C O N T E N T S

II CHAPTER - FRAMES OF REFERENCE

III CHAPTER - OBSERVED KINEMATICS

PREFACE

Proposal.

This paper is the first of a number devoted to an axiomatic approach to classic and relativistic mechanics.

We analyse the foundations of mechanics, trying to reach a new unifying view and to get a systematic exposition of the matter. In these years it is actual in the literature a foundational research, even if along a little different lines.

We study, in a unique context and with a unified language, topics often treated by different authors with different points of view. We try to surpass critically the historical stratification of the matter. In fact, often theories develop under the push of motivations and in a cultural context, which after some time change completely. Nevertheless, the substantial validity of the theories remains. So, while it is historically essential to understand the birth and the development of theories in their real context, from a technical point of view, such an approach can be confusing with respect to the essential synctactical structure of the theory. Moreover, a new synthesis that, even taking into account the historical logic, tries to achieve an independent formulation, can lead to a new phylosophical view.

In these papers we are explicitly concerned only with a theoretical axiomatic treatment.

Phylosophical background.

We want to outline the phylosophical background common to the present and to the subsequent papers, without any claim of rigor and completeness.

We think that a physical theory consists of several mutually connected languages, with different synctaxes, objects and degrees of formal rigor.

There are at least

a) a mathematical synctactical language, which is deductive, selfconsistent, formal, whose object is the theoretical model of the theory;

b) a physical experimental language, which is intuitive, descriptive, whose object is the description of phenomena;

c) an interpretative semantical language, whose object is the relation between the previous two languages.

The appropriate order of exposition of the matter can be different for the mathematical and the physical languages. So the validity of the theory, namely the agreement between the previous languages, must be tested globally and it is meaningless to verify a single axiom or theorem out of their context.

Of course this structure of the theory is not more than an outline.

We are firmly convinced that an omnicomprehensive supertheory cannot exist. We must necessarily deal with a lattice of physical theories, with different physical objects and degrees of validity. The comparison among them is very important and physically expressive. For example the validity of a physical theory is often tested in the context of a more general one.

It is fit to distinguish the language of a theory (even if branched into several sublanguages) from metalanguages which have the theory itself as object. For example, the relativity principles are not part of a phisical theory, as they do not describe physical phenomena, but they are metalinguistic conditions imposed to the theory.

We keep quite distinct the inductive and the deductive construction of a physical theory. In fact, the former has a value more historical than logic, whereas the latter is selfcontained and gives a deep physical insight.

In this way, for each physical question we have to make explicit the experimental and theoretical context, the required degree of approximation and the background accepted as known. As an esample, let us consider inertial frames. We can take into account the geometrical properties of space-time observed by them, the comparison between real and apparent forces observed by them (that involves a theory of interactions), the classical and relativistic approximation, their local and global existence, their experimental determination, etc. The abstract question "what is an inertial frame of reference?" regardless of the previous statements is meaningless.

In the present and in the subsequent papers we deal essentially with the mathematical synctactical language. We follow the actual structuralistic tendency of modern mathematics. Our physical approach is based on a deep analysis of the structure of the fundamental spaces constituting the general framework.
If the good fitting between theory and experiments is not too occasional and limited, but has a deep validity, the choice of basic spaces of the theory cannot be of little relevance and they must contain implicitly all the physical development. We believe that in a good theory all the facts that are mathematically relevant have a great physical interest and viceversa.

We believe that the spirit of Klein's program, of classification of geometrical theories based on their invariance properties, can be surpassed. In fact it was natural in the context of a mathematical language strictly based on coordinates. The situation is quite different now, because we have the intrinsical language of algebra, topology,...

manifolds, fiber bundles,... .
Nevertheless theoretical physics is up to now deeply based on invariance groups and their representation. We think it is time for a change. This proposal requires a large inversion in the traditional sequence and dependence of topics. In such a way, the deep role of mathematics in physical theories gets more relevant and it does not reduce to a computational aspect.

We expect that differential geometry will play a more and more important role in physics. This tendency is present in literature but it does not develop its whole euristic power.

People often say that a high formalization of the theory and a large inversion in its traditional exposition is hard to understand and damages intuition.Not in the long rum, it is our opinion.. In fact we believe that intuition is a process that makes automatic and uncoscious logic proceedings, so that syntax becomes semantic, by means of a long exercise. Then, what to day is abstract to morrow can be intuitive. The more a theory is based on few and well organized axioms, the more the intuitive process will be fast and complete. This believing is supported by many historical examples. The most typical regards elementary geometry. The classic euclidean logic is to day an intuitive description of geometrical daily and familiar physical phenomena. However we make intuitive the description of the same phenomena by means of linear algebra.

Specific criteria.

We try to get a unifying view of classical and special and general relativistic theories. Namely we use the same kind of language and exposition line. We have very similar general frameworks for the three

theories so that the structural differences appear clear and directly comparable. In all cases we have four dimensional "absolute" event spaces, four-dimensional "absolute"motions, velocity and accelerations, four-dimensional forces and so on. In all cases we consider general frames of references and we define the "observed"phenomena by means of the splitting into space and time induced by frames. This point of view is commonly considered as proper to general relativity.
Then for all three theories a general principle of relativity holds! Moreover the constancy of light velocity has not an explicit role and it is completely replaced by the metrical structure of the event space. In all three theories we have well fitted electromagnetic theories, along similar lines.

For classical and special relativistic theories we make a large use of affine spaces.Thatis justified by the physical properties of event spaces. The main peculiarity of affine spaces are free vectors, that is a natural displacement of applied vectors. So we could treat the theory only in terms of free vectors,employng free derivatives Df and D^2f of maps f between affine spaces. But we have also to consider non affine entities, as submanifolds, general frames and coordinates. Then we use a mathematical formalism, which allows a view of affine spaces in terms of free or applied vectors, introducing tangent spaces, affine connection, etc.., hence considering affine spaces as special manifolds.

Affine spaces are the basic element that determines our intrinsical language, permitting a clear and deep distinction among absolute phenomena, frames of reference and coordinate systems.
Galilei's and Lorentz's maps turn out to be of little importance.
In fact these are implicit in the general framework and do not play any basic role in the following exposition. This point of view is upsetting

of current treatment and can influence physical theories based on group representation.

We try to unify in a unique context topics generally exposed in ordinary mechanics, in analytical mechanics, in continuum mechanics, in foundations, etc. Of course we limit ourselves only to a general introductive statement.

I N T R O D U C T I O N

Summary.

The general framework of classical mechanics is the absolute event space $\mathbb{E}$. It is an affine four dimensional space, representing the set of absolute events. A three dimensional subspace $\bar{\mathbb{S}}$ of its vector space $\bar{\mathbb{E}}$ is fixed to represent the set of couples of symultaneous events. Then $\mathbb{E}$ results into the disjoint union of parallel three dimensional affine subspaces $\mathbb{S}_\tau$, generated by $\bar{\mathbb{S}}$, which represent the equivalence classes of symultaneous events. The set $\mathbb{T}$ of these equivalence classes is a one dimensional affine oriented space, which represents absolute time.
The quotient projection $t : \mathbb{E} \to \mathbb{T}$ is the time function. The triple $(\mathbb{E},t,\mathbb{T})$ is an affine trivial bundle (but not canonically a product), whose fibers are the(not canonically isomorphic among themselves) equivalence classes $\mathbb{S}_\tau$. The map $Dt : \bar{\mathbb{E}} \to \mathbb{T}$ associates with each four vector u its absolute time component u°. We have not an absolute projection $\bar{\mathbb{E}} \to \bar{\mathbb{S}}$, or an absolute inclusion $\bar{\mathbb{T}} \hookrightarrow \bar{\mathbb{E}}$. Then we have not an absolute splitting $\bar{\mathbb{E}} = \bar{\mathbb{T}} \oplus \bar{\mathbb{S}}$ (whereas it is induced by a frame of reference). The inclusion $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$ admits the vertical (along $\bar{\mathbb{S}}$,i.e. at a fixed time) derivatives Df of maps defined on $\mathbb{E}$. On the vector space $\bar{\mathbb{S}}$ we have absolute euclidean metric $\check{g}$, defined up to a conformal factor, which describes the classical geometry. For pratical reasons we choose a unit of measure on $\bar{\mathbb{S}}$ and on $\bar{\mathbb{T}}$, selecting the conformal factors. The unit of measure on $\bar{\mathbb{T}}$ determines the identification $\bar{\mathbb{T}} \cong \mathbb{R}$. Then we get the subspace $\mathbb{U} \hookrightarrow \bar{\mathbb{E}}$, constituted by the vectors normalized by $\underline{t} \equiv Dt$, which represents the space of velocities. We define a Poincare's map as a map $G : \mathbb{E} \to \mathbb{E}$, which preserves the structure of $\bar{\mathbb{S}}$.

Its derivative $DG:\bar{\mathbb{E}} \to \bar{E}$ is the associated Galilei's map. At last, for the coordinate description, we define the special charts. These are characterized by the first coordinate function which is cartesian and depends only on time.

The formulation of the theory by means of applied vectors requires the definition of further spaces, namely the spaces of analytical mechanics. To first order, we consider the phase, or vertical, space $\check{T}\mathbb{E} \equiv \mathbb{E}\times\bar{\mathbb{S}} \hookrightarrow T\mathbb{E} \equiv \mathbb{E}\times\bar{\mathbb{E}}$ and the velocity, or unitary, space $\dot{T}\mathbb{E} \equiv \mathbb{E}\times\mathbb{U} \hookrightarrow T\mathbb{E}$, which are (not canonically) isomorphic. To second order, we consider the vertical spaces $\check{T}^2\mathbb{E} \equiv \mathbb{E}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E} \equiv \mathbb{E}\times\bar{\mathbb{E}}\times\bar{\mathbb{E}}\times\bar{\mathbb{E}}$ and $\nu\check{T}^2\mathbb{E} = \mathbb{E}\times\bar{\mathbb{S}}\times 0\times\bar{\mathbb{S}} \to \nu T^2\mathbb{E} \equiv \equiv \mathbb{E}\times\bar{\mathbb{E}}\times 0\times\bar{\mathbb{E}}$ and the bivelocity spaces $\dot{T}^2\mathbb{E} \equiv \mathbb{E}\times\mathbb{U}\times\bar{\mathbb{S}} \to T^2\mathbb{E}$ and $\nu\dot{T}^2\mathbb{E} \equiv \mathbb{E}\times\mathbb{U}\times 0\times\bar{\mathbb{S}} \to \nu T^2\mathbb{E}$, which play an important role in the connection properties of frames of reference, in the calculation of acceleration and in the Coriolis theorem. The natural projections $\Gamma: T^2\mathbb{E} \to \nu T^2\mathbb{E}$ and $\sqcup\!\sqcup: \nu T^2\mathbb{E} \to T\mathbb{E}$ permit the definition of the covarint derivative $\nabla_u v = \sqcup\!\sqcup \circ \Gamma \circ Tv \circ u$, which, after choosing a coordinate system, is expressed by the Christoffel symbols.

The absolute one-body world-line is a (one dimensional) submanifold $M \hookrightarrow \mathbb{E}$, which meets each $\mathbb{S}_\tau$ exactly at one point $M(\tau)$. The world-line is characterized by its absolute motion, that is by the associated map $M : \mathbb{T} \to \mathbb{E}$. The absolute free velocity and acceleration are the maps $DM : \mathbb{T} \to \mathbb{U}$ and $D^2M : \mathbb{T} \to \bar{\mathbb{S}}$. If we need to consider them in terms of applied vectors, we define the velocity and acceleration $dM \equiv (M, DM) : \mathbb{T} \to \dot{T}\mathbb{E}$ and $dM \equiv (M, D^2M) \equiv \sqcup\!\sqcup \circ \Gamma \circ d^2M : \mathbb{T} \to T\mathbb{E}$. After choosing coordinate system, the acceleration is expressed by the Christoffel symbols.

To determine positions, hence to get observed mechanics, we need frames of reference. A frame $\mathcal{P}$ is a continuum filling, in a C^∞ way, the whole event space.

First we can study the absolute kinematics of such a continuum. The continuum $\mathcal{P}$ is constituted by a set $\mathbb{P}$ of disjoint world-lines $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$. In this sense $\mathbb{P}$ can be viewed as the set of the particles of the continuum. For each event $e \in \mathbb{E}$, passes a unique particle $p(e) \in \mathbb{P}$. Then we get a surjective map $p : \mathbb{E} \to \mathbb{P}$. The set of the motions of the particles $\{P_q : \mathbb{T} \to \mathbb{T}_q \to \mathbb{E}\}_{q\in\mathbb{P}}$ determines the motion of the continuum, namely the map $P : \mathbb{T} \times \mathbb{P} \to \mathbb{E}$, which associates with (τ,q) the event touched at the time τ by the particle q. We can define the motion also by the map $\tilde{P} : \mathbb{T} \times \mathbb{E} \to \mathbb{E}$, which associates with (τ,e) the event touched at the time τ by the particle passing through e. Then we obtain a number of fields by deriving the motion. These fields can be expressed in the fundamental form $f:\mathbb{T}\times\mathbb{E} \to \mathbb{F}$, or in the eulerian form $f_\circ : \mathbb{E} \to \mathbb{F}$, or in the lagrangian form $f_{\circ\tau}: \mathbb{S}_\tau \to \mathbb{F}$, where $f(\tau,e)$ is attached to the event $\tilde{P}(\tau,e)$, while $f_\circ(e)$ and $f_\circ(e)$ are attached to e. The three formulations are equivalent, for we have $f_\circ(e) = f(t(e),e)$, $f_{\circ\tau} = f_\circ|_{\mathbb{S}_\tau}$, $f(\tau,e) = f_\circ(\tilde{P}(\tau,e))$. We can also consider two-points fields, for which fundamental, eulerian and lagrangian formu̲tions hold(but the relation among them, with respect to the second point is more complicated, for it involves the derivative of the motion).

Thus we consider the first and second time derivatives of the motion, defining the velocity and the acceleration of the frame $\bar{P} : \mathbb{E} \to \mathbb{U}$ and $\bar{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}$, where $\bar{P}(e)$ and $\bar{\bar{P}}(e)$ are the velocity and the acceleration, at the time $t(e)$, of the particle passing through e. Then we consider the first and second event derivatives of the motion $\hat{P}:\mathbb{E} \to \mathbb{E}^* \otimes \bar{\mathbb{S}}$ and $\hat{\hat{P}}:\mathbb{E} \to \mathbb{E}^* \otimes \mathbb{E}^* \otimes \bar{\mathbb{S}}$, which express the projection and the rate of projection of event intervals into simultaneous event intervals, due to the motion of the continuum.

The motion determines also a diffeomorphism between each couple of simultaneous spaces $\mathbb{S}_{\tau}$ and $\mathbb{S}_{\tau'}$. Then, by a first and second derivatives, we get the jacobian maps $\check{P}_{(\tau',\tau)}:\mathbb{S}_{\tau} \to \bar{\mathbb{S}}^{*}\otimes\bar{\mathbb{S}}$ and $\check{P}_{(\tau',\tau)}:\mathbb{S}_{\tau} \to \bar{\mathbb{S}}^{*}\otimes\bar{S}^{*}\otimes\bar{\mathbb{S}}$. The mixed derivative can be obtained considering the derivative, at a fixed time, of the velocity, namely $D\bar{P}:\mathbb{E} \to \bar{\mathbb{S}}^{*}\otimes\bar{\mathbb{S}}$, or considering the time derivative of $\tilde{P}_{(\tau',\tau)}$.

By means of the spatial metric, we can get the symmetrical and the antisymmetrical parts of the mixed tensor $\check{D}\bar{P}$. The symmetrical part $\epsilon_{\mathcal{P}} \equiv S\check{D}\bar{P}$ gives the rate of change, along the time, of the spatial metric tensor, induced by the motion. The antisymmetrical part

$\omega \equiv \frac{A}{2}\check{D}\bar{P}$, or the associated vector, by means of the Hodge isomorphism $(\check{*}\,\omega \equiv i_{\omega}\check{\eta}), \Omega \equiv \check{*}\,\omega$ represents the absolute angular velocity of the continuum motion.

Then we can consider the frame continuum as a frame of reference. First we define the observed positions. Each position is the set of all the events that touch a unique particle q, namely it is the set $\mathbb{T}_q$. Then we can identify the set of positions with the set of particles $\mathbb{P}$. Then $\mathbb{P}$ is a set of equivalence classes. This set $\mathbb{P}$ has a C^{∞} structure. At each time τ, $\mathbb{P}$ can be represented by the three dimensional affine space $\mathbb{S}_{\tau}$, associating with each position q the event touched at the time τ, by the particle q. Also for $T\mathbb{P}$ and $T^2\mathbb{P}$ we get two interesting representations. We can represent the tangent space $T\mathbb{P}$ as the quotient space $\check{T}\mathbb{E}_{/\mathcal{P}}$, namely as the set of strips of spatial vectors spanned (at the first order) by the motion.

An analogous representation holds for $T^2\mathbb{P}$. We can also represent $T\mathbb{P}$ and $T^2\mathbb{P}$ by $T\mathbb{S}_{\tau}$ and $T^2\mathbb{S}_{\tau}$, taking into account the bijection $\mathbb{P} \to \mathbb{S}_{\tau}$.

The latter representations induce a time-depending metric $g_{\mathcal{P}}: \mathbb{T} \times T\mathbb{P} \to \mathbb{R}$ and a time-depending connection $\check{\Gamma}_{\mathcal{P}}: \mathbb{T} \times T^2\mathbb{P} \to \nu T^2\mathbb{P}$. Taking into account the frame velocity, we can also represent $T\mathbb{P}$, at each time τ, with $\mathbb{S}_\tau \times \mathbb{U}$, hence $T^2\mathbb{P}$ with $\mathbb{S}_\tau \times \mathbb{U} \times \mathbb{U} \times \bar{\mathbb{S}}$. This representation induces a new map $\Gamma'_{\mathcal{P}}: \mathbb{T} \times T^2\mathbb{P} \to \nu T^2\mathbb{P}$, which is the sum of a map $C_{\mathcal{P}}: \mathbb{T} \times T\mathbb{P} \to T\mathbb{P}$ and of a map $D_{\mathcal{P}}: \mathbb{T} \times \mathbb{P} \to T\mathbb{P}$. The latter maps will be interpreted as the generalized Coriolis and the dragging accelerations.

Then we represent the absolute event space $\mathbb{E}$ by the frame-depending splitting into space and time $(t,p) : \mathbb{E} \to \mathbb{T} \times \mathbb{P}$, associating with each event e its absolute time $t(e)$ and its frame position $p(e)$. We get also the splitting $T_e\mathbb{E} \to \bar{\mathbb{T}} \times \bar{\mathbb{S}}$, which associates with each vector u, applied in e, its absolute time component u° and its frame spatial projection $\hat{P}(e)(u) = u - u^\circ \bar{P}(e)$. By means of Tp and T^2p we can associate with each point of $T\mathbb{E}$ and $T^2\mathbb{E}$ the relative observed quantities of $T\mathbb{P}$ and $T^2\mathbb{P}$.

Among all frames, some have a special interest for the peculiar properties of their motion and of the position spaces.
First we consider the affine frames, characterized by the fact that $\check{D}\bar{P}$ depends only on time. Their motion is determined by the motion of one of their particles and by its spatial derivative $\check{P}$, which depends only on time. The sum of strips representing the vectors of $T\mathbb{P}$ results to be independent on the position.
Then $\mathbb{P}$ results into an affine space, with the quotient induced by the motion $\bar{P} \equiv (\mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}}$ as vector space. The affine connection results to be time independent and we can write $\check{\Gamma}_{\mathcal{P}}: T^2\mathbb{P} \to \nu T^2\mathbb{P}$.

Then we consider rigid frames, which are affine frames such that $\epsilon_{\mathcal{P}}$ is zero. The spatial derivative of their motion is unitary. Their motion is determined by the motion of one of their particles and by its spatial unitary derivative(by a time derivation we obtain, from this fact, the classical formula for the velocities of the rigid frame). As the motion preserves, along the time, the spatial metric, $\mathbb{P}$ results into an affine euclidean space,namely $g_{\mathcal{P}}$ results time independent and we can write $g_{\mathcal{P}} : T\mathbb{P} \to \mathbb{R}$. Then we consider translating frames, which are rigid frames such that $\Omega_{\mathcal{P}}$ is zero. The spatial derivative of their motion is zero. Their motion is determined by the motion of one of their particles. As the motion preserves, along the time, the spatial vectors, the vector space $\bar{\mathbb{P}}$ results to be equal to $\bar{\mathbb{S}}$. Finally we consider inertial frames, which are translating frames such that $\bar{\bar{P}}$ is zero. The total derivative of their motion is zero. Their motion is determined by the inertial motion of one of their particles. The projection $\hat{P}$ is constant, hence $\check{\Gamma}_P = \Gamma'_P$.

Now we consider a fixed frame $\mathcal{P}$ and a fixed motion M, we define the quantities of M observed by $\mathcal{P}$ and we make a comparison between absolute and observed quantities. The observed motion is the map $M_{\mathcal{P}}:\mathbb{T} \to \mathbb{P}$, which associates with each time $\tau \in \mathbb{T}$ the position $p(M(\tau))$ touched by M at that time. The observed motion $M_{\mathcal{P}}$ characterizes the absolute motion M, since $M(\tau) = \tilde{P}(\tau, M_{\mathcal{P}},(\tau))$. Then we get the velocity of the observed motion $dM_{\mathcal{P}} : \mathbb{T} \to T\mathbb{P}$, which is the derivative of $M_{\mathcal{P}}$ performed by $\mathcal{P}$, by means of its differentiable structure. We get also the acceleration of the observed motion $\check{\nabla}_{\mathcal{P}} \check{d}M_{\mathcal{P}}:\mathbb{T} \to T\mathbb{P}$, which is the covariant derivative of the velocity of the observed motion, per formed by $\mathcal{P}$, by means of its time depending affine connection $\check{\Gamma}_{\mathcal{P}}$.

