I N T R O D U C T I O N

Summary.

The general framework of classical mechanics is the absolute event space $\mathbb{E}$. It is an affine four dimensional space, representing the set of absolute events. A three dimensional subspace $\bar{\mathbb{S}}$ of its vector space $\bar{\mathbb{E}}$ is fixed to represent the set of couples of symultaneous events. Then $\mathbb{E}$ results into the disjoint union of parallel three dimensional affine subspaces $\mathbb{S}_\tau$, generated by $\bar{\mathbb{S}}$, which represent the equivalence classes of symultaneous events. The set $\mathbb{T}$ of these equivalence classes is a one dimensional affine oriented space, which represents absolute time.
The quotient projection $t : \mathbb{E} \to \mathbb{T}$ is the time function. The triple $(\mathbb{E},t,\mathbb{T})$ is an affine trivial bundle (but not canonically a product), whose fibers are the(not canonically isomorphic among themselves) equivalence classes $\mathbb{S}_\tau$. The map $Dt : \bar{\mathbb{E}} \to \bar{\mathbb{T}}$ associates with each four vector u its absolute time component u°. We have not an absolute projection $\bar{\mathbb{E}} \to \bar{\mathbb{S}}$, or an absolute inclusion $\bar{\mathbb{T}} \hookrightarrow \bar{\mathbb{E}}$. Then we have not an absolute splitting $\bar{\mathbb{E}} = \bar{\mathbb{T}} \oplus \bar{\mathbb{S}}$ (whereas it is induced by a frame of reference). The inclusion $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$ admits the vertical (along $\bar{\mathbb{S}}$,i.e. at a fixed time) derivatives Df of maps defined on $\mathbb{E}$. On the vector space $\bar{\mathbb{S}}$ we have absolute euclidean metric $\check{g}$, defined up to a conformal factor, which describes the classical geometry. For pratical reasons we choose a unit of measure on $\bar{\mathbb{S}}$ and on $\bar{\mathbb{T}}$, selecting the conformal factors. The unit of measure on $\bar{\mathbb{T}}$ determines the identification $\bar{\mathbb{T}} \simeq \mathbb{R}$. Then we get the subspace $\mathbb{U} \hookrightarrow \bar{\mathbb{E}}$, constituted by the vectors normalized by $\underline{t} \equiv Dt$, which represents the space of velocities. We define a Poincare's map as a map $G : \mathbb{E} \to \mathbb{E}$, which preserves the structure of $\bar{\mathbb{S}}$.

Its derivative $DG:\bar{\mathbb{E}} \to \bar{E}$ is the associated Galilei's map. At last, for the coordinate description, we define the special charts. These are characterized by the first coordinate function which is cartesian and depends only on time.

The formulation of the theory by means of applied vectors requires the definition of further spaces, namely the spaces of analytical mechanics. To first order, we consider the phase, or vertical, space $\check{T}\mathbb{E}\equiv\mathbb{E}\times\bar{\mathbb{S}} \hookrightarrow T\mathbb{E}\equiv\mathbb{E}\times\bar{\mathbb{E}}$ and the velocity, or unitary, space $\dot{T}\mathbb{E} \equiv \mathbb{E}\times\mathbb{U}\hookrightarrow T\mathbb{E}$, which are (not canonically) isomorphic. To second order, we consider the vertical spaces $\check{T}^2\mathbb{E}\equiv\mathbb{E}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}\equiv\mathbb{E}\times\bar{\mathbb{E}}\times\bar{\mathbb{E}}\times\bar{\mathbb{E}}$ and $\nu\acute{T}^2\mathbb{E}=\mathbb{E}\times\bar{\mathbb{S}}\times 0\times\bar{\mathbb{S}} \to \nu T^2\mathbb{E} \equiv$ $\equiv\mathbb{E}\times\bar{\mathbb{E}}\times 0\times\bar{\mathbb{E}}$ and the bivelocity spaces $\dot{T}^2\mathbb{E}\equiv\mathbb{E}\times\mathbb{U}\times\bar{\mathbb{S}} \to T^2\mathbb{E}$ and $\nu\dot{T}^2\mathbb{E}\equiv\mathbb{E}\times\mathbb{U}\times 0\times\bar{\mathbb{S}} \to$ $\to \nu T^2\mathbb{E}$, which play an important role in the connection properties of frames of reference, in the calculation of acceleration and in the Coriolis theorem. The natural projections $\Gamma:T^2\mathbb{E} \to \nu T^2\mathbb{E}$ and $\sqcup\!\sqcup: \nu T^2\mathbb{E} \to T\mathbb{E}$ permit the definition of the covarint derivative $\nabla_u v = \sqcup\!\sqcup\circ\Gamma\circ Tv\circ u$, which, after choosing a coordinate system, is expressed by the Christoffel symbols.

