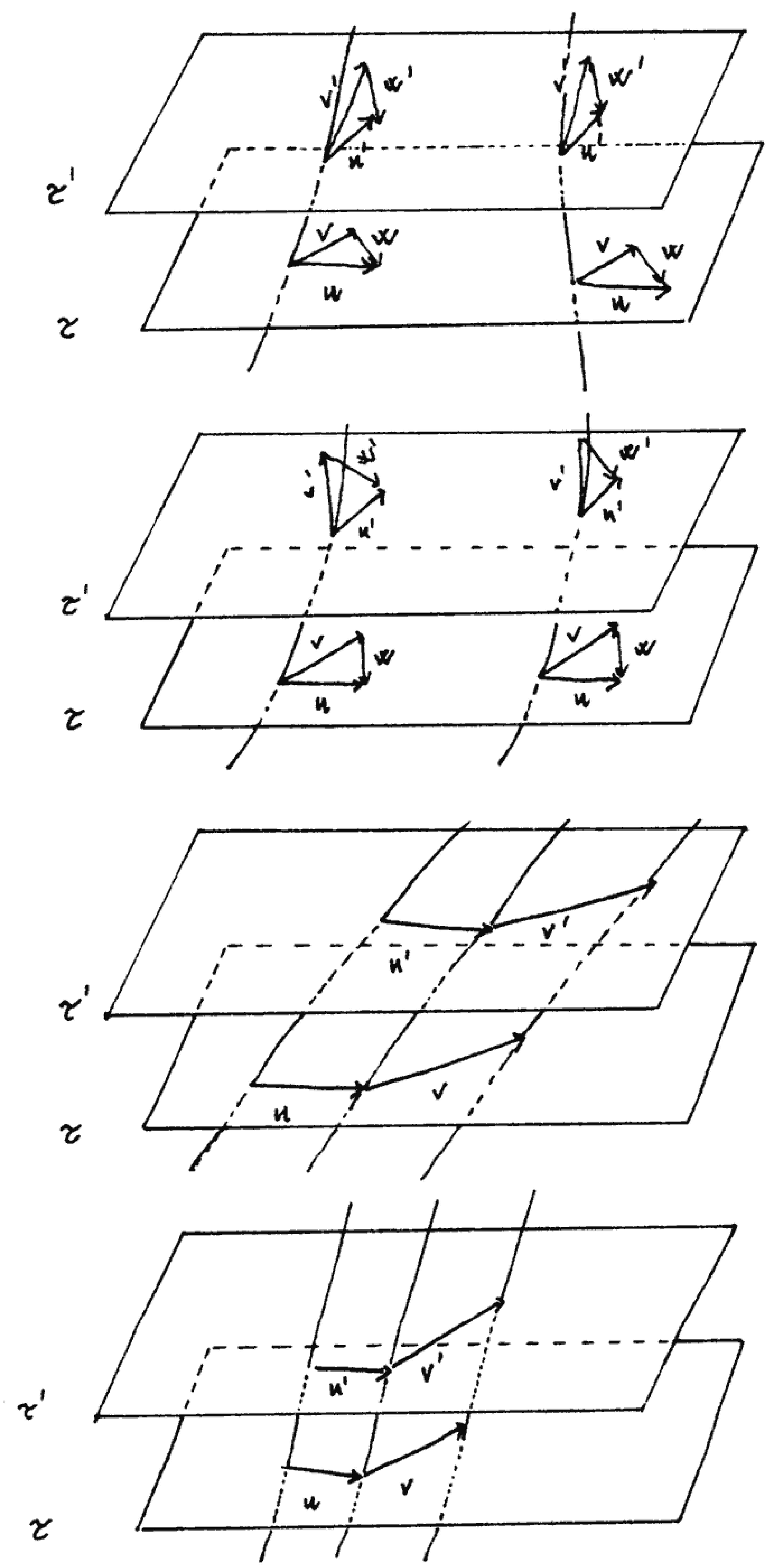
$$T^2P(\tau,\lambda,\mu,v;q,u,v,w) = (P(\tau,q),\lambda\bar{P} + u,\mu\bar{P}+v,\nu\bar{P}+w)$$

c) Γ'_P results time independent and we get

$$\Gamma'_{\mathcal{P}} = \Gamma_{\mathcal{P}}$$

Physical description.

A frame $\mathcal{P}$ is affine if it preserves during the motion the spatial parallelogram rule; it is rigid if moreover it preserves spatial leghts (hence also angles); it is translating if moreover it preserves spatial directions; it is inertial if its world-lines are parallel straight-lines. We can describe the four cases by a picture .



$w' = u' + v'$

$w = u + v$

affine frame

$w' = u' + v'$

$||u|| = ||u'||, ||v|| = ||v'||, ||w|| = ||w'||$

$w = u + v$

rigid frame

$u' = u$

$v' = v$

translating frame

$u' = u$

$v' = v$

inertial frame