The observed velocity of M is the projection on $T\mathbb{P}$ of the velocity dM. The observed velocity and the velocity of the observed motion are equal.

The observed acceleration is the proiection of the sum of the acceleration of the observed motion and of a generalized Coriolis term, plus a dragging term. Namely we can write $D^2M = D^2_{\mathcal{P}}M_{\mathcal{P}}+C_{\mathcal{P}}(D_{\mathcal{P}}M_{\mathcal{P}})+D_{\mathcal{P}}\circ(M_{\mathcal{P}})$.

Finally we consider two frames $\mathcal{P}_1$ and $\mathcal{P}_2$ and a motion M and we make a comparison among quantities observed by $\mathcal{P}_1$ and by $\mathcal{P}_2$. First we consider the quantities of $\mathcal{P}_1$ observed by $\mathcal{P}_2$. Then we find the addition velocities theorem and the generalized Coriolis Theorem. Specializing the kind of frame of references, we get the usual theorems.

Comparison with special and general relativity.

We want to show some surprising and important analogies with the special and general relativity, not involving the light velocity.

In both cases we have a four dimensional event space $\mathbb{E}$, which is affine in the classical and special relativistic case and which has not an absolute splitting into space and time. In the classical case we have a privileged three dimensional subspace $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$, which determines the absolute simultaneity and the absolute time as the quotient space $\mathbb{T}\equiv\mathbb{E}_{/\bar{\mathbb{S}}}$. These facts have not an absolute relativistic counterpart. On the other hand, in the relativistic case we have a Lorentz metric on the whole $T\mathbb{E}$ (it is constant and fixed in special relativity and it is matter depending in general relativity), while in the classical case we have only an euclidean metric on $T\mathbb{E}$. The relativistical Lorentz metric determines, by orthogonality, a frame depending and pointwise (local if the frame is integrable) spatial section which replaces the classical $\bar{\mathbb{S}}$. The classical time orientation is given on $\mathbb{T}$, while the relativistic one is given on the light cone.

In all the three cases we can describe a motion (or its world-line) by a four dimensional map $M : \mathbb{T} \to \mathbb{E}$ (or by a one dimensional submanifold $\mathbb{M} \hookrightarrow \mathbb{E}$), which is absolute, i.e. not depending on any frame of reference. In the relativistic case the condition that $\mathbb{M}$ is time-like replaces the classical condition $t \circ M = id_{\mathbb{T}}$. In the relativistic case $\mathbb{T}$ is not the absolute time, but the proper time of the motion, namely it is $\mathbb{M}$ itself endowed with the affine euclidean structure induced by the metric of $\mathbb{E}$. In the classical case we get $<\underline{t},DM> = 1$ and $<\underline{t},D^2M>=0$.

These conditions are replaced in the relativistic case by $DM^2=-1$ and $DM \circ D^2M = 0$.

In all three cases we can consider the most general kind of frame of reference,while people often consider only rigid frames in classical mechanics and inertial frames in special relativistic. The definition of frame is essentially the same in all three cases: the only differences come from the implicit differences in the definition of the world lines of the continuum particles. Analogous considerations hold for the representation of $\mathbb{P}$, $T\mathbb{P}$ and $T^2\mathbb{P}$, for the time depending metric and connection, the Coriolis and dragging maps and the classification of special frames.

Since we deal with general frames of reference, we get for classical and special relativistic observed kinematics criteria currently used in general relativity. In fact under our statment of the absolute Coriolis Theorem we can recognize usual general relativistic formulas, commonly quoted in other form.

I CHAPTER
ABSOLUTE KINEMATICS

In this paper we study the general event framework constituted by the event space, its partition into the symultaneity spaces, which generate the time and the spetial metric.

We analyse some remarkable spaces and maps connected with the previous ones. Finally we study the one-body absolute motion, velocity and acceleration. All these elements are considered regardless of any frame of reference.

1 - THE EVENT SPACE

First we introduce the general framework for classical mechanics. Event space, simultaneity, spatial metric, future orientation, time.

1 - Basic assumptions on primitive elements of our theory are given by the following definition, which constitutes the framework of classical mechanics.

DEFINITION.

The CLASSICAL EVENT FRAMEWORK is a 4-plet

$$\mathcal{E} \equiv \{E, \bar{\mathbb{S}}, \check{\mathbb{G}}, O\}$$

where

$E \equiv \{\mathbb{E}, \bar{\mathbb{E}}, \sigma\}$ is an affine space, with dimension 4;

$\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$ is a subspace of $\bar{\mathbb{E}}$, with dimension 3;

$\check{\mathbb{G}}$ is a conformal euclidean metric on $\bar{\mathbb{S}}$;

O is an orientation on the quotient space $\mathbb{E}/\bar{\mathbb{S}}$.

$\mathbb{E}$ is the EVENT SPACE; $\bar{\mathbb{E}}$ is the EVENT INTERVAL SPACE;

$\bar{\mathbb{S}}$ is the SIMULTANEOUS INTERVAL SPACE or the SPATIAL INTERVAL SPACE;

$\check{\mathbb{G}}$ is the SPATIAL CONFORMAL METRIC;

O is the FUTURE ORIENTATION,

$-O$ is the PAST ORIENTATION.

Henceforth we assume a classical event framework $\mathcal{E}$ to be given.

2 - The previous definition contains implicitly the notion of absolute time, which we are now giving explicitly .

DEFINITION.

The TIME SPACE is the quotient space

$$T \equiv \mathbb{E} / \bar{\mathbb{S}} .$$

The TIME VECTOR SPACE is the quotient space

$$\bar{\mathbf{T}} \equiv \bar{\mathbf{E}} / \bar{\mathbf{S}} .$$

The TIME PROJECTION is the quotient map

$$t : \mathbf{E} \to \mathbf{T} .$$

The SPACE AT THE TIME $\tau \in \mathbf{T}$ is the subspace

$$\mathbf{S}_\tau \equiv t^{-1}(\tau) \hookrightarrow \mathbf{E}.$$

The TIME BUNDLE is the 3-plet

$$\eta \equiv (\mathbf{E},t,\mathbf{T}) .$$

Hence, each equivalence class is of the type

$$\mathbf{T} \ni \tau \equiv [e] \equiv e + \bar{\mathbf{S}} \equiv \mathbf{S}_\tau \to \mathbf{E}$$

having $$t(e) \equiv \tau.$$

Thus τ and $\mathbf{S}_\tau$ coincide, but τ is viewed as a point of $\mathbf{T}$ and $\mathbf{S}_\tau$ as a subset of $\mathbf{E}$.

Moreover we will denote by j the injective map

$$j \equiv (t,id_{\mathbf{E}}) : \mathbf{E} \to \mathbf{T} \times \mathbf{E}.$$

3 - We get immediate properties for the previous spaces.

PROPOSITION.

a) $(\mathbf{T},\bar{\mathbf{T}})$ results naturally into an affine 1-dimensional oriented space.

b) t is an affine surjective map. We get

$$\bar{\mathbf{S}} = (D\, t)^{-1}(0).$$

c) For each $\tau \in \mathbf{T}$, $(\mathbf{S}_\tau, \bar{\mathbf{S}}, \sigma)$ is an affine 3-dimensional subspace of $\mathbf{E}$;

hence $\{\mathbf{S}_\tau\}_{\tau \in \mathbf{T}}$ is a family of parallel,(not canonically) isomorphic affine subspace of $\mathbf{E}$ and we have

$$\mathbf{E} = \bigsqcup_{\tau\in\mathbf{T}} \mathbf{S}_\tau .$$

d) η is an affine, (not canonically) trivial bundle.

4 - We have the absolute time component of an event interval.

DEFINITION.

The TIME COMPONENT of the vector $u\in\bar{\mathbf{E}}$ is

$$u^\circ \equiv \langle Dt,u\rangle \in \mathbf{T}.$$

u is FUTURE ORIENTED or PAST ORIENTED, according as

$$u^\circ\in\bar{\mathbf{T}}^+ \qquad \text{or} \qquad u \in \bar{\mathbf{T}}^- .$$

Moreover u is spatial if and only if $u^\circ = 0$.

5 - Thus, the equence

$$0 \to \bar{\mathbf{S}} \longrightarrow \bar{\mathbf{E}} \twoheadrightarrow \bar{\mathbf{T}} \to 0$$

is exact, but we have not a canonical splitting of $\bar{\mathbf{E}}$, as we have not a canonical projection $\bar{\mathbf{E}} \twoheadrightarrow \bar{\mathbf{S}}$, or a canonical inclusion $\mathbf{T} \hookrightarrow \bar{\mathbf{E}}$.
However, each vector $v\in\mathbf{E}$, such that $\langle Dt,v\rangle \neq 0$, determines a splitting of $\bar{\mathbf{E}}$.
Namely we get the inclusion

$$\bar{\mathbf{T}} \hookrightarrow \bar{\mathbf{E}},$$

given by

$$\lambda \mapsto \frac{\lambda}{v^\circ}\, v,$$

and the projection

$$\overset{\perp}{p}_v : \bar{\mathbf{E}} \to \bar{\mathbf{S}} ,$$

given by

$$u \mapsto u - \frac{u^\circ}{v^\circ}\, v,$$

which determine the decomposition in the direct sum

$$\bar{\mathbb{E}} = \bar{\mathbb{T}} \oplus \bar{\mathbb{S}} ,$$

given by
$$u = \frac{u^\circ}{v^\circ} v + (u - \frac{u^\circ}{v^\circ} v) \equiv p_v^{\shortparallel}(u) + p_v^{\perp}(u) .$$

6 - According to the bundle structure of $\mathbb{E}$ on $\mathbb{T}$, we can define the vertical derivative of maps, i.e. the derivative along the fibers. Generally we will denote by "v" the quantities connected with η .

DEFINITION.

Let $\mathbb{F}$ be an affine space and let $f : \mathbb{E} \to \mathbb{F}$ be a C^∞ map.

The VERTICAL DERIVATIVE of f is the map

$$\check{D}f \equiv Df_{|\bar{\mathbb{S}}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{F}} \; .$$

Poincaré's and Galilei's maps .

7 - A Poincaré's map is a map $\mathbb{E} \to \mathbb{E}$ which preserves the structure of $\mathcal{E}$ and the associated Galilei's map is its derivative.

DEFINITION.

A POINCARÉ'S MAP is an affine map

$$G : \mathbb{E} \to \mathbb{E},$$

such that

a) $DG(\bar{\mathbb{S}}) = \bar{\mathbb{S}}$

b) $DG \in U(\bar{\mathbb{S}})$,

c) if $G^\circ : \mathbb{T} \to \mathbb{T}$ is the induced map on the quotient space $\mathbb{T} \equiv \mathbb{E}/\bar{\mathbb{S}}$,

then $$DG^\circ = id_{\bar{\mathbb{T}}} .$$

$DG : \bar{\mathbb{E}} \to \bar{\mathbb{E}}$ is the GALILEI'S MAP associated with G.

G is SPECIAL if it preserves the orientations of $\bar{\mathbb{E}}$ and $\bar{\mathbb{S}}$ (hence of $\bar{\mathbb{T}}$) .

8 - PROPOSITION.

Each Poincaré's map G in bijective .

PROOF.

It follows from $DG \in U(\bar{\mathbb{S}})$, $DG^\circ = id_{\bar{\mathbb{T}}}$.

Space and time measure unity.

9 - We have assumed a 1-parameter family $\check{\mathbb{G}}$ of euclidean metrics on $\bar{\mathbb{S}}$. A 1-parameter family $\mathbb{G}^\circ$ of euclidean metrics on $\bar{\mathbb{T}}$ is given a priori, for dim $\bar{\mathbb{T}} = 1$.

An arbitrary choice of one among these makes important simplications in the following.

DEFINITION.

A SPATIAL MEASURE UNITY is a metric $\check{g} \in \check{\mathbb{G}}$.

A TIME MEASURE UNITY is a metric $g^\circ \in \mathbb{G}^\circ$.

The choice of a spatial measure unity $\check{g}$ is equivalent to the choice of the sphere (in the family determined by $\check{\mathbb{G}}$) of $\bar{\mathbb{S}}$, with radius 1 as measured by $\check{g}$.

The choice of a time measure unity g° is equivalent to the choice of the vector

$$\lambda^\circ \in \mathbb{T}^+$$

such that

$$g^\circ(\lambda^\circ, \lambda^\circ) = 1 .$$

Then λ° determines the isomorphism

$$\bar{\mathbb{T}} \to \mathbb{R}$$

given by

$$\lambda \mapsto \frac{\lambda}{\lambda^\circ} .$$

Henceforth we assume a spatial and a time measure unity to be given. Hence we get the identification

$$\bar{\mathbb{T}} \cong \mathbb{R}$$

and the consequent identifications

$$L(\bar{\mathbb{T}}, \bar{\mathbb{E}}) \cong \bar{E}\ ,\ L(\mathbb{T}, \bar{\mathbb{S}}) \cong \bar{\mathbb{S}}\ ,\ L(\bar{\mathbb{E}},\bar{\mathbb{T}}) \cong \bar{\mathbb{E}}^*,\ L(\bar{\mathbb{S}},\bar{\mathbb{T}}) \cong \mathbb{S}^*,\ \ldots$$

In this way, the map $Dt \in L(\bar{\mathbb{E}},\bar{\mathbb{T}})$ is identified with the form

$$\underline{t} \cong Dt \in \bar{\mathbb{E}}^* .$$

10 - Besides the subspace $\bar{\mathbb{S}} \hookrightarrow \bar{\mathbb{E}}$, which results into $\bar{\mathbb{S}} = \underline{t}^{-1}(0)$, an interesting will be played by the subspace of normalized vectors $\underline{t}^{-1}(1)$.

DEFINITION.

The FREE VELOCITY SPACE is

$$\mathbb{U} \equiv \underline{t}^{-1}(1) \hookrightarrow \mathbb{E}.$$

11 - PROPOSITION.

$(\mathbb{U}, \bar{\mathbb{S}})$ results naturally into an affine (not vector) 3-dimensional subspace of $\bar{\mathbb{E}}$.

Of course $\mathbb{U}$ and $\bar{\mathbb{S}}$ are isomorphic as affine spaces, but we have not a canonical affine isomorphism between $\mathbb{U}$ and $\bar{\mathbb{S}}$.

Special charts.

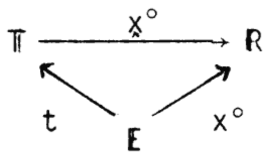
12 - In calculations can be usefull a numerical representation of $\mathbb{E}$, which takes into account its time structure. For simplicity of notations, we consider only diffeomorphisms $\mathbb{E} \to \mathbb{R}^4$, leaving to the reader the obvious generalization to local charts, our considerations being essentially local.

DEFINITION.

A SPECIAL CHART is a C^∞ chart

$$x \equiv \{x^0, x^i\} : \mathbb{E} \to \mathbb{R} \times \mathbb{R}^3 ,$$

such that x^0 factorizes as follows



where $\hat{x}^\circ : \mathbb{T} \to \mathbb{R}$ is a normal oriented cartesian map .

Naturally $\hat{x}^\circ$(hence x°) is determined up an initial time.

We make the usual convention

$$\alpha, \beta, \lambda, \mu, \ldots = 0,1,2,3 \qquad \text{and} \qquad i,j,h,k, \ldots = 1,2,3 \, .$$

We assume in the following a special chart x to be given.

13 - Let us give the coordinate expression of some important quantities.

PROPOSITION.

We have $\qquad Dx^\circ = \underline{t}$;

$\delta x_i : \mathbb{E} \to \bar{\mathbb{S}}$;

if $u \in \bar{\mathbb{E}}$, then $u = u^\circ \, \delta x_o + u^i \, \delta x_i$, where $u^\circ \equiv \langle \underline{t}, u \rangle$;

$$\check{g} = g_{ij} \, \check{D}x^i \otimes \check{D}x^j \, ;$$

$$\Gamma^\circ_{\alpha\beta} \equiv D\delta x_\alpha(\delta x_\beta, Dx^\circ) = - D^2 x^\circ(\delta x_\alpha, \delta x_\beta) = 0,$$

$$\Gamma^k_{ij} \equiv D\delta x_i(\delta x_j, Dx^k) = - D^2 x^k(\delta x_i, \delta x_j) =$$

$$= \frac{1}{2} g^{kh}(\partial_i g_{hj} + \partial_j g_{hi} - \partial_h g_{ij}),$$

$$\Gamma_{i,oj} + \Gamma_{j,oi} = \partial_o g_{ij} \, , \qquad \text{where} \quad \Gamma_{h,\alpha\beta} \equiv g_{hi} \Gamma^i{}_{\alpha\beta}$$.

Moreover $\quad \Gamma^k_{oj} \equiv D\delta x_o(\delta x_j, Dx^k) = - D^2 x^k(\delta x_o, \delta x_j)$

and $\quad \Gamma^k_{oo} \equiv D\delta x_o(\delta x_o, Dx^k) = - D^2 x^k(\delta x_o, \delta x_o)$

can be different from zero, if δx_o is not constant.

Notice that $Dx^\circ = \underline{t}$ is fixed a priori and that the unique conditions imposed a priori on δx_α are

$$<\underline{t},\delta x_0> = 1 \qquad\qquad <\underline{t},\delta x_i> = 0 \ .$$

Physical description.

The event space $\mathbb{E}$ represents the set of all the possible events considered from the point of view of their mutual space-time collocation and without reference to any particular frame of reference. This space $\mathbb{E}$ must be viewed exactl in the same sence as the event space of Special and General Theory of Relativity.

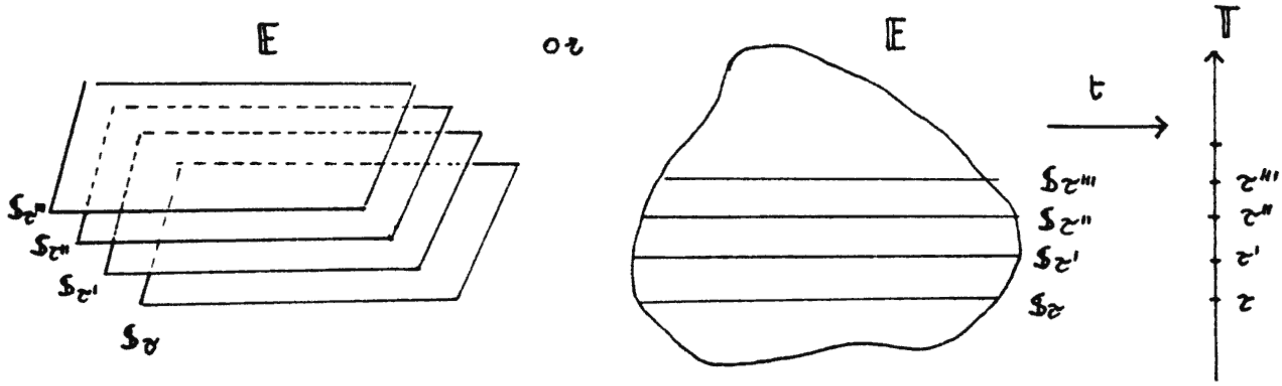
The event space $\mathbb{E}$ is the disjoint union of a family $\{\mathbb{S}_\tau\}_{\tau\in\mathbb{T}}$ of three dimensional affine euclidean, mutually diffeomorphic, spaces. This partition represents the equivalence relation of absolute simultaneity among events. The structure of each space $\mathbb{S}_\tau$ permits all the physical operations considered in the classical time-independent Euclidean Geometry, as stright lines, parallelism, intervals,sum of intervals, by the parallelogram rule, circles, etc. We have not selected a priory a spatial measure unity, for it is not physically significant: by means of rigid rods we can only find ratios between lenghts is all directions and the choice of a particular interval of a rigid rod is a useful but not necessary convenction

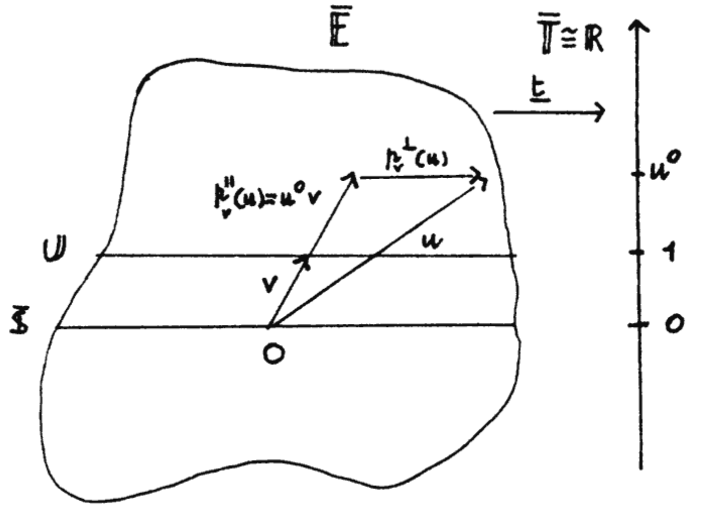
The symultaneity spaces $\mathbb{S}_\tau$ are mutually ,but not canonically,isomorphic, for a particular family of bijections among these leads to a determination of positions, i.e. to a frame of reference, which we have excluded in the general context. Notice that in $\mathbb{S}_\tau$ we have not privileged points or axes.

The required four dimensional affine structure of $\mathbb{E}$ leads to the affine structures of the subspaces $\mathbb{S}_\tau$ and to the one dimensional affine structure of the set $\mathbb{T}$, whose points are the equivalence classes $\mathbb{S}_\tau$.

This space represents the classical absolute time. Its affine structure admits the time intervals, independent of an initial time, and their sum. The one dimensional affine structure of $\mathbb{T}$ leads also to the measure of time intervals with respect to an arbitrary chosen unity. Hence the affine structure of $\mathbb{E}$ contains implicitly the idea of "goodclocks".
The dimension one describes also the total ordinability of times and the assumed orientation o describes the future orientation.
Notice that in $\mathbb{T}$ we have not a privileged initial time.