The absolute one-body world-line is a (one dimensional) submanifold $M \hookrightarrow \mathbb{E}$, which meets each $\mathbb{S}_\tau$ exactly at one point $M(\tau)$. The world-line is characterized by its absolute motion, that is by the associated map $M : \mathbb{T} \to \mathbb{E}$. The absolute free velocity and acceleration are the maps $DM : \mathbb{T} \to \mathbb{U}$ and $D^2M : \mathbb{T} \to \bar{\mathbb{S}}$. If we need to consider them in terms of applied vectors, we define the velocity and acceleration $dM\equiv(M,DM) : \mathbb{T} \to \dot{T}\mathbb{E}$ and $dM\equiv(M,D^2M) \equiv\sqcup\!\sqcup\circ\Gamma\circ d^2M:\mathbb{T} \to T\mathbb{E}$. After choosing coordinate system, the acceleration is expressed by the Christoffel symbols.

To determine positions, hence to get observed mechanics, we need frames of reference. A frame $\mathcal{P}$ is a continuum filling, in a C^∞ way, the whole event space.

First we can study the absolute kinematics of such a continuum. The continuum $\mathcal{P}$ is constituted by a set $\mathbb{P}$ of disjoint world-lines $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$. In this sense $\mathbb{P}$ can be viewed as the set of the particles of the continuum. For each event $e \in \mathbb{E}$, passes a unique particle $p(e) \in \mathbb{P}$. Then we get a surjective map $p : \mathbb{E} \to \mathbb{P}$. The set of the motions of the particles $\{P_q : \mathbb{T} \to \mathbb{T}_q \to \mathbb{E}\}_{q\in\mathbb{P}}$ determines the motion of the continuum, namely the map $P : \mathbb{T} \times \mathbb{P} \to \mathbb{E}$, which associates with (τ,q) the event touched at the time τ by the particle q. We can define the motion also by the map $\tilde{P} : \mathbb{T} \times \mathbb{E} \to \mathbb{E}$, which associates with (τ,e) the event touched at the time τ by the particle passing through e. Then we obtain a number of fields by deriving the motion. These fields can be expressed in the fundamental form $f : \mathbb{T}\times\mathbb{E} \to \mathbb{F}$, or in the eulerian form $f_\circ : \mathbb{E} \to \mathbb{F}$, or in the lagrangian form $f_{\circ\tau} : \mathbb{S}_\tau \to \mathbb{F}$, where $f(\tau,e)$ is attached to the event $\tilde{P}(\tau,e)$, while $f_\circ(e)$ and $f_\circ(e)$ are attached to e. The three formulations are equivalent, for we have $f_\circ(e) = f(t(e),e)$, $f_{\circ\tau} = f_\circ|_{\mathbb{S}_\tau}$, $f(\tau,e) = f_\circ(\tilde{P}(\tau,e))$. We can also consider two-points fields, for which fundamental, eulerian and lagrangian formu tions hold(but the relation among them, with respect to the second point is more complicated, for it involves the derivative of the motion).