To make more evident the described properties of event framework, we can make some pictures using the affine euclidean structure of the paper. We must take care essentially in two things: we must neglect two (or one) dimension of $\mathbb{E}$ and we must partially neglect the euclidean structure of the paper, for we have not a measure of angles between spatial and time vectors. So a time vector orthogonal to a spatial vector is nonsence.





2 - FURTHER SPACES AND MAPS.

Now we introduce some further notions concerning applied vector spaces and maps.

Vertical and unitary spaces.

1 We introduce the spaces of applied vectors relative to $\bar{\mathbb{S}}$, and $\mathbb{U}$.

DEFINITION.

The VERTICAL SPACE WITH RESPECT TO $(\mathbb{E},t,\mathbb{T})$, or the PHASE SPACE, or the ACCELERATION SPACE, is

$$\mathbb{A} \equiv \check{T}\,\mathbb{E} \equiv \mathrm{Ker}\quad Tt \;=\mathbb{E} \times \bar{\mathbb{S}} \hookrightarrow T\,\mathbb{E}\ .$$

The HORIZONTAL SPACE WITH, RESPECT TO $(\mathbb{E},t,T)$ is

$$\mathring{T}\,\mathbb{E} \equiv T\mathbb{E}_{/\check{T}\mathbb{E}} = \mathbb{E} \times \bar{\mathbb{T}}\ .$$

The UNITARY SPACE, or the VELOCITY SPACE, is

$$\mathbb{V} \equiv \overset{\shortmid}{T}\mathbb{E} \equiv (Tt)^{-1}(\mathbb{T}\times 1) = \mathbb{E} \times \mathbb{U} \hookrightarrow \mathbb{E}\ .$$

2 Let us remember that $T\mathbb{E}$ has two bundle structures, namely

$$(T\mathbb{E},Tt,T\,\mathbb{T}) \qquad \text{and} \qquad (T\mathbb{E},\ \pi_{\mathbb{E}}\ ,\ \mathbb{E})\ .$$

PROPOSITION.

a) $\check{T}\mathbb{E}$ is the submanifold of $T\mathbb{E}$ characterized by $\dot{x}^{\circ} = 0$

$\overset{\shortmid}{T}\mathbb{E}$ is the submanifold of $T\mathbb{E}$ characterized by $\dot{x}^{\circ} = 1$.

b) $\check{T}\mathbb{E}$ and $\overset{\shortmid}{T}\mathbb{E}$ have two natural bundle structures, namely

$$(\check{T}\mathbb{E},\check{t},T) \qquad \text{and} \qquad (\check{T}\mathbb{E},\check{\pi}_{\mathbb{E}},\mathbb{E})\ .$$

$$(\overset{|}{T}\mathbb{E}, t, \mathbb{T}) \qquad \text{and} \qquad (T\mathbb{E}, \pi_{\mathbb{E}}, \mathbb{E}).$$

c) The sequence $\quad 0 \to \acute{T}\mathbb{E} \hookrightarrow T\mathbb{E} \to T\mathbb{E} \to 0 \quad$ is exact .

We have not a canonical splitting of $T\mathbb{E}$, as we have not a canonical projection $T\mathbb{E} \to T\mathbb{E}$, or a canonical inclusion $T\mathbb{E} \hookrightarrow T\mathbb{E}$.

In the same way we have not a canonical isomorphism $\acute{T}\mathbb{E} \to \overset{|}{T}\mathbb{E}$.

3 We can extend the vertical derivative in terms of applied vectors.

DEFINITION.

Let F be a C^∞ manifold and $f : \mathbb{E} \to \mathbb{F}$ a C^∞ map.

The VERTICAL TANGENT MAP of f, WITH RESPECT TO $(\mathbb{E}, t, \mathbb{T})$, is the map

$$\acute{T}f \equiv Tf_{\acute{T}\mathbb{E}} : \acute{T}\mathbb{E} \to T\mathbb{E} \ .$$

4 We can view the metric as a function on $\check{T}\mathbb{E}$, which will become the kinetic energy in dynamics.

DEFINITION.

The METRIC FUNCTION is the funtion

$$\check{g} : \acute{T}\mathbb{E} \to \mathbb{R},$$

given by $\quad (e,u) \mapsto \frac{1}{2} u^2 \quad$.

5 PROPOSITION.

We have $\quad \check{g} = \frac{1}{2} \check{g}_{ij} \dot{x}^i \dot{x}^j \quad$.

<u>Second order spaces, affine connection and canonical projection.</u>

6 We consider now the second order tangent spaces.

DEFINITION.

The VERTICAL SPACE, WITH RESPECT TO $(\check{T}\mathbb{E},\check{t},\mathbb{T})$, is

$$\check{T}^2\mathbb{E} \equiv \operatorname{Ker} \; T\check{t} = \mathbb{E}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E} .$$

The VERTICAL SPACE, WITH RESPECT TO $(\grave{T}\mathbb{E},\grave{t},\mathbb{T})$ and $(\grave{T}\mathbb{E},\pi_{\mathbb{E}},\mathbb{E})$, is

$$\nu\check{T}^2\mathbb{E} \equiv \operatorname{Ker} \; T\grave{t} \quad \operatorname{Ker} \; T\pi_{\mathbb{E}} = \mathbb{E}\times\bar{\mathbb{S}}\times 0\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E}.$$

The BIUNITARY SPACE or BIVELOCITY SPACE, is

$$\mathbb{V}^2 \equiv \acute{T}\mathbb{E} \equiv sT\acute{T}\mathbb{E} \equiv \mathbb{E}\times\mathbb{U}\times\bar{\mathbb{S}} \xhookrightarrow{\text{diagonal}} \mathbb{E}\times\mathbb{U}\times\mathbb{U}\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E} .$$

The VERTICAL BIUNITARY SPACE, WITH RESPECT TO $(\acute{T}\mathbb{E},\acute{\pi}_{\mathbb{E}},\mathbb{E})$, is

$$\nu\mathbb{V}^2 \equiv \nu\acute{T}^2\mathbb{E} \equiv \nu T\acute{T}\mathbb{E} = \mathbb{E}\times\mathbb{U}\times 0\times\bar{\mathbb{S}} \hookrightarrow T^2\mathbb{E} .$$

7 PROPOSITION.

$\check{T}^2\mathbb{E}$ is the submanifold of $T^2\mathbb{E}$ characterized by $\check{\dot{x}}^0=\dot{\check{x}}^0 = \ddot{x}^0 = 0$.

$\nu\check{T}^2\mathbb{E}$ " " " " " " " $\check{\dot{x}}^0=\dot{\check{x}}^\alpha = \ddot{x}^\alpha = 0$.

$\acute{T}^2\mathbb{E}$ " " " " " " " $\check{\dot{x}}^0=\dot{\check{x}}^0=1,\check{\dot{x}}^i=\dot{\check{x}}^i,\ddot{x}^0= 0$.

$\nu\acute{T}^2\mathbb{E}$ " " " " " " " $\check{\dot{x}}^0= 1,\dot{\check{x}}^\alpha=0,\ddot{x}^0 = 0$.

8 Let us consider some important canonical maps, which are used to define the covariant derivatives.

DEFINITION.

a) The AFFINE CONNECTION MAP

$$\Gamma : T^2\mathbb{E} \to \nu T^2\mathbb{E} ,$$

given by $(e,u,v,w) \mapsto (e,u,o,w)$,

induces naturally the maps

$$\check{\Gamma} : \check{T}^2\mathbb{E} \to \nu \check{T}^2\mathbb{E}$$

and
$$\overset{\shortmid}{\Gamma} : \overset{\shortmid}{T}^2\mathbb{E} \to \nu \overset{\shortmid}{T}^2\mathbb{E} .$$

b) The CANONICAL PROJECTION (which is an isomorphism on fibers).

$$\sqcup : \nu T^2\mathbb{E} \to T\mathbb{E} ,$$

given by $(e,u,o,w) \mapsto (e,w)$,

induces naturally the maps

$$\check{\sqcup} : \nu \check{T}^2\mathbb{E} \to \check{T}\mathbb{E}$$

and
$$\overset{\shortmid}{\sqcup} : \nu \overset{\shortmid}{T}^2\mathbb{E} \to \check{T}\mathbb{E} \quad \underline{\cdot}$$

9 PROPOSITION.

We have
$$\begin{cases} \check{\ddot{x}}^{\alpha} \circ \Gamma = \ddot{\check{x}}^{\alpha} \\ \check{\dot{x}}^{\alpha} \circ \Gamma = \dot{\check{x}}^{\alpha} \\ \dot{\check{\dot{x}}}^{\alpha} \circ \Gamma = 0 \\ \ddot{x}^{0} \circ \Gamma = \ddot{x}^{0} \; ; \; \ddot{x}^{k} \circ \Gamma = \ddot{x}^{k} + \check{\dot{\Gamma}}^{k}_{\alpha\beta} \, \bar{\dot{x}}^{\alpha} \, \dot{\breve{x}}^{\beta} . \end{cases}$$

We have
$$\begin{cases} \check{x}^{\alpha} \circ \sqcup = \check{\ddot{x}}^{\alpha} \\ \dot{x}^{\alpha} \circ \sqcup = \ddot{x}^{\alpha} \end{cases} \quad \underline{\cdot}$$

10 Then we can introduce the covariant derivative in a way that, not making an essential use of free vectors, can be extended to manifolds.

DEFINITION.

Let $u \equiv (id_{\mathbb{E}}, \breve{u}) : \mathbb{E} \to T\mathbb{E}$ and $v \equiv (id_{\mathbb{E}}, \breve{v}) : \mathbb{E} \to T\mathbb{E}$ be C^{∞} vector fields.

The COVARIANT DERIVATIVE of v with respect to u is

$$\nabla_u v \equiv \coprod \circ \Gamma \circ T v \circ u = (id_{\mathbb{E}}, D\tilde{u}(\tilde{v})) : E \to T\mathbb{E} \quad .$$

3 - ASSOLUTE KINEMATICS.

Here we introduce the basic elements of one-body kinematics independent of any frame of reference.

Absolute world-line and motion.

1 The basic definition of kinematics is the following. Here we consider a C^∞ world-line extending along the whole $\mathbb{T}$. We leave to the reader the easy generalization to the case when it is C^2 almost every where, or when it extends along an interval of $\mathbb{T}$.

DEFINITION.

A WORLD-LINE is a connected C^∞ submanifold

$$\mathbb{M} \hookrightarrow \mathbb{E}$$

such that $\mathbb{S}_\tau \cap \mathbb{M}$ is a singleton, $\forall \tau \in \mathbb{T}$.

The MOTION, RELATIVE TO THE WORLD LINE $\mathbb{M}$, is the map

$$M : \mathbb{T} \to \mathbb{E}$$

given by $\tau \mapsto$ the unique element $\in \mathbb{S}_\tau \cap \mathbb{M}$.

Henceforth in this section we suppose a world-line $\mathbb{M}$, or its motion M, to be given.

2 PROPOSITION.

$\mathbb{M}$ is an ambedded 1-dimensional submanifold of $\mathbb{E}$, diffeomorphic to $\mathbb{R}$.

M is a section of $(\mathbb{E},t,\mathbb{T})$, namely it is a C^∞ embending, such that

$$t \circ M = id_{\mathbb{T}} ,$$

i.e. such that

$$M^\circ \equiv x^\circ \circ M = \hat{x}^\circ .$$

Hence the map

$$M : T \to \mathbb{M} \qquad \text{is a } C^\infty \text{ diffeomorphism} .$$

The world line $\mathbb{M}$ is characterized by its motion M .

3 The affine structures of $\mathbb{T}$ and $\mathbb{E}$ admit a Kind of privileged world-lines.

DEFINITION.

$\mathbb{M}$ is INERTIAL if it is an affine subspace of $\mathbb{E}$

4 PROPOSITION.

$\mathbb{M}$ is inertial if and only if M is an affine map, i.e.

$$M(\tau') = M(\tau) + DM(\tau'-\tau), \quad \text{with} \quad DM \in U .$$

Absolute velocity and acceleration.

5 Previously we introduce useful notations .

a) Let $\mathbb{F}$ be an affine space and let

$$\phi : \mathbb{T} \to \mathbb{F}$$

be a C^{∞} map.

Then we put $\qquad d\phi \equiv (\phi, D\phi) : \mathbb{T} \to T\mathbb{F}$.

In particular, if $\qquad \phi \equiv f : \mathbb{T} \to \mathbb{E}$,

we get $\qquad df \equiv (f, Df) : \mathbb{T} \to T\mathbb{E}$

and $\qquad d^2 f \equiv (f, Df, Df, D^2 f) : \mathbb{T} \to T^2\mathbb{E}$.

b) We put $\qquad \nabla df \equiv \coprod \circ \Gamma \circ d^2 f = (f, D^2 f) : \mathbb{T} \to T\mathbb{E}$.

The coordinate expressions are

$$df = Df^{\alpha}(\partial x_{\alpha} \circ f)$$

$$d^2 f = Df^{\alpha}(\partial x_{\alpha} \circ df) + D^2 f^{\alpha}(\partial \dot{x}_{\alpha} \circ df)$$

$$\nabla df = (D^2 f^{\alpha} + (\Gamma^{\alpha}_{\beta\gamma} \circ f) Df^{\beta} Df^{\gamma})(\partial x_{\alpha} \circ f) \ .$$

6 We can view the absolute velocity in terms of free or of applied vectors, equivalently.

DEFINITION.

The FREE VELOCITY of M is the map

$$DM : \mathbb{T} \to \bar{\mathbb{E}} \ .$$

The VELOCITY of M is the map

$$d\, M \equiv (M,\, D\, M) : \mathbb{T} \to T\mathbb{E} \quad .$$

7 PROPOSITION.

We have $$< \underline{t}\, ,\, D\, M > = 1 \; . \qquad (*)$$

Hence, we can write

$$D\, M : T \to \mathbb{U}$$

and $$d\, M : \mathbb{T} \to \mathbb{V} \equiv \overset{\prime}{T}\mathbb{E}$$

and we get $$D\, M^{\circ} = 1$$

$$D\, M = \delta x_{o} \circ M + D\, M^{k}(\delta x_{k} \circ M)$$

$$d\, M = \partial x_{o} \circ M + D\, M^{k}(\partial x_{k} \circ M) \; .$$

PROOF. (*) follows from $t \circ M = id_{\mathbb{T}}$.

8 We can view the absolute acceleration in terms of free or of applied vectors, equivalently and second order tangent space may intervene esplicitly or not .

DEFINITION.

The FREE ACCELERATION of M is the map

$$D^{2}M : \mathbb{T} \to \bar{\mathbb{E}} \; .$$

The LIFTED ACCELERATION of M is the map

$$\Gamma \circ d^{2}M = (M, DM \; ; \; 0,\, D^{2}M) : \mathbb{T} \to \curlyvee T^{2}\mathbb{E} \; .$$

The ACCELERATION of M is the map

$$\nabla\, d\, M \equiv \underline{||} \circ \Gamma \circ d^{2}M = (M, D^{2}M) : \mathbb{T} \to T\,\mathbb{E} \quad .$$

9 PROPOSITION.

We have $\langle \underline{t}, D^2M \rangle = 0$ (*)

Hence, we can write

$$D^2M : \mathbf{T} \to \bar{\mathbb{S}} ,$$

$$\Gamma \circ d^2M : \mathbf{T} \to \nu \dot{T}^2\mathbb{E}$$

and $$\nabla d M : \mathbf{T} \to \mathbb{A} \equiv \check{T}\, \mathbb{E}$$

and we get $$D^2M^\circ = 0$$

$$D^2M = (D^2M^k + (\Gamma^k_{ij} \circ M) DM^i DM^j + (\Gamma^k_{oj} \circ M) DM^j + \Gamma^k_{oo} \circ M)\ \delta x_k$$

$$\Gamma \circ d^2M = (D^2M^k + (\Gamma^k_{ij} \circ M) DM^i DM^j + (\Gamma^k_{oj} \circ M) DM^j + \Gamma^k_{oo} \circ M)\ \partial \dot{x}_k$$

$$\nabla d M = (D^2M^k + (\Gamma^k_{ij} \circ M) DM^i DM^j + (\Gamma^k_{oj} \circ M) Dm^j + \Gamma^k_{oo} \circ M)\ \partial x_k$$

Geometrical analysis.

Here we give some further element of analysis of $\mathbb{M}$, not essential from a kinematical point of view.

10 M has two structures: the C^∞ structure induced by $\mathbb{E}$ and the oriented euclidean affine structure induced by T (but, in general, $\mathbb{M}$ is not an affine subspace of $\mathbb{E}$) .

The embending $$TM : T\,\mathbb{T} \to T\mathbb{M} \hookrightarrow T\mathbb{E}$$

is given by $$(\tau,\lambda) \mapsto (M(\tau)), \lambda DM(\tau)) .$$

The embending $$T^2M : T^2\mathbf{T} \to T^2\mathbb{M} \hookrightarrow T^2\mathbb{E}$$

is given by $$(\tau,\lambda;\mu,v) \to (M(\tau),\lambda DM(\tau);\mu DM(\tau), vDM(\tau)+\lambda\mu D^2M(\tau)).$$

Now, let us consider the two fields

$$\bar{\mathbf{M}} \equiv d\,M \circ M^{-1} : \mathbb{M} \to T\,\mathbb{M}$$

and $$\bar{\bar{\mathbf{M}}} \equiv \nabla dM \circ M^{-1} : \mathbb{M} \to \check{T}\,\mathbb{E}_{|\mathbb{M}} .$$

11 PROPOSITION.

$\bar{\mathbf{M}}$ results into the unitary oriented constant field, with respect to the oriented euclidean affine structure of $\mathbb{M}$ induced by $\mathbb{T}$.

Moreover, each vector field $X : \mathbb{M} \to T\,\mathbb{M}$ can be written as

$$X = X^\circ\,\mathbb{M} , \qquad \text{where} \qquad X^\circ \equiv \langle \underline{t}, x \rangle .$$

12 PROPOSITION.

Let $X : \mathbb{M} \to T\,\mathbb{M}$ and $y : \mathbb{M} \to T\,\mathbb{M}$ be two C^∞ fields.
Then the covariant derivative

$$\nabla_X Y \equiv \coprod \circ \Gamma \circ T\,Y \circ X : \mathbb{M} \to T\,\mathbb{E}_{|\mathbb{M}}$$

is given by $$\nabla_X Y = p''_{\bar{M}} \circ \nabla_X Y + p^\perp_{M} \circ \nabla_X Y ,$$

where $$p''_{\bar{M}} \circ \nabla_X Y = X^\circ\ D\ Y^\circ\ \bar{\mathbf{M}}$$

results into the covariant derivative with respect to the affine structure of $\mathbb{M}$ and

$$p^\perp_{M} \circ \nabla_X Y = X^\circ\ Y^\circ\ \bar{\bar{\mathbf{M}}}$$

shows that the tensor

$$\bar{\bar{\mathbf{M}}} \otimes \underline{t} \otimes \underline{t} : \mathbb{M} \to T_{(1,2)}\mathbb{E}_{|\mathbb{M}}$$

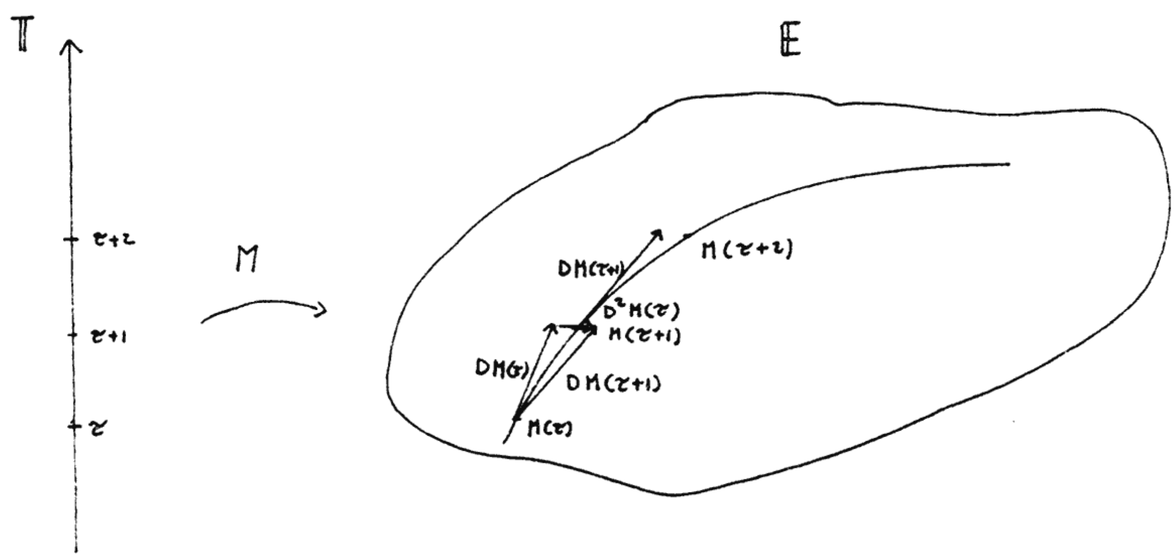
can be considered as the second fundamental form of $\mathbb{M}$ $\underline{.}$

Physical description.

The world-line $\mathbb{M}$ of a particle represents the set of all the events "touched" by the particle and the motion M is the map that associates with each time the relative event. Of course the events being absolute, i.e. independent of any frame of reference, the same occurs for the world line and the motion. The affine structure of $\mathbb{E}$ allows a privileged type of motions, namely the inertial ones.

As we have the absolute motion M, we have the absolute velocity DM and acceleration D^2M. These contain all the information necessary to derive the velocity and acceleration observed by any frame of reference, when it is chosen. The fact that DM is a unitary vector and D^2M is a spatial vector will put in evidence how the observed velocity changes and that the observed acceleration does not change from an inertial frame of reference to an other.

We can describe the previous facts by pictures.



II CHAPTER

FRAMES OF REFERENCE

Here we study the absolute kinematics of a continuum, which, viewed as a frame of reference, determines positions, the splitting of event space into space-time and the consequent splitting of velocity space. We analyse the positions space and its structures as the time-depending metric, the time-depending affine connection and the Coriolis map. Finally we make a classification of frames.

I FRAMES AND THE REPRESENTATION OF $\mathbb{E}$.

Frames, positions and adapted charts.

1 The basic elements of observed kinematics are frames, constituted by a reference continuum, whose particles determine positions on $\mathbb{E}$.

For simplicity of notations, we consider only global frames, leaving to the reader the obvious generalization to local frames.

DEFINITION.

A FRAME (OF REFERENCE) is a couple

$$\mathcal{P} \equiv \{\mathbf{P}, \{\mathbf{T}_q\}_{q\in\mathbf{P}}\}$$

where $\mathbf{P}$ is a set and, $\forall\, q \in \mathbf{P}$, $\mathbf{T}_q$ is a world line, such that

a)(*)
$$\mathbb{E} = \bigsqcup_{q\in\mathbf{P}} \mathbf{T}_q \quad ;$$

b) $\forall\, e \in \mathbb{E}$, there exists a neighbourhood U of e and a C^∞ chart

$$x \equiv \{x^\circ, x^i\} : U \to \mathbb{R} \times \mathbb{R}^3$$

adapted to the family of submanifolds $\{\mathbf{T}_q\}_{q\in\mathbf{P}}$.

$\mathbf{P}$ is the POSITION SPACE; each $q\in\mathbf{P}$ is a POSITION; the map

$$p : \mathbb{E} \to \mathbf{P}$$

given by $e \mapsto$ the unique $q\in\mathbf{P}$, such that $e \in \mathbf{T}_q$,

is the POSITION MAP; if $e \in \mathbb{E}$, then $p(e) \in \mathbf{P}$ is the POSITION of e .

(*) It sufficies to assume $E = \bigcup_{p\in P} \mathbf{T}_q$. In fact, taking into account b) and that each $\mathbf{T}_q$ is open and connected, we see that, if $q \neq q'$, then $\mathbf{T}_q \cap \mathbf{T}_{q'} = \emptyset$.

Henceforth we assume a frame $\mathcal{P}$ to be given.

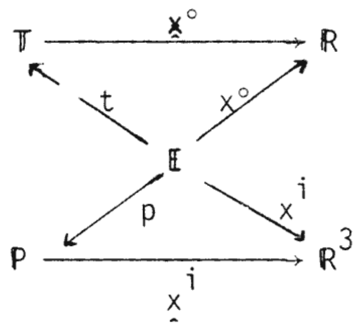
2 Calculations develop in an easier way if performed with respect to a chart adapted to $\mathcal{P}$. For simplicity of notations, we consider only global charts, leaving to the reader the obvious generalization to local charts, our considerations being essentially local.

DEFINITION.

A CHART ADAPTED TO $\mathcal{P}$ is a chart

$$\{x^\circ, x^i\} : \mathbb{E} \to \mathbb{R} \times \mathbb{R}^3 ,$$

such that it is special and it factorizes through $\mathbb{P}$, i.e. such that the following diagram is commutative



where $\underset{\frown}{x}^\circ : \mathbb{T} \to \mathbb{R}$ is a normal oriented cartesian chart

Charts adapted to $\mathcal{P}$ exist by definition 1.

Hencefort we assume a chart x adapted to $\mathcal{P}$ to be given.

<u>Representation of the position space $\mathbb{P}$</u> .

3 $\mathbb{P}$ results naturally into a C^∞ manifold.

PROPOSITION.

There is a unique C^∞ structure on $\mathbb{P}$, such that the map $p : \mathbb{E} \to \mathbb{P}$ is C^∞ . Namely it is induced by the charts adapted to $\{\mathbb{T}_q\}_{q \in \mathbb{P}}$.

PROOF.

<u>Unicity</u>. If $y : V \subset \mathbb{P} \to \mathbb{R}^3$ is a chart which makes p C^∞

and if $x : U \subset \mathbb{E} \to \mathbb{R} \times \mathbb{R}^3$ is a C^∞ chart adapted to $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$, then the map (defined locally)

$$\mathbb{R}^3 \hookrightarrow \mathbb{R} \times \mathbb{R}^3 \xrightarrow{x^{-1}} \mathbb{E} \xrightarrow{p} \mathbb{P} \xrightarrow{y} \mathbb{R}^3 ,$$

which is the change from x to y, is C^∞.

Existence. The change of charts on $\mathbb{P}$ induced by charts adapted to $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$ is C^∞ $\dot{-}$

4 We get a first immediate representation of $\mathbb{P}$.

The frame $\mathcal{P}$ determines a partition of $\mathbb{E}$ into the equivalence classes $\{\mathbb{T}_q\}_{q\in\mathbb{P}}$.

Then we get the natural identification of $\mathbb{P}$ with the quotient space $\mathbb{E}/\mathcal{P}$

$$\mathbb{P} \cong \mathbb{E}/\mathcal{P},$$

by writing $$q \cong p^{-1}(q) \equiv \mathbb{T}_q$$

and $$[e] = q = [e'] \iff p(e) = q = p(e') .$$

We will ofter identify $\mathbb{P}$ and $\mathbb{E}/\mathcal{P}$.

5 Choicing a time $\tau \in \mathbb{T}$ and taking, for each equivalence class, its representative, at the time τ, we get a second interesting representation of $\mathcal{P}$.

For this purpose, let us introduce three maps related with $\mathcal{P}$.

DEFINITION.

Let $\tau, \tau' \in \mathbb{T}$.

Then we define the three maps

a) $$P_\tau : \mathbb{P} \to \$_\tau$$

given by $q \mapsto$ the unique $e \in \$_\tau \cap \mathbb{T}_q$;

b) $$p_\tau \equiv p_{|\$_\tau} : \$_\tau \to \mathbb{P} ;$$

c) $$\tilde{P}_{(\tau',\tau)} \equiv P_{\tau'} \circ p_\tau : \$_\tau \to \$_{\tau'} \ .$$

6 Then we see that $\mathbb{P}$ is diffeomorphic (not canonically) to a 3-dimensional affine space.

PROPOSITION.

The maps $\tilde{P}_\tau$ and p_τ are inverse C^∞ diffeomorphisms:

$$\tilde{P}_\tau : \mathbb{P} \to \$_\tau \qquad , \qquad p_\tau : \$_\tau \to \mathbb{P} .$$

Moreover we have

$$\tilde{P}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)} = P_{(\tau'',\tau)}$$

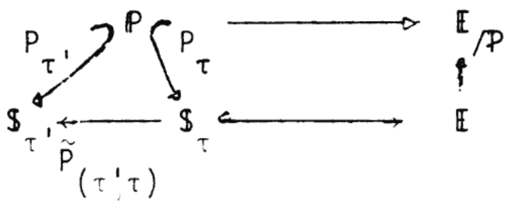
and $$\tilde{P}_{(\tau,\tau)} = id_{\$_\tau}$$

hence $\tilde{P}_{(\tau',\tau)}$ is a C^∞ diffeomorphism .

PROOF.

P_τ and p_τ ave inverse bijections. Moreover, p_τ , which is the composition $\$_\tau \hookrightarrow \mathbb{E} \to \mathbb{P}$, is C^∞ and $\det DP_\tau = \det(\partial y_i \circ \hat{x}^j \circ p_\tau) \neq 0$, where y is a special chart .

7 The relation among the different representation of $\mathbb{P}$ is shown by the following commutative diagram



Frame motion.

8 We need a further map given by the motions associated to the world lines of $\mathcal{P}$.

DEFINITION.

The MOTION of $\mathcal{P}$ is the map

$$P : \mathbb{T} \times \mathbb{P} \to \mathbb{E}$$

given by $(\tau, q) \mapsto$ the unique $e \in \mathbb{S}_\tau \cap \mathbb{T}_q$.

Thus P is the union of the family of maps $\{P_\tau\}_{\tau \in \mathbb{T}}$ previously introduced; on the other hand, P is the union of the family of maps $\{P_q\}_{q \in \mathbb{P}}$, constituted by the motions associated with the world-lines of $\mathcal{P}$.

The motion P characterizes the frame $\mathcal{P}$.

9 For calculations it is more advantageous a further map, substantially equivalent to P, which relates affine spaces.

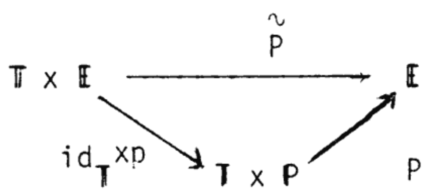
DEFINITION.

We define the map

$$\tilde{P} \equiv P \circ (id_{\mathbb{T}} \times p) : \mathbb{T} \times \mathbb{E} \to \mathbb{E} ,$$

given by $(\tau, e) \mapsto P(\tau, p(e))$.

Thus the following diagram is commutative by definition



10 The following immediate formulas will be used in calculations.

PROPOSITION.

We have

a) $t(\tilde{P}(\tau,e)) = \tau$, i.e. $t \circ \tilde{P} = id_{\mathbb{T}}$;

b) $\tilde{P}\,(t(e),e) = e$, i.e. $P \circ j = id_{\mathbb{E}}$;

c) $\tilde{P}\,(\tau,\tilde{P}\,(\sigma,e)) = \tilde{P}(\tau,e)$.

$\tilde{P}$ characterizes the frame $\mathbb{P}$.

We have $x^{\circ} \circ \tilde{P} = \underset{\wedge}{x}^{\circ}$, $x^{i} \circ \tilde{P} = x^{i}$.

Rappresentation of $\mathbb{E}$.

11 The frame $\mathbb{P}$ determines the splitting of the event space in space-time.

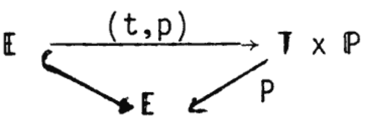
THEOREM.

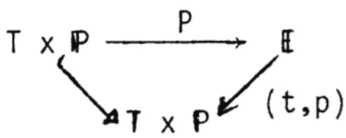
The maps

$(t,p) : \mathbb{E} \to \mathbb{T} \times \mathbb{P}$ and $P : \mathbb{T} \times \mathbb{P} \to \mathbb{E}$

are inverse C^{∞} diffeomorphisms.

Namely the following diagrams are commutative





Hence $(\mathbb{E},p,\mathbb{P})$ results into a C^{∞} bundle, with fiber $\mathbb{T}$.

PROOF.

P and (t,p) are inverse bijections. Moreover (t,p) is C^∞ and det $D(t,p) \neq 0$.

12 DEFINITION.

The FRAME BUNDLE is

$$\equiv (\mathbb{E},p,\mathbb{P}) \quad .$$

Thus we have two bundle structures on $\mathbb{E}$, namely

$\eta \equiv (\mathbb{E},t,\mathbb{T})$, wich has an absolute basis $\mathbb{T}$ and a non canonical fiber diffeomorfic to $\mathbb{P}$ or to $\mathbb{S}_\tau$, $\forall\ \tau\in\mathbb{T}$,

$\pi \equiv (\mathbb{E},p,\mathbb{P})$, wich has a frame depending basis $\mathbb{P}$, diffeomorfic to $\mathbb{S}_\tau$, $\forall\ \tau\in\mathbb{T}$, and an absolute fiber $\mathbb{T}$.

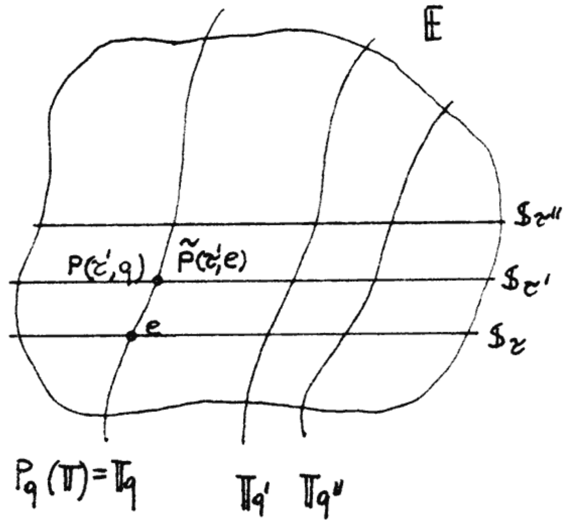
The frame bundle π characterizes the frame $\mathcal{P}$.

Physical description.

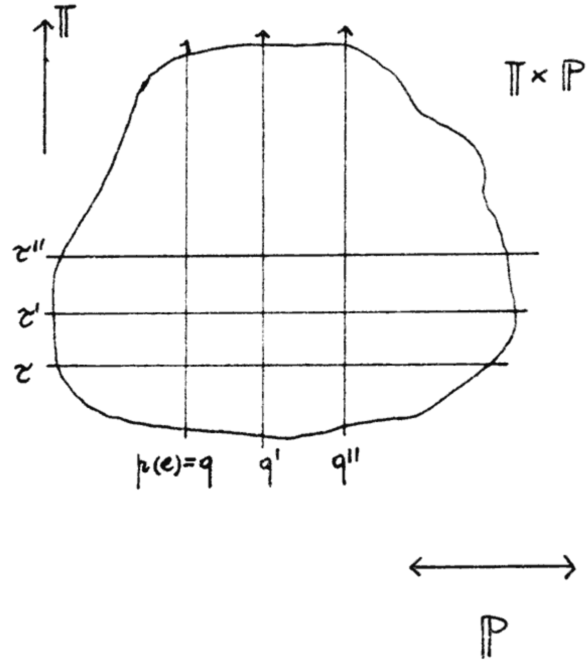
A frame $\mathcal{P}$ is a set $\mathbb{P}$ of particles, never meeting, filling, at each time $\tau\in\mathbb{T}$, the whole space $\mathbb{S}_\tau$, with a C^∞ flow, hence first a frame is a continuum and we study the absolute kinematics of its particles.

Such a continuum can be viewed as a frame of reference. In fact it determines a partition of $\mathbb{E}$ in positions. Each position in the setof all events touched by the same frame particle. Under this aspect we can identify the set of positions with the setof particles $\mathbb{P}$.

We can describe the frame, its motion, the positions and the splitting of $\mathbb{E}$ into the space-time $\mathbb{T} \times \mathbb{P}$, by a picture. Notice that we can consider only the differentiable properties induced by the paper to $\mathbb{P}$, in the picture : in fact the affine and metrical properties of $\mathbb{P}$ are time depending.



E
$S_{\tau''}$
$P(\tau',q)$ $\tilde{P}(\tau',e)$
$S_{\tau'}$
e
S_{τ}
$P_q(T) = T_q$
$T_{q'}$ $T_{q''}$



T
T × P
τ''
τ'
τ
p(e) = q
q'
q''
P

2 - FRAMES AND THE REPRESENTATION OF $T\mathbb{E}$.

In this section we are dealing with the first order derivatives of the frame and tangent spaces.

Frame velocity and jacobians.

1 The velocity of the frame is the vector field on $\mathbb{E}$ constituted by the velocities of the world-lines of the frame. Hence it is the first derivative of the notion with respect to time. On the orther hand, the jacobians are the first derivatives with respect to event. We consider only free entities, for simplicity of notions, leaving to the reader to write them in the complete form.

DEFINITION.

a) The (FREE) VELOCITY-FUNDAMENTAL FORM - of $\mathcal{P}$ is the map

$$D_1\tilde{P} : T \times \mathbb{E} \to \bar{\mathbb{E}} .$$

The (FREE) VELOCITY-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\bar{P} \equiv D_1\tilde{P} \circ j : \mathbb{E} \to \mathbb{E}$$

b) The (FREE) JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathcal{P}$ is the map

$$D_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} .$$

The (FREE) JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\hat{P} \equiv D_2\tilde{P} \circ j : \mathbb{E} \to \bar{\mathbb{E}}^* \times \bar{\mathbb{E}} .$$

The (FREE) SPATIAL JACOBIAN-FUNDAMENTAL- EULERIAN FORM - of $\mathcal{P}$ is the map

$$\check{P} \equiv \check{D}_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{E}}$$

The (FREE) SPATIAL JACOBIAN-LAGRANGIAN-LAGRANGIAN FORM, RELATIVE TO THE INITAL TIME $\tau \in \mathbb{T}$ AND TO THE FINAL TIME $\tau' \in \mathbb{T}$, of P - is the map

$$\check{P}_{(\tau'\tau)} \equiv D\tilde{P}_{(\tau'\tau)} : \mathbb{S}_{\tau} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} .$$

We will denote by

$\bar{\mathbf{P}} : \mathbb{E} \to T\mathbb{E}$, $\hat{\mathbf{P}} : T\mathbb{E} \to T\mathbb{E}$ $\check{\mathbf{P}} : \mathbb{T} \times \check{T\mathbb{E}} \to \check{T\mathbb{E}}$

the maps associated with $\bar{P}$, $\hat{P}$, $\check{P}$.

We will write

$$\check{x}_p \equiv \hat{\mathbf{P}}(x) , \qquad \forall\ x \in T\mathbb{E} ,$$

$$\check{u}_p \equiv \hat{P}(e)(u), \qquad \forall\ x \in T_e\mathbb{E} .$$

2 We get immediate important properties of these maps

PROPOSITION.

We have

a) $\underline{t} \circ D_1\tilde{P} = 1$

b) $\underline{t} \circ D_2\tilde{P} = 0$

Hence we can write

$$D_1\tilde{P} : \mathbb{T} \times \mathbb{E} \to \mathbb{U} \quad , \quad \bar{D}_2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$$

$$\bar{P} : \mathbb{E} \to \mathbb{U} \qquad \check{P} : E \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \quad , \quad \hat{P}: \mathbb{E} \to \mathbb{E}^* \otimes \bar{\mathbb{S}}$$

Moreover, all the previous maps are expressible by $\tilde{P}$, $\bar{P}$ and $\check{P}$:

c) $$D_1\tilde{P} = \bar{P} \circ \tilde{P} \ ;$$

e) $$\hat{P} = id_{\bar{\mathbb{E}}} - t \otimes \bar{P} \ ,$$

hence $\ddot{P}$ is a projection operator $\bar{\mathbb{E}} \to \bar{\mathbb{S}}$;

d) $$D_2\tilde{P} = \check{P} \circ (\hat{P} \circ \pi^2) \ ;$$

f) $$\check{D}_2\tilde{P}_{\tau'}|\mathbb{S}_\tau = \check{P}_{(\tau',\tau)}$$

We have also the group properties

g) $$(\check{P}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)}) \circ \check{P}_{(\tau',\tau)} = \check{P}_{(\tau'',\tau)} \ ;$$

h) $$\check{P}_{(\tau,\tau)} = id_{\bar{\$}} \quad ;$$

hence $\check{P}_{(\tau',\tau)}$ preserves the orientation of $\$$

i) $$\det \check{P}_{(\tau',\tau)} > 0.$$

We have

e) $$\bar{P} = \delta x_o \ , \qquad \hat{P} = Dx^i \otimes \delta x_i$$

PROOF.

a) and b) follow from (II,1,10 a), by derivation with respect to τ and e.

c) follows from (II,1,10 c), by derivation with respect to τ and taking $\sigma \equiv \tau$,

d) follows from (II,1,10 b), by derivation with respect to e.

e) follows from (II,1,10 c), by derivation with respect to e.

f) follows from definitions.

g) and h) follows from (II,1,6).

i) $\check{P}_{(\tau',\tau)}(e)$ is an isomorphism, hence $\det \check{P}_{(\tau',\tau)}(e) \neq 0$;

$\det \check{P}_{(\tau,\tau)}(e) = 1$, for (h), and $\check{P}_{(\tau',\tau)}(e)$ is continuous with respect to τ', for (f) .

Representation of $T\mathbb{P}$.

3 In order to get the space $T\mathbb{P}$ handy, it is useful to regard it as a quotient. In this way we could view $T\mathbb{P}$ as a quotient space $T\mathbb{E}_{/\mathbb{P}}$. But a reduced representation by means of $\check{T\mathbb{E}}_{/\mathbb{P}}$ is more simple, for the equivalence classes have a unique representative for each time $\tau \in \mathbb{T}$.

PROPOSITION.

Let $v \in T\mathbb{P}$. Then

$$\mathcal{C}_v \equiv \check{T}\, p^{-1}(v) = (T_2P)_v\,(\mathbb{T}) \hookrightarrow \check{T}\mathbb{E}$$

is a C^∞ submanifold.

Then we get a partition of $\check{T}\mathbb{E}$, given by

$$\check{T}\mathbb{E} = \bigsqcup_{v \in T\mathbb{P}} \mathcal{C}_v \ ,$$

and the quotient space $\check{T}\mathbb{E}_{/\mathbb{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

$$[e,u] = [\,e',u'] \Longleftrightarrow p(e) = p(e')\ ,\ \check{P}(t(e'),e)(u) = u'\ . \qquad \text{(b)}$$

We get a natural C^∞ diffeomorphism between $T\mathbb{P}$ and $\check{T}\mathbb{E}_{/\mathbb{P}}$ given by the unique maps

$$T\mathbb{P} \to \check{T}\mathbb{E}_{/\mathbb{P}} \qquad \text{and} \qquad \check{T}\mathbb{E}_{/\mathbb{P}} \to T\mathbb{P}\ ,$$

which make commutative the two folloving diagrams, respectively,

$$\begin{array}{ccc} \mathbb{T} \times T\,\mathbb{P} & \xhookrightarrow{T_2P} & \check{T}\,\mathbb{E} \\ \downarrow{\scriptstyle \mathbb{T}^2} & & \downarrow \\ T\,\mathbb{P} & \hookrightarrow & \check{T}\,\mathbb{E}_{/\mathbb{P}} \end{array} \qquad\qquad \begin{array}{ccc} \check{T}\,\mathbb{E} & \xrightarrow{Tp} & T\,\mathbb{P} \\ & \searrow \quad \nearrow & \\ & T\mathbb{E}_{/\mathbb{P}} & \end{array}$$

PROOF.

(a) follows from (II,1,11).

(b) follows from

$$[e,u] = [e',u'] \Longleftrightarrow (e',u') \in (T_2P)_{Tp(e,u)} \Longleftrightarrow (e'u') = TP_{(t(e),t(e'))}\,(e,u).$$

The C^∞ structure on $T\mathbb{E}_{/\mathbb{P}}$ is induced by the charts adapted to $(\mathbb{T}_q)_{q \in \mathbb{P}}$.