Thus we consider the first and second time derivatives of the motion, defining the velocity and the acceleration of the frame $\bar{P} : \mathbb{E} \to \mathbb{U}$ and $\bar{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}$, where $\bar{P}(e)$ and $\bar{\bar{P}}(e)$ are the velocity and the acceleration, at the time $t(e)$, of the particle passing through e. Then we consider the first and second event derivatives of the motion $\hat{P} : \mathbb{E} \to \mathbb{E}^* \otimes \bar{\mathbb{S}}$ and $\hat{\hat{P}} : \mathbb{E} \to \mathbb{E}^* \otimes \mathbb{E}^* \otimes \bar{\mathbb{S}}$, which express the projection and the rate of projection of event intervals into simultaneous event intervals, due to the motion of the continuum.

The motion determines also a diffeomorphism between each couple of simultaneous spaces $\mathbb{S}_\tau$ and $\mathbb{S}_{\tau'}$. Then, by a first and second derivatives, we get the jacobian maps $\check{P}_{(\tau',\tau)}:\mathbb{S}_\tau \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$ and $\check{\check{P}}_{(\tau',\tau)}:\mathbb{S}_\tau \to \bar{\mathbb{S}}^* \otimes \bar{S}^* \otimes \bar{\mathbb{S}}$. The mixed derivative can be obtained considering the derivative, at a fixed time, of the velocity, namely $D\bar{P}:\mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$, or considering the time derivative of $\tilde{P}_{(\tau',\tau)}$.

By means of the spatial metric, we can get the symmetrical and the antisymmetrical parts of the mixed tensor $\check{D}\bar{P}$. The symmetrical part $\epsilon_{\mathcal{P}} \equiv S\check{D}\bar{P}$ gives the rate of change, along the time, of the spatial metric tensor, induced by the motion. The antisymmetrical part

$\omega \equiv \frac{A}{2}\check{D}\bar{P}$, or the associated vector, by means of the Hodge isomorphism $(\check{*}\,\omega \equiv i_\omega \check{\eta}), \Omega \equiv \check{*}\,\omega$ represents the absolute angular velocity of the continuum motion.

Then we can consider the frame continuum as a frame of reference. First we define the observed positions. Each position is the set of all the events that touch a unique particle q, namely it is the set $\mathbb{T}_q$. Then we can identify the set of positions with the set of particles $\mathbb{P}$. Then $\mathbb{P}$ is a set of equivalence classes. This set $\mathbb{P}$ has a C^∞ structure. At each time τ, $\mathbb{P}$ can be represented by the three dimensional affine space $\mathbb{S}_\tau$, associating with each position q the event touched at the time τ, by the particle q. Also for $T\mathbb{P}$ and $T^2\mathbb{P}$ we get two interesting representations. We can represent the tangent space $T\mathbb{P}$ as the quotient space $\check{T}\mathbb{E}_{/\mathcal{P}}$, namely as the set of strips of spatial vectors spanned (at the first order) by the motion.

An analogous representation holds for $T^2\mathbb{P}$. We can also represent $T\mathbb{P}$ and $T^2\mathbb{P}$ by $T\mathbb{S}_\tau$ and $T^2\mathbb{S}_\tau$, taking into account the bijection $\mathbb{P} \to \mathbb{S}_\tau$.

The latter representations induce a time-depending metric $g_{\mathcal{P}}:\mathbb{T}\times T\mathbb{P}\to\mathbb{R}$ and a time-depending connection $\check{\Gamma}_{\mathcal{P}}:\mathbb{T}\times T^2\mathbb{P}\to \nu T^2\mathbb{P}$. Taking into account the frame velocity, we can also represent $T\mathbb{P}$, at each time τ, with $\mathbb{S}_\tau \times\mathbb{U}$, hence $T^2\mathbb{P}$ with $\mathbb{S}_\tau\times\mathbb{U}\times\mathbb{U}\times\bar{\mathbb{S}}$. This representation induces a new map $\Gamma'_{P}:\mathbb{T}\times T^2\mathbb{P}\to\nu T^2\mathbb{P}$, which is the sum of a map $C_P:\mathbb{T}\times T\mathbb{P}\to T\mathbb{P}$ and of a map $D_{\mathcal{P}}:\mathbb{T}\times\mathbb{P}\to T\mathbb{P}$. The latter maps will be interpreted as the generalized Coriolis and the dragging accelerations.