We will often make the identification

$$T\mathbb{P} \cong \check{T}\,\mathbb{E}_{/\mathbb{P}}\ ,$$

which is very useful in calculations.

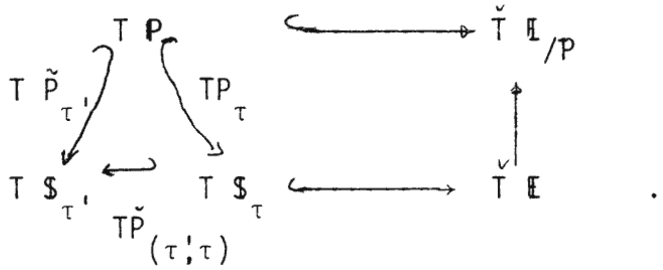
4 Choicing a time $\tau \in \mathbb{T}$ and taking, for each equivalence class, its representative at the time τ, we get a second interesting representation of $T\mathbb{P}$.

PROPOSITION.

The maps $\widetilde{TP}_\tau$ and Tp_τ are inverse C^∞ diffeomorphisms

$$\widetilde{TP}_\tau : T\mathbb{P} \to T\mathbb{S}_\tau \equiv T\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} , \quad T\, p_\tau : T\, \mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \to T\mathbb{P}$$.

5 The relation between the different representations of $T\mathbb{P}$ is shown by the following commutative diagram



6 Taking into account the identification $T\mathbb{P} \cong T\mathbb{E}_{/P}$, we get the following expression of T_p and TP.

PROPOSITION.

a) $T\, p\, (e,u) = [e, \hat{P}(e)(u)]$

b) $T\, P\, (\tau,\lambda;[e,u]) = (\check{P}(\tau,e),\lambda\, \bar{P}(\tilde{P}(\tau,e) + \check{P}(\tau,e)(u))$

PROOF.

a) The following diagram is commutative

$$\begin{array}{ccccc} & & \overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{T_2 \tilde{P}} & & \\ \mathbb{T} \times T\mathbb{E} & \xrightarrow{id_{\mathbb{T}} \times Tp} & \mathbb{T} \times T\mathbb{P} & \xrightarrow{T_2 P} & \check{T}\mathbb{E} \\ \uparrow & & \downarrow & & \downarrow \\ T\mathbb{E} & \xrightarrow[Tp]{} & T\mathbb{P} & \xrightarrow[\cong]{} & \check{T}\mathbb{E}_{/P} \end{array}$$

Hence we get

$$Tp(e,u) = (T_2 \tilde{P})(t(e);e,u) \; .$$

b) The following diagram is commutative

$$\begin{array}{ccccc} T\mathbb{T} \times \check{T}\mathbb{E} & \hookrightarrow & T\mathbb{T} \times T\mathbb{E} & \xrightarrow{T\tilde{P}} & T\mathbb{E} \\ \downarrow & & \downarrow T(id_{\mathbb{T}} \times p) & \nearrow_{TP} & \\ T\mathbb{T} \times \check{T}\mathbb{E}_{/P} & \xrightarrow[\cong]{} & T\mathbb{T} \times T\mathbb{P} & & \end{array}$$

Hence, we get

$$TP(\tau,\lambda; [e,u]) = T\check{P}(\tau,\lambda; TP_{(\tau,t(e))}(e,u)) \; \dot{-}$$

<u>Frame vertical and horizontal spaces</u>.

8 The bundle $\Pi \equiv (\mathbb{E},p,\mathbb{P})$ induces two useful spaces.

DEFINITION.

The FRAME VERTICAL TANGENT SPACE is

$$\check{T}_p\mathbb{E} \equiv \text{Ker}\; Tp \hookrightarrow T\mathbb{E} \; .$$

The FRAME HORIZONTAL TANGENT SPACE, or FRAME PHASE SPACE is

$$\mathring{T}_p\mathbb{E} \equiv T\mathbb{E}_{/\check{T}_p\mathbb{E}} \qquad \dot{-}$$

9 We have several representations of these spaces.

PROPOSITION.

a) We have $\check{T}_{\mathcal{P}}\mathbb{E} \equiv \operatorname{Ker} Tp = \operatorname{Im} T_1\mathcal{P}$.

Hence $\check{T}_{\mathcal{P}}\mathbb{E}$ is the subspace of $T\mathbb{E}$ generated by the velocity of $\mathcal{P}$

$$\check{T}_{\mathcal{P}}\mathbb{E} = \{(e,u) \in T\mathbb{E} \mid u = \lambda\, \bar{P}(e)\} .$$

$\check{T}_{\mathcal{P}}\mathbb{E}$ is the C^∞ submanifold of $T\mathbb{E}$ characterized by

$$\dot{x}^i = 0 .$$

Moreover, the maps

$$T_1 P : T\mathbb{T} \times \mathbb{P} \to \check{T}_{\mathcal{P}}\mathbb{E} \qquad \text{and} \qquad (Tt, \check{p}) : \check{T}_{\mathcal{P}}\mathbb{E} \to T\mathbb{T} \times \mathbb{P}$$

$$(\tau,\lambda;q) \mapsto (\mathcal{P}(\tau,q), \lambda\bar{P}(P(\tau,q))) \qquad\qquad (e, \lambda\bar{P}(e)) \mapsto (t(e), \lambda; p(e))$$

are inverse C^∞ diffeomorphisms;

the maps

$$\check{T}_{\mathcal{P}}\mathbb{E} \to \mathring{T}\,\mathbb{E} \qquad \text{and} \qquad \mathring{T}\,\mathbb{E} \to \check{T}_{\mathcal{P}}\mathbb{E} ,$$

given by $(e, \lambda\bar{P}(e)) \mapsto (e,\lambda)$ and $(e,\lambda) \mapsto (e, \lambda\bar{P}(e))$

are inverse C^∞ diffeomorphisms;

the following diagram is commutative

$$\begin{array}{ccc} \check{T}_{\mathcal{P}}\mathbb{E} & \xrightarrow{(Tt,\check{p})} & T\mathbb{T} \times \mathbb{P} \\ & \searrow \quad \mathring{T}\,\mathbb{E} \quad \swarrow_{T_1 P} & \end{array}$$

b) The charts adapted to $\{\mathbb{T}_q\}_{q \in \mathbb{P}}$ induce a C^∞ atlas on $\mathring{T}_{\mathcal{P}}\mathbb{E}$.

Hence $\mathring{T}_{\mathcal{P}}\mathbb{E}$ is the space

$$\mathring{T}_{\mathcal{P}}\mathbb{E} = \{[e,v]\}_{(e,v) \in T\mathbb{E}} = \{[e,u]\}_{(e,u) \in \check{T}\mathbb{E}} = \{(e, u + \check{T}_{\mathcal{P}e}\mathbb{E})\} .$$

Moreover, the maps

$$\mathbb{T} \times T\mathbb{P} \to \mathring{T}_{\mathcal{P}}\mathbb{E} \qquad \text{and} \qquad \mathring{T}_{\mathcal{P}}\mathbb{E} \to \mathbb{T} \times T\mathbb{P}$$

induced by the diagrams

$$\mathbb{T} \times T\mathbb{P} \xrightarrow{T_2 P} T\,\mathbb{E} \to \mathring{T}_p\mathbb{E} \qquad \text{and} \qquad \begin{array}{ccc} & T\,\mathbb{E} & \\ \swarrow & & \searrow (\check{t}, Tp) \\ \mathring{T}_p\mathbb{E} & \longrightarrow & \mathbb{T} \times T\mathbb{P} \end{array}$$

and given by

$$(\tau, [e,u]) \mapsto [\tilde{P}(\tau,e), \check{P}(\tau,e)(u)] \quad \text{and} \quad [e,v] \mapsto (t(e), [e, \hat{P}(e,(v)]))$$

are inverse C^∞ diffeomorphisms;

the maps

$$\mathring{T}_p\mathbb{E} \to \check{T}\,\mathbb{E} \quad \text{and} \quad \check{T}\,\mathbb{E} \to \mathring{T}_p\mathbb{E}$$

given by $[e,v] \mapsto (e, \hat{P}(e)(v))$ and $(e,u) \mapsto [e,u]$

are inverse C^∞ diffeomorphisms;

the maps

$$\mathring{T}_p\mathbb{E} \to \dot{T}\,\mathbb{E} \quad \text{and} \quad \dot{T}\,\mathbb{E} \to \mathring{T}_p\mathbb{E}$$

given by $[e,v] \to (e, \hat{P}(e)(v) + \bar{P}(e))$ and $(e,w) \mapsto [e,w]$

are inverse C^∞ diffeomorphisms;

the following diagram is commutative

$$\begin{array}{ccccc} & & \dot{T}\mathbb{E} & \xleftarrow{(T\,P)_1} & \\ & \nearrow & & & \\ \mathring{T}_p\,\mathbb{E} & & \longrightarrow & & \mathbb{T} \times T\mathbb{P} \\ & \searrow & & & \\ & & \check{T}\,\mathbb{E} & \xleftarrow{(TP)_0 \equiv T_2 P} & \end{array} \qquad \dot{-}$$

We will often make the identifications

$$\check{T}_{p}\mathbb{E} \overset{\sim}{=} T\,\mathbb{T} \times \mathbb{P} \overset{\sim}{=} \overset{\circ}{T}\,\mathbb{E}$$

$$\overset{\circ}{T}_{p}\mathbb{E} \overset{\sim}{=} \mathbb{T} \times T\,\mathbb{P} \cong \check{T}\,\mathbb{E} \cong \overset{\prime}{T}\,\mathbb{E}$$

Frame metric function.

10 We get a "time depending" Riemannian structure on $\mathbb{P}$, induced by the family of diffeomorphisms

$$T\,\mathbb{P} \longrightarrow T\,\mathbb{S}_{\tau}$$

DEFINITION.

The FRAME TIME DEPENDING METRIC FUNCTION is the function

$$g_{\mathcal{P}} : \mathbb{T} \times T\,\mathbb{P} \longrightarrow \mathbb{R}$$

given by the composition

$$\mathbb{T} \times T\,\mathbb{P} \to \check{T}\,\mathbb{E} \xrightarrow{g} \mathbb{R} ,$$

i.e. $$g_{\mathcal{P}}(\tau,[e,u]) \equiv \frac{1}{2}(\check{P}(\tau,e)(u))^{2}$$ ∸

Taking into account $\mathbb{T} \times T\,\mathbb{P} \overset{\sim}{=} \mathbb{V}$, we will write also

$$g_{\mathcal{P}} : \mathbb{V} \longrightarrow \mathbb{R} .$$

11 PROPOSITION.

We have $$g_{\mathcal{P}} = \frac{1}{2}\, g_{ij}\, \dot{x}^{i}\dot{x}^{j}$$ ∸

Representation of $T\mathbb{E}$.

12 Most of the previous results can be sumarized in the following fundamental theorem, which gives the representation of $T\mathbb{E}$ induced by the frame.

THEOREM.

The map $\qquad T\mathbb{E} \to \overset{\circ}{T}_{P}\mathbb{E} \times_{\mathbb{E}} \overset{\circ}{T}\,\mathbb{E}$,

given by the natural projections, is a C^{∞} diffeomorphism.

The map $\qquad \check{T}\mathbb{E} \oplus_{\mathbb{E}} \check{T}_{P}\mathbb{E} \to T\mathbb{E}$,

given by the natural inclusions, is a C^{∞} diffeomorphism.

The maps

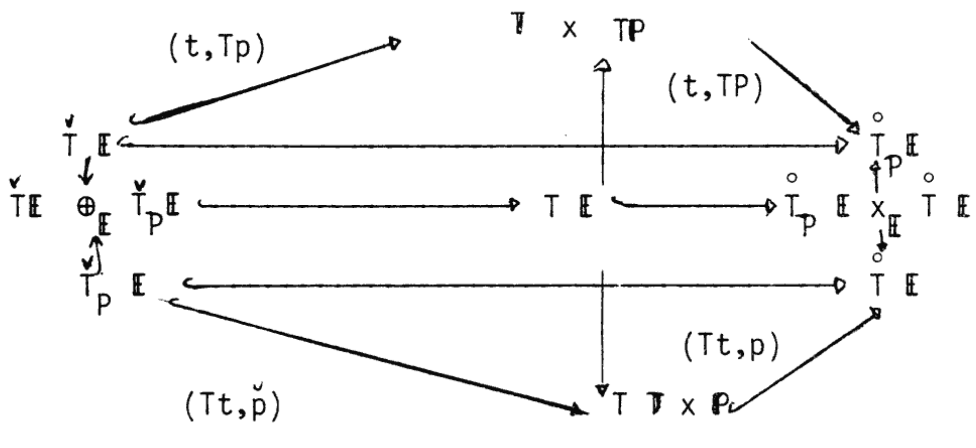
$t(t,p) : T\mathbb{E} \to T\,\mathbb{T} \times T\mathbb{P}$ $\qquad$ and $\qquad$ $TP : T\,\mathbb{T} \times T\mathbb{P} \to T\mathbb{E}$

are inverse C^{∞} diffeomorphisms.

Moreover we have the C^{∞} diffeomorphisms

$\overset{\circ}{T}_{P}\mathbb{E} \to \mathbb{T} \times TP \to T\mathbb{E}$ $\qquad$ and $\qquad$ $\overset{\circ}{T}\mathbb{E} \to T\,T \times P \to T_{P}\mathbb{E}$.

Hence, the relation among the previous three representations of $T\mathbb{E}$ is given by the following commutative diagram



The maps

$$T\mathbb{E} \to \check{T}\mathbb{E} \to \mathbb{T} \times T\mathbb{P} \to \overset{\circ}{T}_{P}\mathbb{E}$$

are given by

$$(e,u) \mapsto (e,\hat{P}(e)(u)) \mapsto (t(e),[e,\hat{P}(e)(u)]) \mapsto [e,u]$$

The map

$$T\mathbb{E} \to \check{T}_{\mathcal{P}}\mathbb{E} \to T\,\mathbb{T} \times P \to \mathring{T}\,\mathbb{E}$$

are given by

$$(e,u) \to (e,u^{\circ}\bar{P}(e)) \to (t(e),u^{\circ},[e]) \to (e,u^{\circ}) \quad \underline{.}$$

The choice of the most convenient representation depends on circumstances. Some of these have a theoretical relevance, other a computational advantage. So, for esplicite calculations, we will generally use the following identifications

$$T\,\mathbb{P} \overset{\sim}{=} \acute{T}\mathbb{E}_{/\mathcal{P}} \quad , \qquad \text{and} \qquad \mathbb{T} \times T\,\mathbb{P} \overset{\sim}{=} T_{\mathcal{P}}\mathbb{E} \quad .$$

Notice that in the decomposition of the vector field $x : \mathbb{E} \to T\mathbb{E}$

$$x = x^{\circ}\,\bar{P} + \check{x}_{\mathcal{P}} : E \to \check{T}_{P}\mathbb{E} + \check{T}\,\mathbb{E}$$

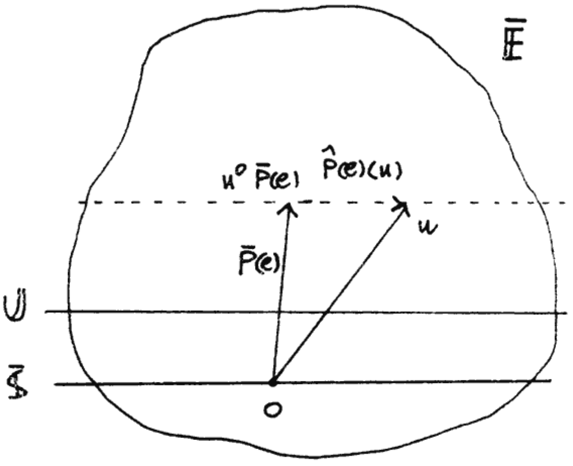
the component x° is absolute, but the space $\check{T}_{\mathcal{P}}\mathbb{E}$ is frame depending, and the space $\check{T}\mathbb{E}$ is absolute, but the component $\check{x}_{P}$ is frame depending.

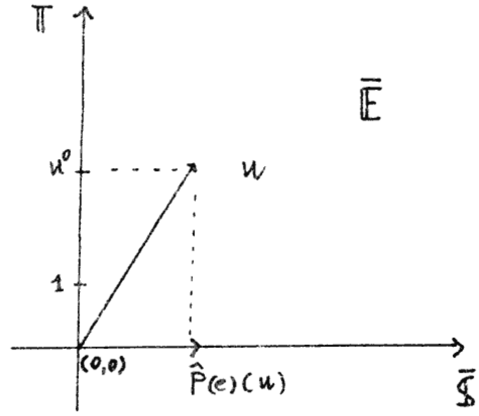
Physical description.

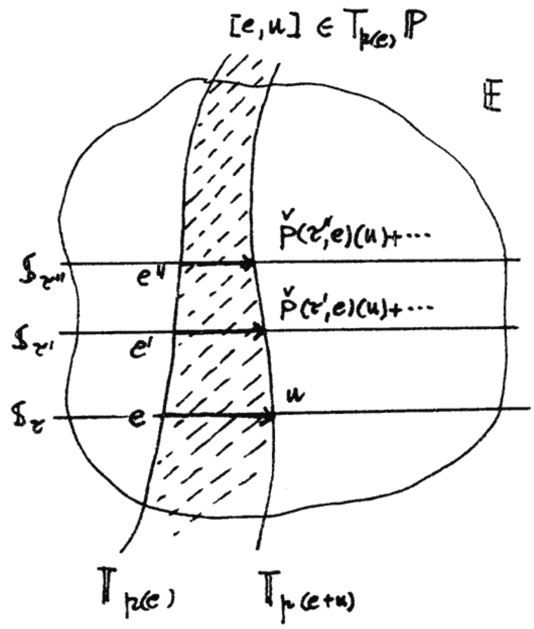
$\bar{P}$ is the field of velocity the frame continuum. $\hat{P}$ is the spatial projection operator induced by the velocity and $\check{P}$ is the infinitesimal displacement generated by the continuum motion on spatial vectors.

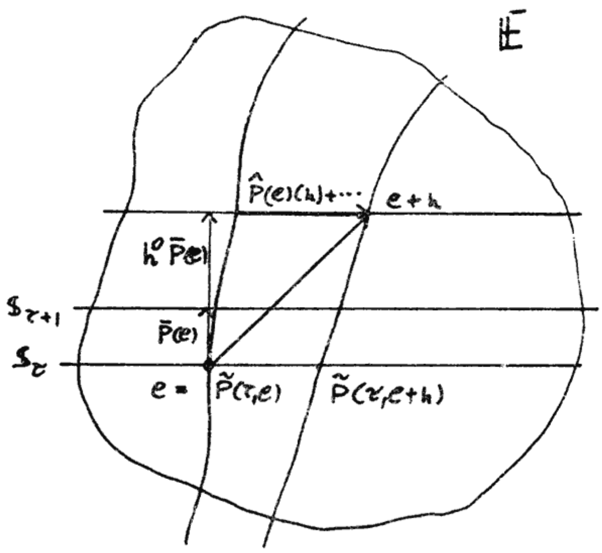
We identify (at the first order) each vector of $T_{q}\mathbb{P}$ with a strip having as first side the world-line q and as second side another world-line.

We can describe the situation by a picture.


Ē
u° P̄(e)
P̂(e)(u)
P̄(e)
u
U
S̄
0


T
Ē
u°
u
1
(0,0)
P̂(e)(u)
S̄


[e,u] ∈ T_{p(e)} P
E
P̌(τ'',e)(u)+⋯
P̌(τ',e)(u)+⋯
e''
e'
e
u
T_{p(e)}
T_{p(e+u)}


Ē
P̂(e)(h)+⋯
e+h
h° P̄(e)
P̄(e)
S_{τ+1}
S_τ
e = P̃(τ,e)
P̃(τ,e+h)

3 FRAMES AND THE REPRESENTATION OF $T^2\mathbb{E}$.

In this section we are dealing with the second order derivatives of the frame and tangent spaces.

Frame acceleration, second jacobians, strain and spin .

1 The acceleration of the frame is the vector field on $\mathbb{E}$ constituted by the accelerations of the world-lines of the frame. Hence it is the second derivative of the motion with respect to time. On the other hand,the second and mixed jacobians are the second derivatives with respect to event- event and time- event. We consider only free entities.

DEFINITION.

For simplicity of notations, leaving to the reader to write them in the complete from .

a) The (FREE) ACCELERATION-FUNDAMENTAL FORM - of $\mathcal{P}$ is the map

$$D_1^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}} .$$

The (FREE) ACCELERATION-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\bar{\bar{P}} \equiv D_1^2 \, P \circ j : \mathbb{E} \to \bar{\mathbb{E}}$$

b) The (FREE) SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathcal{P}$ is the map

$$D_2^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} .$$

The (FREE) SECOND JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\hat{\hat{P}} \equiv D_2^2 \, \tilde{P} \circ j : E \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}} .$$

The (FREE) SPATIAL SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\check{\check{P}} \equiv \check{D}_2^2\tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} .$$

The (FREE) SPATIAL SECOND JACOBIAN-LAGRANGIAN - LAGRANGIAN FORM WITH RESPECT

TO THE INITIAL TIME $\tau \in T$ AND THE FINAL TIME $\tau' \in T$ - of $\mathcal{P}$ is the map

$$\check{\check{P}}_{(\tau',\tau)} \equiv D^2 P_{(\tau',\tau)} : \mathbb{S}_\tau \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \quad .$$

c) The (FREE) MIXED SECOND JACOBIAN-FUNDAMENTAL-EULERIAN FORM of $\mathcal{P}$ is the map

$$D_2 D_1 \tilde{P} : \mathbb{T} \times E \to \mathbb{E}^* \otimes \bar{\mathbb{E}}$$

The (FREE) MIXED SECOND JACOBIAN-EULERIAN-EULERIAN FORM of $\mathcal{P}$ is the map

$$\hat{\bar{P}} \equiv D_2 D_1 \tilde{P} \circ j : \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}$$

The (FREE) MIXED SPATIAL SECOND JACOBIAN-EULERIAN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\check{\bar{P}} \equiv \check{D}_2 D_1 P \circ j : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{E}}$$

d) The (FREE) STRAIN-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\epsilon_{\mathcal{P}} \equiv S \circ \check{\bar{P}} : \mathbb{E} \to \mathbb{S} \otimes \bar{\mathbb{S}} \; .$$

The (FREE) SPIN - EULERIAN FORM - of $\mathcal{P}$ is the map

$$\omega_{\mathcal{P}} \equiv \frac{A}{2} \circ \check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$$

The (FREE) ANGULAR VELOCITY-EULERIAN FORM - of $\mathcal{P}$ is the map

$$\Omega_{\mathcal{P}} \equiv \check{*} \frac{A}{2} \circ \check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}} \; .$$

2 We get immediate important properties of these maps.

PROPOSITION.

We have

a) $$\underline{t} \circ D_1^2 \tilde{P} = 0$$

b) $\underline{t} \circ D_2^2 \tilde{P} = 0$,

c) $\underline{t} \circ D_2 D_1 \tilde{P} = 0$,

hence we can write

$D_1^2\tilde{P} : \mathbb{T} \times E \to \bar{\mathbb{S}}$ $\qquad$ $D_2^2 \tilde{P} : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{E}}^* \times \bar{\mathbb{S}}$ $\qquad$ $D_2 D_1 \tilde{P} : \mathbb{T} \times E \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$

$\bar{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}$ $\qquad$ $\hat{\hat{P}} : \mathbb{E} \to \mathbb{E}^* \otimes \bar{E}^* \otimes \bar{\mathbb{S}}$ $\qquad$ $\hat{\bar{P}} : \mathbb{E} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$

$\check{\bar{P}} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$

Moreover all the previous maps are expressible by $\tilde{P}, \bar{P}, DP$ and $\check{\check{P}}$:

d) $D_1^2\tilde{P} = \bar{\bar{P}} \circ \tilde{P}$

e) $\hat{\hat{P}} = - \bar{\bar{P}} \otimes \underline{t} \otimes \underline{t} - (\check{D}\bar{P} \circ \hat{P}) \otimes \underline{t} - \underline{t} \otimes (\check{D}\bar{P} \circ \hat{P})$

f) $\hat{\bar{P}} = \check{D}\bar{P} \circ \hat{P}$

g) $\bar{\bar{P}} = D\bar{P}(\bar{P})$

h) $\check{\bar{P}} = \check{D}\bar{P}$

i) $(D_2^2\tilde{P})_{\tau'|\mathbb{S}_\tau} = \check{\check{P}}_{(\tau',\tau)} \circ \hat{P}_{|\mathbb{S}_\tau}$

l) $(D_1 \check{P}) \circ j = \check{D}\bar{P}$.

If $u \equiv u^\circ \bar{P} + u_P : \mathbb{E} \to \bar{\mathbb{E}}$, we can write

m) $D\bar{P}(u) = u^\circ \bar{\bar{P}} + \frac{1}{2} \varepsilon_{\mathcal{P}}(\check{u}_P) + \Omega_{\mathcal{P}} \times \check{u}_P$.

n) We have $\underline{\varepsilon}_{\mathcal{P}} = L_{\bar{P}}\, g$.

o) $\bar{\bar{P}} = \Gamma^i_{oo}\, \delta x_i$

$$\bar{\bar{P}} = \Gamma^k_{io}\, Dx^i \otimes \delta x_k$$

$$\hat{\hat{P}} = - \Gamma^k_{oo}\, Dx^o \otimes Dx^o - \Gamma^k_{io}(Dx^i \otimes Dx^o + Dx^o \otimes Dx^i) \otimes \delta x_k$$

$$\underline{\varepsilon}_{\mathcal{F}} = \Gamma_{j,o}{}^{i} + \Gamma^{i}{}_{oj})\, Dx^i \otimes Dx^j = \partial_o g_{ij}\, Dx^i \otimes Dx^j$$

$$\underline{\omega}_{\mathcal{F}} = \frac{1}{2} (\Gamma_{j,oi} - \Gamma_{i,oj})\, Dx^i \otimes Dx^j$$

$$\Omega_{\mathcal{F}} = \frac{1}{2} \sqrt{\det(g^{ij})}\ \epsilon^{kij}\Gamma_{j,oi}\, \delta x_k$$

PROOF.

a),b) and c) follow from (II,1,10 a) by double derivation with respect to τ,τ; e.e. and τ,e.

d) follows from (II,1,10 c) by double derivation with respect to τ and taking $\sigma \equiv \tau$.

e) follows from (II,1,10 b) by double derivation with respect to e.

f) follows from (II,1,10 c) by double derivation with respect to τ and with respect to τ and e and taking $\sigma \equiv \tau$.

g) follows from (II,2,2c) by derivation with respect to e.

h) follows from f).

i) follows from (II,1,10 c) by double derivation with respect to e and taking $\tau \equiv t(e)$, $\sigma \equiv \tau$.

l) follows from $D_2 D_1 \tilde{P} = D_1 D_2 \tilde{P}$.

m) follows from g and f)

n) follows from $(L_{\bar{P}}\, g)_{ij} = \partial_o g_{ij} = \Gamma_{j,oi} + \Gamma_{i,oj}$ $\dot{-}$

Representation of $T^2\mathbf{P}$ and $\nu T^2\mathbf{P}$.

3 In order to get the space $T^2\mathbf{P}$ handy, it is useful to regard it as a quotient. In this way we could view $T^2\mathbf{P}$ as a quotient space $T^2\mathbf{E}_{/\mathcal{P}}$. But a

reduced representation by means of $T^2\mathbb{E}_{/\mathcal{P}}$ is more simple for the equivalence classes have a unique representative for each time $\tau \in T$.

PROPOSITION.

Let $v \in T^2\mathbb{P}$. Then

$$\mathbb{C}_v \equiv T^2p^{-1}(v) = (T^2_2\, P)_v(\mathbb{T}) \to T^2\mathbb{E} \qquad \text{(a)}$$

is a C^∞ submanifold.

Then we get a partition of $T^2\mathbb{E}$, given by

$$T^2\mathbb{E} = \bigsqcup_{v \in T^2\mathbb{P}} \mathbb{C}_v \ ,$$

and the quotient space $T^2\mathbb{E}_{/\mathcal{P}}$, which has a natural C^∞ structure and whose equivalence classes are characterized by

$$[e,u,v,w] = [e',u',v',w'] \Longleftrightarrow p(e)=p(e'),\ \check{P}(t(e'),e)(u) = u' ,$$
$$\check{P}(t(e'),e)(v) = v',\ \check{\check{P}}(t(e'),e)(u,v) + \check{P}(t(e'),e)(w) = w' \qquad \text{(b)}$$

We get a natural C^∞ diffeomorphism between $T^2\mathbb{P}$ and $\check{T}^2\mathbb{E}_{/\mathcal{P}}$ given by the unique maps

$$T^2\mathbb{P} \to \check{T}^2\mathbb{E}_{/\mathcal{P}} \qquad \text{and} \qquad \check{T}^2\mathbb{E}_{/\mathcal{P}} \to T^2\mathbb{P}$$

which make commutative the following diagrams, respectively,

$$\begin{array}{ccc} \mathbb{T} \times T^2\,\mathbb{P} & \xrightarrow{T^2_2 P} & \check{T}^2\mathbb{E} \\ \pi^2 \downarrow & & \downarrow \\ T^2\,\mathbb{P} & \longrightarrow & \check{T}^2\mathbb{E}_{/\mathcal{P}} \end{array} \qquad\qquad \begin{array}{ccc} \check{T}^2\mathbb{E} & \longrightarrow & T^2\mathbb{P} \\ & \searrow \quad \nearrow & \\ & T^2\mathbb{E}_{/\mathcal{P}} & \end{array}$$

PROOF.

Analogous to (II,2,3) $\underline{\cdot}$

4 Choicing a time $\tau \in T$ and taking, for each equivalence class, its representative at the time τ, we get a second interesting representation of $T^2\mathbb{P}$.

PROPOSITION.

The maps $T^2 P_\tau$ and $T^2 p_\tau$ are inverse C^∞ diffeomorphisms

$$T^2\tilde{P}_\tau : T^2 P \to T^2\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \times \bar{\mathbb{S}}\ , \quad T^2 p_\tau : T^2\mathbb{S}_\tau \equiv \mathbb{S}_\tau \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \times \bar{\mathbb{S}} \to T^2 P\ .$$

5 The relation among the different representations of $T^2\mathbb{P}$ is shown by the following commutative diagram

$$\begin{array}{ccccc}
 & T^2\mathbb{P} & \hookrightarrow & T^2\mathbb{E}_{/P} \\
T^2\tilde{P}_\tau \swarrow & & \nwarrow T^2\tilde{p}_\tau & & \uparrow \\
T^2\mathbb{S}_\tau & \longleftarrow & T^2\mathbb{S}_\tau & \hookrightarrow & T\,\mathbb{E}
\end{array}$$

6 The previous representations of $T^2\mathbb{P}$ reduce to analogous representations of $v\,T^2\mathbb{P}$.

COROLLARY.

The quotient space $(vT^2\mathbb{E})_{/\mathcal{P}}$ is a C^∞ submanifold of $T^2\mathbb{E}_{/\mathcal{P}}$ and its equivalence classes are characterized by

$[e,u,o,w] = [e',u',o,w] \Longleftrightarrow p(e) = p(e'),\ P(t(e'),e)(u) = u',$
$P(t(e')e)(w) = w'.$

The diffeomorphism $T^2 P \to T^2\mathbb{E}_{/\mathcal{P}}$ induces a diffeomorphism

$$vT^2\mathbb{P} \to (vT^2\mathbb{E})_{/\mathcal{P}}$$

and the diffeomorphism $T^2\mathbb{E}_{/\mathcal{P}} \to T^2\mathbb{P}$ induces the inverse diffeomorphism

$$(vT^2\mathbb{E})_{/\mathcal{P}} \to vT^2\mathbb{P}\ .$$

Moreover, the following diagrams are commutative

$$\begin{array}{ccc} \nu T^2\,\mathbb{E} & \xrightarrow{\ \amalg\ } & T\,\mathbb{E} \\ {\scriptstyle T^2 p}\downarrow & & \downarrow{\scriptstyle T\,p} \\ \nu T^2\mathbb{P} & \xrightarrow[\amalg_{\mathcal{P}}]{} & T\,\mathbb{P} \end{array} \qquad\qquad \begin{array}{ccc} \nu T^2\mathbb{S}_\tau & \xrightarrow{\ \amalg\ } & T\,\mathbb{S}_\tau \\ \downarrow & & \downarrow \\ \nu T^2\mathbb{P} & \xrightarrow[\amalg_{\mathcal{P}}]{} & T\,\mathbb{P} \end{array}$$

$\dot{-}$

7 Taking into account the identification $T^2\mathbb{P} \overset{\sim}{=} \check{T}^2\mathbb{E}_{/\mathcal{P}}$, we get the following expression of T^2p and T^2P.

PROPOSITION.

a) $T^2p(e,u,v,w) = [e, P(e)(u), \hat{P}(e)(v), \hat{\hat{P}}(e)(u,v) + \hat{P}(e)(w)]$

b) $T^2P(\tau,\lambda,\mu,\nu;\ [e,u,v,w]) =$

$= (\tilde{P}(\tau,e),\ \lambda\bar{P}(\tilde{P}(\tau,e)) + P(\tau,e)(u),\ \mu\bar{P}(P^2(\tau,e)) + P(\tau,e)(v)\ ,$

$\lambda\mu\ \bar{\bar{P}}(P(\tau,e) + \lambda D\bar{P}(P(\tau,e))(P(\tau,e)(v)) + \lambda D\bar{P}(P(\tau,e)(P(\tau,e)(u)) + \nu\bar{P}(e^2(\tau,e) +$

$+ P(\tau,e)(u,v) + P(\tau,e)(w)$

PROOF.

Analogous to (II,2,6) $\dot{-}$

<u>Frame connection and Cariolis map.</u>

8 For each $\tau \in T$, we can view $\mathbb{P}$ as an affine space ,depending on τ, taking into account the isomorphism $\mathbb{T} \times T\mathbb{P} \to T\mathbb{E}$. Hence we get a "time depending" affine connection on $\mathbb{P}$

$$\check{\Gamma}_P : \mathbb{T} \times s\ T^2\,P \to \nu T^2\,\mathbb{P}.$$

THEOREM.

There is a unique map

$$\check{\Gamma}_{\mathcal{P}} : \mathbb{T} \times s\, T^2\, \mathbb{P} \to \nu T^2\, \mathbb{P} .$$

such that the following diagram is commutative

$$\begin{array}{ccc} s\, T^2\, \mathbb{E} & \xrightarrow{\Gamma} & \nu T^2\, \mathbb{E} \\ (t, T^2 p) \downarrow & & \downarrow T^2 p \\ \mathbb{T} \times s T^2 \mathbb{P} & \xrightarrow{\check{\Gamma}_{\mathcal{P}}} & \nu T^2 \mathbb{P} \end{array} .$$

Such a map is given by the following commutative diagram

$$\begin{array}{ccc} s\, \check{T}^2\, \mathbb{E} & \xrightarrow{\Gamma} & \nu \check{T}^2\, \mathbb{E} \\ \uparrow (T^2 P)_{o,o,o} & & \downarrow T^2 p \\ \mathbb{T} \times s\, T^2\, \mathbb{P} & \xrightarrow{\check{\Gamma}_{\mathcal{P}}} & \nu T^2\, \mathbb{P} \end{array}$$

Namely we get

$$\check{\Gamma}_p(\tau, [e,u,u,w]) = [\tilde{P}(\tau,e), \check{P}(\tau,e)(u);\ 0, \check{\check{P}}(\tau,e)(u,u) + \check{P}(\tau,e)(w)] ,$$

hence, if $t(e) \equiv \tau$

$$\check{\Gamma}_p(\tau, [e,u,u,w]) = [e,u,o,w] .$$

PROOF.

$(t, T^2 p)$ is $T^2 \mathbb{E} \to \mathbb{T} \times s\, T^2 \mathbb{P}$ and $(T^2 P)_{(o,o,o)} : \mathbb{T} \times s\, T^2 \mathbb{P} \to s\, \check{T}^2 \mathbb{E}$

are inverse C^∞ diffeomorphisms.

9 Then we can introduce the "following map", that will be used (III,1) to define the covariant derivative of maps $\mathbb{T} \to T\mathbb{P}$, hence the acceleration of observed motion .

DEFINITION.

The FRAME TIME DEPENDING AFFINE CONNECTION is the map

$$\acute{\Gamma}_{\mathbb{P}} : \mathbb{T} \times s\, T^2\mathbb{P} \rightarrow \nu T^2\mathbb{P} ,$$

given by $(\tau, [e,u,u,w]) \mapsto [\tilde{P}(\tau,e), \check{P}(\tau,e)(u); 0, \check{\acute{P}}(\tau,e)(u,u\ ) + \check{P}(\tau,e)(w)]$

10 The time depending affine connection $\check{\Gamma}_{\mathbb{P}}$ does not sufficies for Kinematics. Coriolis theorem, (III,1) which makes a comparison between the acceleration of an observed motion and the observed acceleration of a motion, requires a further map $\dot{\Gamma}_{\mathbb{P}} : \mathbb{T} \times s\ T^2\ \mathbb{P} \rightarrow \nu\ T^2\ \mathbb{P}$, which is obtei_ned taking into account the isomorphism $\mathbb{T} \times T\mathbb{P} \rightarrow \dot{T}\mathbb{E}$.

THEOREM.

There is a unique map

$$\dot{\Gamma}_{\mathbb{P}} : \mathbb{T} \times s\, T^2\mathbb{P} \rightarrow \nu T^2\mathbb{P}$$

such that the following diagram is commutative

$$\begin{array}{ccc} \dot{T}^2\mathbb{E} & \xrightarrow{\Gamma} & \nu \dot{T}^2\mathbb{E} \\ {\scriptstyle (\ddot{t}, T^2 p)} \downarrow & & \downarrow {\scriptstyle T^2 p} \\ \mathbb{T} \times s\, T^2\mathbb{P} & \xrightarrow{\dot{\Gamma}_{\mathbb{P}}} & \nu T^2\mathbb{P} \end{array}$$

Such a map is given by the following commutative diagram

$$\begin{array}{ccc} \dot{T}^2\mathbb{E} & \xrightarrow{\dot{\Gamma}} & \nu \dot{T}^2\mathbb{E} \\ {\scriptstyle (T^2\mathbb{P})_{(1,1,0)}} \uparrow & & \downarrow {\scriptstyle T^2 p} \\ \mathbb{T} \times s T^2\mathbb{P} & \xrightarrow{\dot{\Gamma}_{\mathbb{P}}} & \nu T^2\mathbb{P} \end{array}$$

Namely we get

$\dot{\Gamma}_{\mathbb{P}}(\tau, [e,u,u,w]) =$

$= [\tilde{P}(\tau,e), \check{P}(\tau,e)(u), 0, \acute{P}(\tau,e)(w) + 2\acute{\bar{P}}(\tau,e)(P(\tau,e)(u)) + \bar{\bar{P}}(\grave{P}(\tau,e))]$

hence, if $t(e) \equiv \tau$,

$$\acute{\Gamma}_{\mathcal{P}}(\tau,[e,u,u,w]) = [e,u,0,w + 2\,\check{\bar{P}}(e)(u) + \bar{\bar{P}}(e)] \;.$$

Thus we have

$$\acute{\Gamma}_{\mathcal{P}} = \check{\Gamma}_{\mathcal{P}} + \acute{C}_{\mathcal{P}} + \check{\acute{D}}_{\mathcal{P}} \;,$$

where

$C_{\mathcal{P}} : \mathbb{T} \times T\,\mathbb{P} \to T\,\mathbb{P}$ and $D_{\mathcal{P}} : \mathbb{T} \times \mathbb{P} \to T\,\mathbb{P}$ are given by

$$C_{\mathcal{P}}(\tau,[e,u]) \equiv [\tilde{P}(\tau,e)\;,\; 2\check{\bar{P}}(\tilde{P}(\tau,e))(u)]$$

$$D_{\mathcal{P}}(\tau,e) \equiv [\;\tilde{P}(\tau,e)\;,\; \bar{\bar{P}}\;(\tilde{P}(\tau,e))]$$

hence

$\check{C}_{\mathcal{P}} : \mathbb{T} \times s\,T^2\mathbb{P} \to \nu T^2\mathbb{P}$ and $\check{D}_{\mathcal{P}} : \mathbb{T} \times s\,T^2\mathbb{P} \to \nu T^2\mathbb{P}$

are given by

$$\check{C}_{\mathcal{P}}(\tau,[e,u,u,w]) \equiv [\tilde{P}(\tau,e),\check{P}(\tau,e)(u),0,\; 2\,\check{\bar{P}}(\tilde{P}(\tau,e))(\check{P}(\tau,e)(u))\;]$$

$$\check{\acute{D}}_{\mathcal{P}}(\tau,[e,u,u,w]) \equiv [\tilde{P}(\tau,e),\check{P}(\tau,e)(u),0,\; \bar{\bar{P}}(\tilde{P}(\tau,e)]$$

PROOF.

$(\check{\acute{t}},T^2p) : \acute{T}^2\mathbb{E} \to \mathbb{T} \times sT^2\mathbb{P}$ and $(T^2\mathbb{P})_{(1,1,0)} : \mathbb{T} \times sT^2\mathbb{P} \to \acute{T}^2\mathbb{E}$

are inverse C^∞ diffeomorphisms.

11 Then we can give the following definition

DEFINITION.

The FRAME CORIOLIS MAP is the map

$$C_{\mathcal{P}} : \mathbb{T} \times T\mathbb{P} \to T\mathbb{P}$$

given by

$$(\tau,[e,u]) \mapsto [\tilde{P}(\tau,e),\; 2\,\check{\bar{P}}(P(\tau,e))(u)]$$

The FRAME DRAGGING MAP is the map

$$D_{\mathcal{P}} : \mathbb{T} \times \mathbb{P} \to T\,\mathbb{P}$$

given by $$(\tau, [e]) \to [\tilde{P}(\tau, e), \bar{\bar{P}}(P(\tau, e))] \quad .$$

Physical description.

$\bar{\bar{P}}$ is the field of acceleration of the field continuum. $\mathcal{E}_{\mathcal{P}}$ is the rate of change, during time, of the spatial metric; $\Omega_{\mathcal{P}}$ describes the rate of change, during time, of the spatial directions. This facts are implicitly proved in the next section.

It is not easy to describe by picture the fundamental ,but not straight forward,results of this section.

4 SPECIAL FRAMES.

A classification of the most important types of frames can be performed taking into account the vanishing of quantities occurring, in $D\bar{P}$.
So we get a chain of four types, characterized by a more and more rich structure of the fosition space $\mathbb{P}$.

<u>Affine frames</u>.

1 DEFINITION

The frame $\mathcal{P}$ is AFFINE if

$$\check{D}^2 \bar{P} = 0 \; .$$

2 We have interesting characterizations of affine frames.

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is affine.

b) $\check{D}\bar{P}$ depends only on time, i.e. $\check{D}\bar{P}$ is factorizable as follows

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\check{D}\bar{P}} & \\ \downarrow t & & \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \\ \mathbb{T} & \nearrow & \end{array}$$

c) We have $\quad \check{\check{P}} = 0$

d) $\check{P}$ depends only on time, i.e. $\check{P}$ is factorizable as follows

$$\begin{array}{ccc} \mathbb{T} \times \mathbb{E} & \xrightarrow{\check{P}} & \\ \downarrow id_T x t & & \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \\ \mathbb{T} \times \mathbb{T} & \nearrow & \end{array}$$

e) Let $\sigma \in \mathbb{T}$; then , $\forall \tau \in T$, the map

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S}_\sigma \to \mathbb{S}_\tau$$

is affine, i.e.

$$\tilde{P}_{e'}(\tau) = \check{P}_e(\tau) + \check{P}_{(\tau,\sigma)} \quad (e' - e) \ .$$

f) $\forall\ \tau \in \mathbb{T}$, the map

$$\bar{P}_{|\$_\tau} : \$_\tau \to \mathbb{U}$$

is affine, i.e.

$$\bar{P}(e') = \bar{P}(e) + \frac{1}{2}\,\epsilon_{P}(\tau)(e'-e) + \Omega_{P}(\tau) \times (e'-e) \ .$$

PROOF.