Then we represent the absolute event space $\mathbb{E}$ by the frame-depending splitting into space and time $(t,p):\mathbb{E}\to\mathbb{T}\times\mathbb{P}$, associating with each event e its absolute time $t(e)$ and its frame position $p(e)$. We get also the splitting $T_e\mathbb{E}\to\bar{\mathbb{T}}\times\bar{\mathbb{S}}$, which associates with each vector u, applied in e, its absolute time component u° and its frame spatial projection $\hat{P}(e)(u) = u-u^\circ\bar{P}(e)$. By means of Tp and T^2p we can associate with each point of $T\mathbb{E}$ and $T^2\mathbb{E}$ the relative observed quantities of $T\mathbb{P}$ and $T^2\mathbb{P}$.

Among all frames, some have a special interest for the peculiar properties of their motion and of the position spaces.
First we consider the affine frames, characterized by the fact that $\check{D}\bar{P}$ depends only on time. Their motion is determined by the motion of one of their particles and by its spatial derivative $\check{P}$, which depends only on time. The sum of strips representing the vectors of $T\mathbb{P}$ results to be independent on the position.
Then $\mathbb{P}$ results into an affine space, with the quotient induced by the motion $\bar{P}\equiv(\mathbb{T}\times\bar{\mathbb{S}})_{/\mathcal{P}}$ as vector space. The affine connection results to be time independent and we can write $\check{\Gamma}_{\mathcal{P}}:T^2\mathbb{P}\to\nu T^2\mathbb{P}$.

Then we consider rigid frames, which are affine frames such that $\epsilon_{\mathcal{P}}$ is zero. The spatial derivative of their motion is unitary. Their motion is determined by the motion of one of their particles and by its spatial unitary derivative(by a time derivation we obtain, from this fact, the classical formula for the velocities of the rigid frame). As the motion preserves, along the time, the spatial metric, $\mathbb{P}$ results into an affine euclidean space,namely $g_{\mathcal{P}}$ results time independent and we can write $g_{\mathcal{P}} : T\mathbb{P} \to \mathbb{R}$. Then we consider translating frames, which are rigid frames such that $\Omega_{\mathcal{P}}$ is zero. The spatial derivative of their motion is zero. Their motion is determined by the motion of one of their particles. As the motion preserves, along the time, the spatial vectors, the vector space $\bar{\mathbb{P}}$ results to be equal to $\bar{\mathbb{S}}$. Finally we consider inertial frames, which are translating frames such that $\bar{\bar{P}}$ is zero. The total derivative of their motion is zero. Their motion is determined by the inertial motion of one of their particles. The projection $\hat{P}$ is constant, hence $\check{\Gamma}_P = \Gamma'_P$.

Now we consider a fixed frame $\mathcal{P}$ and a fixed motion M, we define the quantities of M observed by $\mathcal{P}$ and we make a comparison between absolute and observed quantities. The observed motion is the map $M_{\mathcal{P}}:\mathbb{T} \to \mathbb{P}$, which associates with each time $\tau \in \mathbb{T}$ the position $p(M(\tau))$ touched by M at that time. The observed motion $M_{\mathcal{P}}$ characterizes the absolute motion M, since $M(\tau) = \tilde{P}(\tau, M_{\mathcal{P}},(\tau))$. Then we get the velocity of the observed motion $dM_{\mathcal{P}} : \mathbb{T} \to T\mathbb{P}$, which is the derivative of $M_{\mathcal{P}}$ performed by $\mathcal{P}$, by means of its differentiable structure. We get also the acceleration of the observed motion $\check{\nabla}_{\mathcal{P}}\check{d}M_{\mathcal{P}}:\mathbb{T} \to T\mathbb{P}$, which is the covariant derivative of the velocity of the observed motion, per