It sufficies to prove f) $\Longrightarrow$ e), the other implications being immediate. f) $\Longrightarrow$ e).

Let $$D_1\tilde{P}(\tau,e') = D_1\tilde{P}(\tau,e) + \check{D}_2 D_1 \tilde{P}(\tau)\ (e'-e),$$

with $$t(e) \equiv \sigma \equiv t(e') \ .$$

Then, by integration, we get

$$\tilde{P}(\tau,e') = \tilde{P}(\tau,e) + A(\tau)(e'-e) + B(\tau,e-e')$$

where

$$A(\tau) : \bar{\$} \to \bar{\$}$$

is a linear map.

Moreover, for (II,1.10 a) and (II.1.10 b) also B is linear with respect to $(e-e')$.

Then $$\tilde{P}(\tau,e') = \tilde{P}(\tau,e) + \check{D}_2\tilde{P}(\tau)(e'-e) \ .$$

Here by abuse of notation we have written

$$\check{D}\,\bar{P} : \mathbb{T} \to \bar{\$}^* \otimes \bar{\$} \ ; \ \check{P} : \mathbb{T} \times \mathbb{T} \to \bar{\$}^* \otimes \$ \ , \ \dots$$

as $\check{D}\bar{P}$, $\check{\check{P}}$, ... depend only on time.

Hence the motion of an affine frame $\mathcal{P}$ is characterized by the motion of one of its particles

$$P_q : \mathbb{T} \to \mathbb{E} \qquad \text{and by} \qquad \epsilon_{\mathcal{P}} : \mathbb{T} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}} \ , \ \bar{\Omega} : \mathbb{T} \to \bar{\mathbb{S}} \ .$$

3 Let $\mathcal{P}$ be affine. since $\check{P}$ depends only on time, we can get a reduction of the representation of $T\,\mathbb{P}$ by $\check{T}\mathbb{E}_{/\mathcal{P}}$, writing

$$(\mathbb{E} \times \bar{\mathbb{S}})_{/\mathcal{P}} \stackrel{\sim}{=} (\mathbb{P} \times \mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} = \mathbb{P} \times (\mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} \ .$$

THEOREM.

a) Let $\bar{\mathbb{P}}$ be the quotient space

$$\bar{\mathbb{P}} \equiv (\mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} \ ,$$

given by $\qquad [\tau, u] = [\tau', u'] \iff u' = \check{P}_{(\tau', \tau)}(u)$.

Then $\bar{\mathbb{P}}$ results into a vector space, putting

$$\lambda\,[\tau, u] \equiv [\tau, \lambda u]$$

$$[\tau, u] + [\tau', u'] \equiv [\tau, u + \check{P}_{(\tau, \tau')}(u)]$$

For each $\tau \in \mathbb{T}$, the map

$$\begin{aligned} \bar{\mathbb{P}} &\longrightarrow \bar{\mathbb{S}} \\ [\tau', u] &\longrightarrow \check{P}_{(\tau, \tau')}(u) \ , \end{aligned}$$

is an isomorphism.

b) Let $\sigma_{\mathcal{P}}$ be the map

$$\sigma_{\mathcal{P}} : \mathbb{P} \times \bar{\mathbb{P}} \to \mathbb{P},$$

given by $(q,[\tau,u]) \mapsto p(\mathcal{P}(\tau,q)+u)$.

Then the triple $(\mathbb{P},\bar{\mathbb{P}},\sigma_{\mathcal{P}})$ is a three dimensional affine space.

c) For each $\tau \in \mathbb{T}$, the maps

$$p_\tau : \mathbb{S}_\tau \to \mathbb{P} \qquad \text{and} \qquad \mathcal{P}_\tau : \mathbb{P} \to \mathbb{S}_\tau$$

are affine isomorphisms.

d) We get the splittings $T\,\mathbb{P} = \mathbb{P} \times \mathbb{P}$ and $T^2\mathbb{P} = \mathbb{P} \times \bar{\mathbb{P}} \times \bar{\mathbb{P}} \times \bar{\mathbb{P}}$,
writing

$$[e,u] = (p(e),[t(e),u]) \quad \text{and} \quad [e,u,v,w] = (p(e),[t(e)u],[t(e),v][t(e)w]))$$

$\Gamma_{\mathcal{P}}$ results to be time independent and it is the affine connection of $\mathbb{P}$

$$\check{\Gamma}_{\mathcal{P}} : s\,T^2\mathbb{P} \to {}_{\nu}T^2\,\mathbb{P}$$

$$(q,[\tau,u],[\tau,u],[\tau,w]) \mapsto (q,[\tau,u],0,[\tau,w]) \ .$$

PROOF.

It follows from the fact that, $\forall\ \tau',\tau \in \mathbb{T}$, the map

$$\tilde{P}_{(\tau',\tau)} : \mathbb{S}_\tau \to \mathbb{S}_{\tau'}$$

is affine and from the properties

$$\tilde{P}_{(\tau'',\tau')} \circ \tilde{P}_{(\tau',\tau)} = \tilde{P}_{(\tau'',\tau)} \quad , \quad \tilde{P}_{(\tau,\tau)} = id_{\mathbb{S}_\tau}$$

4 We get simplified formulas for $T\,p$, T^2p, $T\,P$, T^2P and $\acute{\Gamma}_{\mathcal{P}}$.

COROLLARY.

We have

a) $T\,p(e,u) = (p(e),\ [t(e),\ \hat{P}(e)(u)])$.

b) $T P(\tau,\lambda;q,[\tau',u]) = (P(\tau,q), \lambda\bar{P}(P(\tau,q) + P_{(\tau,\tau')}(u))$

c) $\dot{\Gamma}_{\mathcal{P}}(\tau;q,[\tau,u],[\tau,v],[\tau,w]) =$

$$= (q,[\tau,u],0,[\tau,w + \epsilon_p(\tau)\ (u) + 2\Omega_p(\tau) \times u + \bar{\bar{P}}(P(\tau,q))]).$$

<u>Rigid frames</u>.

5 DEFINITION.

The frame $\mathcal{P}$ is RIGID if it is affine and

$$\epsilon_{\mathcal{P}} = 0 \quad .$$

6 We have interesting characterizations of rigid frames.

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is rigid.

b) Let $\sigma \in \mathbb{T}$; then, $\forall\ \tau \in \mathbb{T}$, the map

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S} \to \mathbb{S}_\tau$$

preserves the distances, i.e.

$$||\tilde{P}_{(\tau,\sigma)}(e) - \tilde{P}_{(\tau,\sigma)}(e')\ || = ||e-e'||$$

c) $\forall\ \sigma\in\mathbb{T}$, the map

$$\bar{P}_{|\mathbb{S}_\sigma} : \mathbb{S}_\sigma \to \mathbb{U}$$

is affine and

$$\bar{P}(e') = \bar{P}(e) + \Omega_p(\sigma) \times (e'-e) \quad .$$

d) We have $\check{\check{P}} = 0$ and $\check{P} : \mathbb{T} \times \mathbb{E} \to \mathbb{S}\,U(\bar{\mathbb{S}})$.

PROOF.

a)<==>c) trivial.

a) ==> b) $\epsilon_{\mathcal{P}}$ is the Lie derivative of g with respect $\bar{P}$, i.e. the derivative with respect to time of the deformations tensor $g \circ (\check{P},\check{P}) - g$. Then $\epsilon_{\mathcal{P}} = 0$, by integration with respect to time, gives the result.

b) ==>d) It is known (the proof is a purely algebric computation, making use of an orthogonal basis) that, if A is an affine euclidean space and $f : A \to A$ is a map which preserves the norm, then f is an affine map with unitary derivative. Then we see that $P(\tau,\sigma)$ is affine adn $\tilde{D}P_{(\tau,\sigma)} \in U(\bar{\mathbb{S}})$.

d) ==>a) $\check{P}_{(\tau',\tau)} \in U(\mathbb{S})$ gives

$$\check{P}_{(\tau,\tau')} = \check{P}^t_{(\tau',\tau)}$$

hence, deriving respect to τ'

$$\check{P}_{(\tau',\tau)} \circ \check{P}^t_{(\tau',\tau)} = id_{\bar{\mathbb{S}}} ,$$

we get

$$D_1\check{P}_{(\tau',\tau)} \circ \check{P}^t_{(\tau',\tau)} + \check{P}_{(\tau',\tau)} \circ D_1 \check{P}^t_{(\tau',\tau)} = 0$$

and, for $\tau' \equiv \tau$,

$$\epsilon_{\mathcal{P}}(\tau) = S\, D_1 \check{P}_{(\tau,\tau)} = D_1 \check{P}_{(\tau,\tau)} + D_1\check{P}^t_{(\tau,\tau)} = 0 \quad \dot{-}$$

Hence the motion of a rigid frame $\mathcal{P}$ is characterized by the motion of one of its particles $\mathcal{P}_q : \mathbb{T} \to \mathbb{E}$ and by $\Omega_{\mathcal{P}} : \mathbb{T} \to \bar{\mathbb{S}}$.

7 Let $\mathcal{P}$ be rigid.

THEOREM.

$\mathcal{P}$ results into an affine euclidean space. In fact $g_{\mathcal{P}}$ results to be time independent and we can define the map

$$g_{\mathcal{P}} : \bar{\mathbb{P}} \to \mathbb{R}$$

which is given by $\qquad [\tau, u] \mapsto \frac{1}{2} u^2 \quad .$

The affine connection $\check{\Gamma}_{\mathcal{P}}$ results into the Riemannian connection of $\mathcal{P}$.

Translating frames.

8 DEFINITION

A frame $\mathcal{P}$ is TRANSLATING if it is rigid and

$$\Omega_{\mathcal{P}} = 0 \quad .$$

9 We have interesting characterizations of translating frames .

PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is translating

b) Let $\sigma \in \mathbb{T}$; then $\forall \ \tau \in T$, the map,

$$\tilde{P}_{(\tau,\sigma)} : \mathbb{S}_{\sigma} \to \mathbb{S}_{\tau}$$

is affine, with derivative $\quad D\,\tilde{P}_{(\tau,\sigma)} = id_{\bar{\mathbb{S}}}$, i.e.

$$\tilde{P}_{e'}(\tau) = \tilde{P}_{e}(\tau) + (e'-e) \quad .$$

c) $\forall \ \tau \in \mathbb{T}$, the map

$$\bar{P}_{|\mathbb{S}_{\tau}} : \mathbb{S}_{\tau} \to \mathbb{U}$$

is constant, i.e. $\quad \bar{P}(e') = \bar{P}(e) \quad .$

Hence the motion of a translating frame is characterized by the motion of one of its particles $P_q : \mathbb{T} \to E$.

We will write $\bar{P} : \mathbb{T} \to \mathbb{U}$, $\bar{\bar{P}} = D\bar{P} : \check{\mathbb{T}} \to \bar{\mathbb{S}}$. $\hat{P} = d_{\mathbb{E}} - \underline{t} \otimes \bar{P} : \mathbb{T} \to \bar{\mathbb{E}}^* \otimes \bar{\mathbb{S}}$.

10 Let $\mathcal{P}$ be translating. Since $\check{P} = id_{\bar{\mathbb{S}}}$, we can get a further reduction of the representation of $T\mathbb{P}$ by $\check{T}\mathbb{E}_{/\mathcal{P}}$, writing

$$(\mathbb{E} \times \bar{\mathbb{S}})_{/\mathcal{P}} \cong (\mathbb{P} \times \mathbb{T} \times \bar{\mathbb{S}})_{/\mathcal{P}} = \mathbb{P} \times \bar{\mathbb{S}} \quad .$$

THEOREM.

Let $\mathcal{P}$ be translating.

a) The map

$$\bar{\mathbb{P}} \to \bar{\mathbb{S}} ,$$

given by

$$[\tau,u] \to u ,$$

is well defined and it is an isomorfism.

Then the map

$$\sigma_P : \mathbb{P} \times \bar{\mathbb{S}} \to \mathbb{P} ,$$

given by

$$(q,u) \to p(P(\tau,q)+u) ,$$

does not depend on the choice of $\tau \in \mathbb{T}$.

b) The triple $(\mathbb{P}, \mathbb{S}, \epsilon_{\mathcal{P}})$ is an affine euclidean space,

11 We get simplified formulas for $T\,p$, $T^2 p$, $T\,P$, $T^2 P$, $\Gamma_{\mathcal{P}}$.

PROPOSITION

Let P translating

a) $T\,p(e,u) = (p(e), u - u^\circ\, \bar{P}(t(e)))$

$$T^2 p\,(e,u,v,w) = (p(e), u-u^\circ\, \bar{P}(t(e))\ v-v^\circ\, \bar{P}(t(e)), w-w^\circ\, \bar{P}(t(e)) + \bar{\bar{P}}(t(e))).$$

b) $TP(\tau,\lambda;q,u) = (P(\tau,q),\lambda\bar{P}(\tau) + u$,

$$T^2P(\tau,\lambda,\mu,\nu;q,u,v,w) = (P(\tau,q),\lambda\bar{P}(\tau)+u,\mu\bar{P}(\tau)+\nu,\lambda\bar{\bar{P}}(\tau)+\nu\bar{P}(\tau)+w)$$

c) $\dot{\Gamma}_{\mathcal{P}}(\tau;q,u,u,w) = (q,u,0,w + \bar{\bar{P}}(\tau))$.

<u>Inertial frames.</u>

12 DEFINITION.

A frame P is inertial if it is translating and

$$\bar{\bar{P}} = 0.$$

13 PROPOSITION.

The following conditions are equivalent.

a) $\mathcal{P}$ is inertial,

b) $\mathcal{P}$ is translating and $D\,\bar{P} = 0$.

c) $\tilde{P}$ is are affine map, i.e. (taking into account the properties (II.1.10)

$$\tilde{P}(\tau,e) = e + \bar{P}(\tau-t(e)), \qquad \text{with } \bar{P} \in U .$$

d) $\bar{P} : \mathbb{E} \to \mathbb{U}$ is a constant map .

Hence an inertial frame is characterized by its constant velocity.

14 PROPOSITION.

We have

a) $T\,p(e,u) = (p(e),u-u^\circ\bar{P})$

$$T^2p(e,u,v,w) = (p(e),u-u^\circ\bar{P},v-v^\circ\bar{P}, w-w^\circ\bar{P})$$

b) $T\,P(\tau,\lambda;q,u) = (P(\tau,q),\lambda\bar{P}+u)$

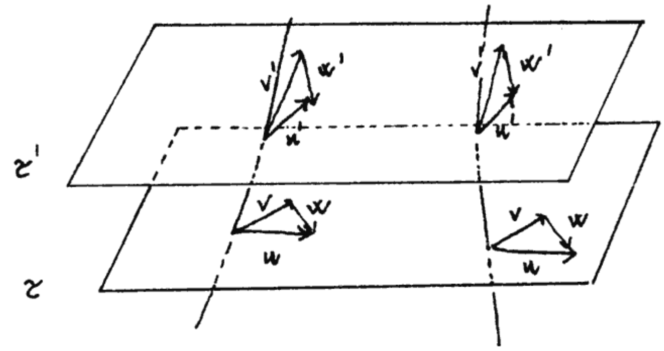
$$T^2P(\tau,\lambda,\mu,v;q,u,v,w) = (P(\tau,q),\lambda\bar{P} + u,\mu\bar{P}+v,\nu\bar{P}+w)$$

c) $\dot{\Gamma}_P$ results time independent and we get

$$\dot{\Gamma}_{\mathcal{P}} = \Gamma_{\mathcal{P}}$$

Physical description.

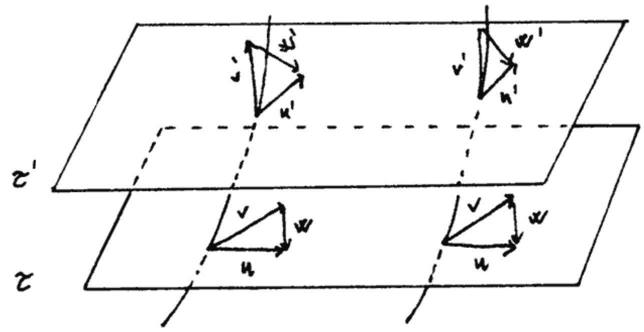
A frame $\mathcal{P}$ is affine if it preserves during the motion the spatial parallelogram rule; it is rigid if moreover it preserves spatial leghts (hence also angles); it is translating if moreover it preserves spatial directions; it is inertial if its world-lines are parallel straight-lines. We can describe the four cases by a picture .



$w' = u'+v'$

affine frame

$w = u + v$

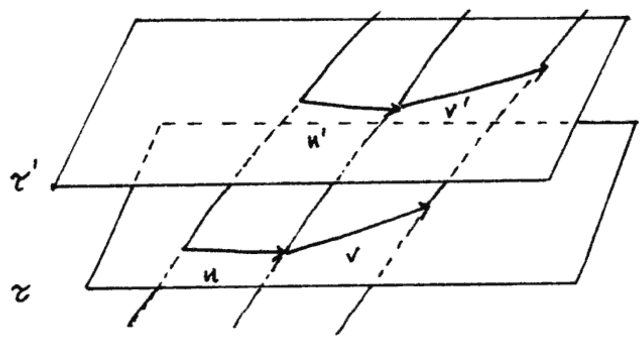


$w' = u' + v'$

$||u||=||u'||, ||v||= ||v'||, ||w||= ||w'||$

rigid frame

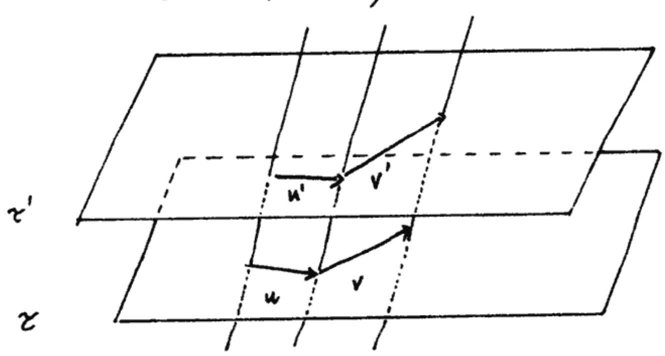
$w = u + v$



$u' = u$

$v' = v$

translating frame



$u' = u$

$v' = v$

inertial frame

III CHAPTER
OBSERVED KINEMATICS

Here we analyse the one- body kinematics in terms of the positions determined by a frame , introducing the observed motion and its velocity and acceleration. By comparison between the absolute and the observed motion we get the "absolute" velocity addition and Coriolis theorem. Finally we make the comparison between the observed motions relative to two frames, getting the velocity addition and Coriolis theorem.

1 OBSERVED KINEMATICS.

Let $\mathcal{P}$ a fixed frame and let M be a fixed motion. We analyse M as viewed by $\mathcal{P}$.

1 We first introduce useful notations.

Let $f : \mathbb{T} \to \mathbb{P}$ be a C^∞ map.

a) We put
$$\check{f} \equiv (id_{\mathbb{T}}, f) : \mathbb{T} \to \mathbb{T} \times \mathbb{P}$$
$$\check{d}f \equiv (id_{\mathbb{T}}, df) : \mathbb{T} \to \mathbb{T} \times T\mathbb{P},$$
$$\check{d}^2 f \equiv (id_{\mathbb{T}}, d^2 f) : \mathbb{T} \to \mathbb{T} \times T^2\mathbb{P} .$$

b) df and d^2f being functions on $\mathbb{T}$, we can choose a natural representative of the equivalence classes of $T\mathbb{P}$ and $T^2\mathbb{P}$. So we put

$$df \equiv [f, D_{\mathcal{P}} f]$$

and we get
$$d^2 f \equiv [f, D_{\mathcal{P}} f, D_{\mathcal{P}} f, D^2_{\mathcal{P}} f]$$

where
$$D_{\mathcal{P}} f : \mathbb{T} \to \bar{\mathbb{S}} \quad \text{and} \quad D^2_{\mathcal{P}} f : \mathbb{T} \to \bar{\mathbb{S}}$$

resemble derivatives of affine spaces, but are note properly such.

c) We put
$$\check{\nabla}_p df \equiv \amalg_p \circ \check{\Gamma}_p \circ \check{d}^2 f : \mathbb{T} \to T\mathbb{P}$$
$$\dot{\nabla}_p df \equiv \amalg_p \circ \dot{\Gamma}_p \circ \check{d}^2 f : \mathbb{T} \to T\mathbb{P}.$$

Observed motion and absolute velocity addition and Coriolis theorem.