formed by $\mathcal{P}$, by means of its time depending affine connection $\check{\Gamma}_{\mathcal{P}}$.
The observed velocity of M is the projection on $T\mathbb{P}$ of the velocity dM. The observed velocity and the velocity of the observed motion are equal.
The observed acceleration is the proiection of the sum of the acceleration of the observed motion and of a generalized Coriolis term, plus a dragging term. Namely we can write $D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}}+C_{\mathcal{P}}(D_{\mathcal{P}}M_{\mathcal{P}})+D_{\mathcal{P}}\circ(M_{\mathcal{P}})$.

Finally we consider two frames $\mathcal{P}_1$ and $\mathcal{P}_2$ and a motion M and we make a comparison among quantities observed by $\mathcal{P}_1$ and by $\mathcal{P}_2$. First we consider the quantities of $\mathcal{P}_1$ observed by $\mathcal{P}_2$. Then we find the addition velocities theorem and the generalized Coriolis Theorem. Specializing the kind of frame of references, we get the usual theorems.

Comparison with special and general relativity.

We want to show some surprising and important analogies with the special and general relativity, not involving the light velocity.

In both cases we have a four dimensional event space $\mathbb{E}$, which is affine in the classical and special relativistic case and which has not an absolute splitting into space and time. In the classical case we have a privileged three dimensional subspace $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$, which determines the absolute simultaneity and the absolute time as the quotient space $\mathbb{T} \equiv \mathbb{E}_{/\bar{\mathbb{S}}}$.
These facts have not an absolute relativistic counterpart. On the other hand, in the relativistic case we have a Lorentz metric on the whole $T\mathbb{E}$ (it is constant and fixed in special relativity and it is matter depending in general relativity), while in the classical case we have

only an euclidean metric on $T\mathbb{E}$. The relativistical Lorentz metric determines, by orthogonality, a frame depending and pointwise (local if the frame is integrable) spatial section which replaces the classical $\bar{\mathbb{S}}$. The classical time orientation is given on $\mathbb{T}$, while the relativistic one is given on the light cone.

In all the three cases we can describe a motion (or its world-line) by a four dimensional map $M : \mathbb{T} \to \mathbb{E}$ (or by a one dimensional submanifold $\mathbb{M} \hookrightarrow \mathbb{E}$), which is absolute, i.e. not depending on any frame of reference. In the relativistic case the condition that $\mathbb{M}$ is time-like replaces the classical condition $t \circ M = id_{\mathbb{T}}$. In the relativistic case $\mathbb{T}$ is not the absolute time, but the proper time of the motion, namely it is $\mathbb{M}$ itself endowed with the affine euclidean structure induced by the metric of $\mathbb{E}$. In the classical case we get $<\underline{t},DM> = 1$ and $<\underline{t},D^2M>=0$.

These conditions are replaced in the relativistic case by $DM^2=-1$ and $DM \circ D^2M = 0$.

In all three cases we can consider the most general kind of frame of reference,while people often consider only rigid frames in classical mechanics and inertial frames in special relativistic. The definition of frame is essentially the same in all three cases: the only differences come from the implicit differences in the definition of the world lines of the continuum particles. Analogous considerations hold for the representation of $\mathbb{P}$, $T\mathbb{P}$ and $T^2\mathbb{P}$, for the time depending metric and connection, the Coriolis and dragging maps and the classification of special frames.

Since we deal with general frames of reference, we get for classical and special relativistic observed kinematics criteria currently used in general relativity. In fact under our statment of the absolute Coriolis Theorem we can recognize usual general relativistic formulas, commonly quoted in other form.