2 The basic definition of observed kinematics is the following.

DEFINITION.

a) The MOTION OF M OBSERVED BY $\mathcal{P}$ is the map

$$M_{\mathcal{P}} \equiv p \circ M : \mathbb{T} \to \mathbb{P} .$$

b) The VELOCITY OF M OBSERVED BY $\mathcal{P}$ is the map

$$(d\, M)_{\mathcal{P}} \equiv T p \circ d M : \mathbb{T} \to T \mathbb{P}.$$

The VELOCITY OF THE OBSERVED MOTION $M_{\mathcal{P}}$ is the map

$$d\, M_{\mathcal{P}} : T \to T\, \mathbb{P}\ .$$

c) The ACCELERATION OF M OBSERVED BY $\mathcal{P}$ is the map

$$(\nabla\, d\, M)_{\mathcal{P}} \equiv T\, p \circ \nabla\, d\, M : T \to T\, \mathbb{P}\ .$$

The ACCELERATION OF THE OBSERVED MOTION $M_{\mathcal{P}}$ is the map

$$\check{\nabla}_{\mathcal{P}}\, \check{d}\, M_{\mathcal{P}} \equiv \perp\!\!\perp_{\mathcal{P}} \circ \check{\Gamma}_{\mathcal{P}} \circ \check{d}^2 M_{\mathcal{P}} : T \to T\, \mathbb{P}$$

3 We can make the comparison between the observed entities and the entities of the observed motion.

THEOREM. " ABSOLUTE VELOCITY ADDITION AND CORIOLIS THEOREM"

a) $$M = P \circ \check{M}_{\mathcal{P}}$$

i.e., putting $$\mathbb{E} \overset{\sim}{=} \mathbb{T} \times \mathbb{P},$$
$$M \overset{\sim}{=} \check{M}_{\mathcal{P}}\ .$$

b) $[M, DM - \bar{P} \circ M] = (\, d\, M)_{\mathcal{P}} = d\, M_{\mathcal{P}} \equiv [M, D_{\mathcal{P}} M_{\mathcal{P}}]$

i.e. $$D\, M - \bar{P} \circ M = D_{\mathcal{P}} M_{\mathcal{P}}\ .$$

c) $[M, D^2 M] = (\nabla\, d\, M)_{\mathcal{P}} = \check{\nabla}_{\mathcal{P}} \check{d}\, M_{\mathcal{P}} =$

$$= [M, D^2_{\mathcal{P}} M_{\mathcal{P}} + \epsilon_{\mathcal{P}} \circ \check{M}_{\mathcal{P}} (D_{\mathcal{P}} M_{\mathcal{P}}) + 2 \Omega_{\mathcal{P}} \circ \check{M}_{\mathcal{P}} \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ M_{\mathcal{P}})$$

i.e. $$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} + (\epsilon_{\mathcal{P}} \circ \check{M}_{\mathcal{P}})(D_{\mathcal{P}} M_{\mathcal{P}}) + 2(\Omega_{\mathcal{P}} \circ \check{M}_{\mathcal{P}}) \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ \bar{M}_{\mathcal{P}}\ .$$

PROOF.

a) $M = P \circ (t,p) \circ M = P \circ (id_{\mathbb{T}}, M_{\mathcal{P}}) \equiv P \circ \check{M}_{\mathcal{P}}\ .$

b) $(d\,M)_{\mathcal{P}} = T\,p \circ d\,M = d(p \circ M) = d\,M_{\mathcal{P}}$.

c) $(\nabla\, d\,M)_{\mathcal{P}} = T\,p \circ \sqcup \circ \Gamma \circ d^2 M = \sqcup_{\mathcal{P}} \circ T^2 p \circ \Gamma' \circ d^2 M =$

$$= \sqcup_{\mathcal{P}} \circ \Gamma'_{\mathcal{P}} \circ \overleftarrow{d}^2 M \quad \dot{-}$$

4 COROLLARY.

We have

$$\underset{\frown}{x}^k \circ M_{\mathcal{P}} = M^k \equiv x^k \circ M$$

$$\underset{\frown}{\dot{x}}^k \circ d\,M_{\mathcal{P}} = D\,M^k$$

$$\underset{\frown}{\dot{x}}^k \circ \nabla'_{\mathcal{P}} d\,M_{\mathcal{P}} = D^2 M^k + (\Gamma^k_{ij} \circ \check{M}_{\mathcal{P}})\,D\,M^i\,D\,M^j \quad \dot{-}$$

5 COROLLARY.

a) if $\mathcal{P}$ is affine, we have

$$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} + \epsilon_{\mathcal{P}}(D_{\mathcal{P}} M_{\mathcal{P}}) + 2\Omega_{\mathcal{P}} \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ \check{M}_{\mathcal{P}} \ .$$

b) If $\mathcal{P}$ is rigid, we have

$$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} + 2\Omega_{\mathcal{P}} \times D_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \circ \check{M}_{\mathcal{P}}$$

c) If $\mathcal{P}$ is translating, we have

$$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} + \bar{\bar{P}} \ .$$

d) If $\mathcal{P}$ is inertial, we have

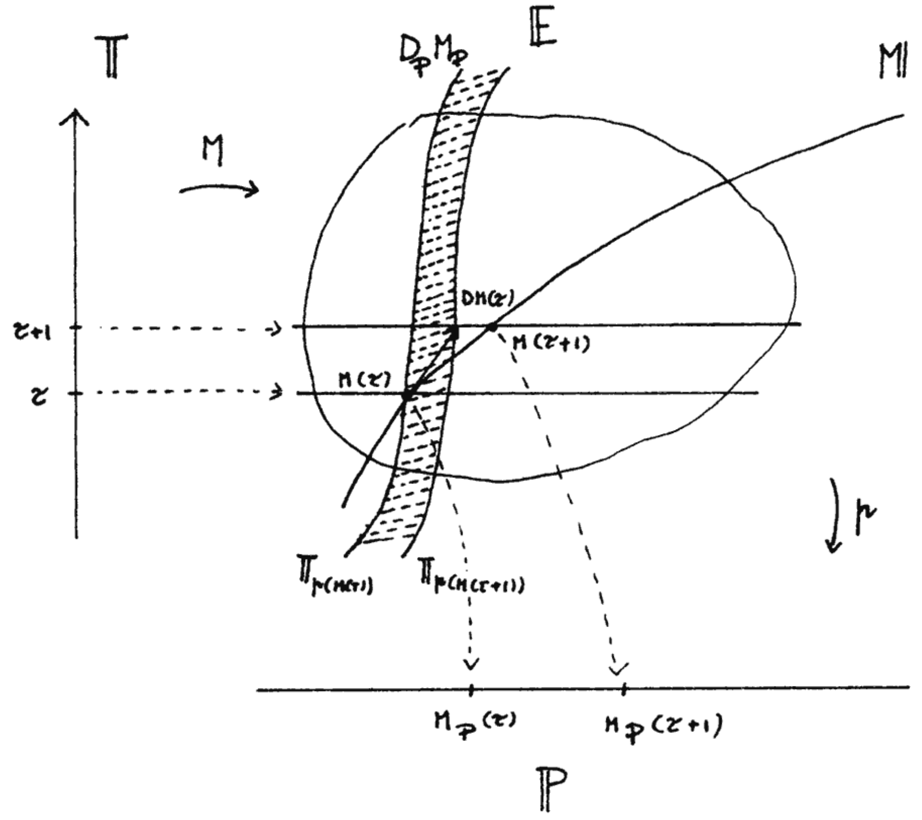
$$D^2 M = D^2_{\mathcal{P}} M_{\mathcal{P}} \qquad \dot{-}$$

Physical description.

The observed motion $M_{\mathbb{P}}$ is the map that associates, with each time $\tau \in \mathbb{T}$, the position constituted by the world-line of the frame, passing throung $M(\tau)$.

The observed velocity and acceleration are the map that associate with each time $\tau \in \mathbb{T}$, the strips touched by the absolute velocity and acceleration.

The difference between the observed acceleration and the acceleration of the observed motion takes into account the variation, during the time, of the affine properties of $T\mathbb{P}$ and of the projection $T\,\mathbb{E} \to T\mathbb{P}$.



2 RELATIVE KINEMATICS.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two fixed frames and let the subfixes "1" an "2" denote quantities relative to $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Let M be a fixed motion. We make a comparison between the kinematics observed by $\mathcal{P}_1$ and $\mathcal{P}_2$.

<u>Motion of a frame observed by a frame.</u>

1 If we consider $\mathcal{P}_1$ as a set of world-lines and $\mathcal{P}_2$ as observing $\mathcal{P}_1$, we are led naturally to the following definition by (III,1,2).

We consider only free velocity and acceleration for simplicity of notations, leaving to the reader to write them in the complete form.

Here $D_{1\mathcal{P}_2}$ and $D^2_{1\mathcal{P}_2}$ are the derivative in the sense of (III,1,1,b) with respect to P_2 and the suffix 1 denote partial derivative with respect to the first variable, i.e. the time.

DEFINITION.

a) The MOTION OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\tilde{P}_{12} \equiv p \circ \tilde{P}_1 : \mathbb{T} \times \mathbb{E} \to \mathbb{P}_2$$

The MUTUAL MOTION of (P_1,P_2) is the map

$$\tilde{P}_{(1,2)} \equiv \tilde{P}_1 - \tilde{P}_2 : \mathbb{T} \times \mathbb{E} \to \bar{\mathbb{S}} \quad .$$

b) The (FREE) VELOCITY OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\bar{P}_{12} \equiv (D_{1\mathcal{P}_2}\tilde{P}_{12})\circ j : \mathbb{E} \to \bar{\mathbb{S}}$$

The (FREE) VELOCITY OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\bar{P}_{12} \equiv \hat{P}_2 \circ \bar{P}_1 : \mathbb{E} \to \bar{\mathbb{S}}$$

The (FREE) VELOCITY OF THE MUTUAL MOTION $\tilde{P}_{(1,2)}$ is the map

$$\bar{P}_{(1,2)} \equiv D_1 \tilde{P}_{(1,2)} \circ j : \mathbb{E} \to \bar{\mathbb{S}} .$$

c) The (FREE) ACCELERATION OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\bar{\bar{P}}_{12} \equiv (D^2_{1P_2} \tilde{P}_{12}) \circ j : \mathbb{E} \to \bar{\mathbb{S}} .$$

The (FREE) ACCELERATION OF $\mathcal{P}_1$ OBSERVED BY $\mathcal{P}_2$ is the map

$$\bar{\bar{P}}_{1,2} \equiv \hat{P}_2 \circ \bar{\bar{P}}_1 : \mathbb{E} \to \bar{\mathbb{S}} .$$

The (FREE) ACCELERATION OF THE MUTUAL MOTION $\tilde{P}_{(1,2)}$ in the map

$$\bar{\bar{P}}_{(1,2)} = D^2_1 \tilde{P}_{(1,2)} \circ j : \mathbb{E} \to \bar{\mathbb{S}}.$$

d) The (FREE) STRAIN OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\epsilon_{12} = \check{S}\, D\, \bar{P}_{12} : \mathbb{E} \to \bar{\mathbb{S}}^* \otimes \bar{\mathbb{S}}$$

The (FREE) SPIN OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\omega_{12} = \frac{A}{2}\, \check{D}\, \bar{P}_{12} : \mathbb{E} \to \mathbb{S}^* \otimes \bar{\mathbb{S}} .$$

The (FREE) ANGULAR VELOCITY OF THE OBSERVED MOTION $\tilde{P}_{12}$ is the map

$$\Omega_{12} = * \frac{A}{2}\, \check{D}\, \bar{P}_{12} \quad \dot{-}$$

2 We can make the comparison between the observed entities and the entities of the observed motion, as shown by (III,1,3) .

PROPOSITION.

a) $$\tilde{P}_1 = (\pi_1, \tilde{P}_{12}) \; ; \mathbb{T} \times \mathbb{E} \to \mathbb{T} \times \mathbb{P}_2 \overset{\sim}{=} \mathbb{E} .$$

b) $$\bar{P}_{(1,2)} = \bar{P}_{1,2} = \bar{P}_1 - \bar{P}_2 = \bar{P}_{12}$$

c) $$\bar{\bar{P}}_{1,2} = \bar{\bar{P}}_1 = \bar{\bar{P}}_{12} + \epsilon_{\mathcal{P}_2}(\bar{P}_{12}) + 2\Omega_{\mathcal{P}_2} \times \bar{P}_{12} + \bar{\bar{P}}_2 .$$

d) $$\bar{\bar{P}}_{(1,2)} = \bar{\bar{P}}_1 - \bar{\bar{P}}_2 , \; \epsilon_{12} = \epsilon_1 - \epsilon_2 , \; \omega_{12} = \omega_1 - \omega_2 , \; \Omega_{12} = \Omega_1 - \Omega_2 \quad \dot{-}$$

3 We get an immediate comparison between the quantities "12" and "21".

COROLLARY.

a) $\tilde{P}_{(1,2)} = - \tilde{P}_{(2,1)}$, $\bar{P}_{(1,2)} = - \bar{P}_{(2,1)}$, $\bar{\bar{P}}_{(1,2)} = - \bar{\bar{P}}_{(2,1)}$

$\epsilon_{12} = - \epsilon_{21}$, $\omega_{12} = - \omega_{21}$, $\Omega_{12} = -\Omega_{21}$

b) $\bar{P}_{11} = \epsilon_{11} = \omega_{11} = \Omega_{11} = 0$ $\dot{-}$

4 We have time depending diffeomorphism between spaces concerning $\mathcal{P}_1$ and $\mathcal{P}_2$.

PROPOSITION.

Let $\tau \epsilon T$.

The maps

$$p_{12\tau} \equiv p_2 \circ P_{1_\tau} : \mathbb{P}_1 \to \mathbb{P}_2$$

given by

$$[e]_1 \to [P_1(\tau,e)]_2,$$

and

$$T\, p_{12\tau} : \mathbb{TP}_1 \to \mathbb{TP}_2,$$

given by

$$[e,u]_1 \to [\tilde{P}_1(\tau,e), \check{P}_1(\tau,e)(u)]_2 ,$$

are C^∞ diffeomorphisms $\dot{-}$

Velocity addition and generalized Coriolis theorems.

5 As conclusion, we get the comparison between velocity and acceleration of the motion M observed by $\mathcal{P}_1$ and $\mathcal{P}_2$.

THEOREM. "VELOCITY ADDITION AND GENERALIZED CORIOLIS THEOREMS".

a) $$M_{\mathcal{P}_2} = p_2 \quad \bar{M}_{\mathcal{P}_1} \quad .$$

b) $$D_{\mathcal{P}_2} M_{\mathcal{P}_2} = D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{P}_{12} \circ M \; .$$

c) $$D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{12} \circ M(D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2\Omega_{12} \circ M \times D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_{12} \circ M.$$

PROOF.

It follows from (II,5,3) and (II,6,2) .

6 COROLLARY.

Let $\mathcal{P}_2$ be inertial. Then we get

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{\mathcal{P}_1} \circ M(D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2\Omega_{\mathcal{P}_1} \circ M \times D_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_1 \circ M \quad .$$

If $\mathcal{P}_1$ is affine, we have

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \epsilon_{\mathcal{P}_1} (D_{\mathcal{P}_1} M_{\mathcal{P}_1}) + 2 \, \Omega_{\mathcal{P}_1} \times D_{\mathcal{P}_1} M_{\mathcal{P}_2} + \bar{\bar{P}}_1 \; ;$$

if $\mathcal{P}_1$ is rigid, we have

$$D^2 M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + 2 \, \Omega_{\mathcal{P}_1} \times D_{\mathcal{P}_1} M_{\mathcal{P}_2} + \bar{\bar{P}}_1 \; ;$$

if $\mathcal{P}_1$ is translating we have

$$D^2 M = D^2_{P_2} M_{P_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} + \bar{\bar{P}}_1 \quad ;$$

if $\mathcal{P}_1$ is inertial, we have

$$D^2M = D^2_{\mathcal{P}_2} M_{\mathcal{P}_2} = D^2_{\mathcal{P}_1} M_{\mathcal{P}_1} \quad \dot{-}$$

Physical description.

The observed motion $\tilde{P}_{12e} : \mathbb{T} \to P_2$ gives the position in $\mathbb{P}_2$ touched during time, by the particle of $\mathcal{P}_1$ passing through e. The velocity and the acceleration of $\tilde{P}_{12}$ are calculed by $\mathcal{P}_2$ by its differential structure and by its time depending affine structure, in the same way of any observed motion.

The velocity and acceleration of $\mathcal{P}_1, \bar{P}_{1,2}(e)$ and $\bar{\bar{P}}_{(1,2)}(e)$, are the spatial projections, performed by $\mathcal{P}_2$, of the absolute velocity and acceleration of the particle of $\mathcal{P}_1$, passing through e.

Notice that, in all the previous quantities, $\mathcal{P}_1$ is involued only through the motion of its only particle $\hat{P}_{1e}$, while $\mathcal{P}_2$ can use also its spatial derivative, which take into account the mutual motion of its particles.

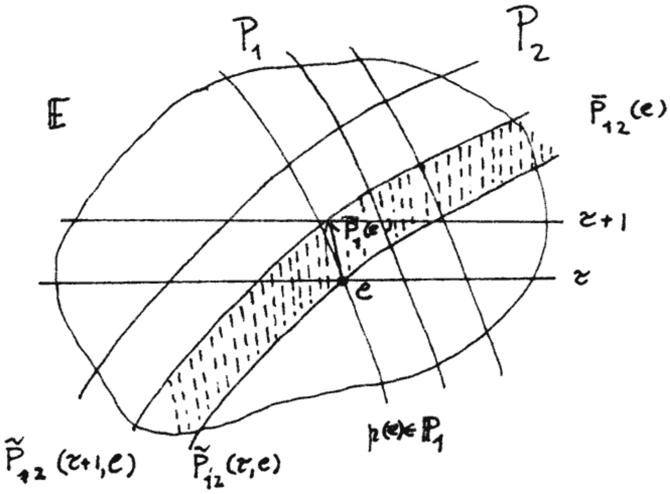
The nutual motion, velocity and acceleration $P_{(1,2)}(e) : T \to \bar{\$}$, $\bar{P}_{(1,2)}(e) \in \bar{\$}$, $\bar{\bar{P}}_{(1,2)}(e) \in \bar{\$}$ are the absolute spatial distance and its time first and second derivatives between the two particles, one of $\mathcal{P}_1$ and one $\mathcal{P}_2$, passing through e.

So it is not surprising if $\bar{\bar{P}}_{1,2} \neq - \bar{\bar{P}}_{2,1}$, $\bar{\bar{P}}_{12} \neq - \bar{\bar{P}}_{21}$.

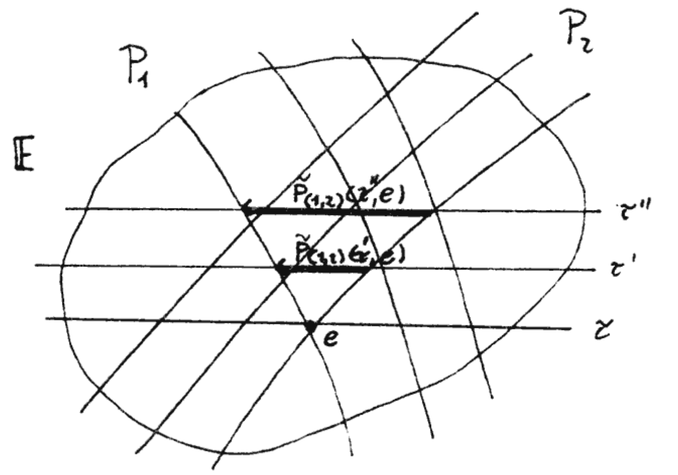
The velocity addition theorem, relative to a motion M, gives the classical result that the velocity of the observed motion by $\mathcal{P}_2$ is the sum of the velocity of the observed motion by $\mathcal{P}_1$, plus the veloci_ ty of $\mathcal{P}_1$ observed by P_2 .

The generalized Coriolis theorem says that the acceleration of the observed motion by $\mathcal{P}_2$ is the sum of the acceleration of the observed motion by $\mathcal{P}_2$, plus the acceleration of $\mathcal{P}_1$ observed by $\mathcal{P}_2$ plus the classical angular velocity term, plus a strain term. When we consider rigid frames, we get, as a particular case, the classical result.

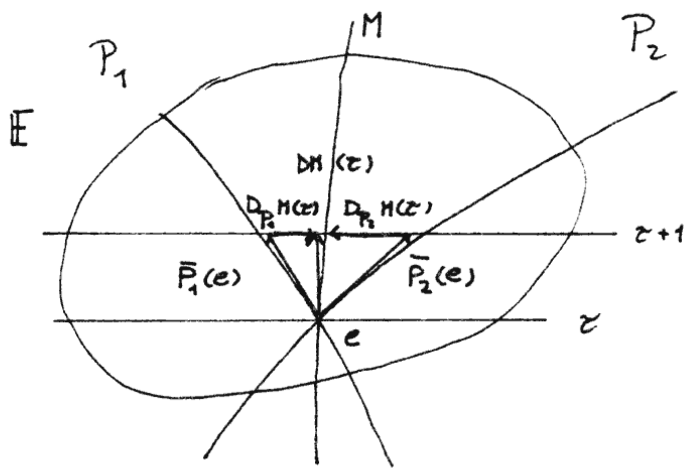
We can describe such results by a picture.



motion and velocity of $\mathcal{P}_1$ observed by $\mathcal{P}_2$



mutual motion



velocity addition

R E F E R E N C E S

E. CARTAN, Ann. scient. Éc. Norm. Sup. t. 40,1923, p.328

E. CARTAN, Ann. scient. Éc. Norm. Sup. t. 41,1924,p.1

P. HAVAS, Rev. Mod. Physic.,t. 36,1964, p. 938.

H.P.KÜNZLE, Ann. Inst. Henri Poincaré,A. Vol. XVII, n° A, 1972,p.337.

C.TRUSDELL-R.A.TOUPIN, The classical field theories, in FLÜGGE (ed.) Handbuch der Physik, Bd. III/I, Springer, Berlin, 1960

K.ABRAHAM-J.E.MARDEN, Foundation of mechanics, Benjamin, New York, 1967

C.CODBILLON- Géométrie différentialle et mécanique analytique - Hermann - Coll.Méthods - Paris 1969

M.MODUGNO-G.MONTESANO - Elementi di analisi differenziale sugli spazi affini-Quad.dell'Ist.Mat.Univ. Lecce - Q.7 - 1977

M.MODUGNO-G.MONTESANO - Le sottovarietà degli spazi affini Quad. dell'Ist. Mat. Univ. -Lecce - Q. 8 - 1977

M. MODUGNO - Formulation of analytical mechanics in General Relativity - Ann. Inst. Henri Poincaré, Vol. XXI,n°2,p.142-174 1974

D.CANARUTTO M.MODUGNO - On the structure of special relativistic kinematics- To appear.

M.MODUGNO-R.RAGIONIERI - On the structure of "classical electromagnetic field" - To appear.