5. Non-standard methods.

It is well-known that ultrafilter masses (or measures: this stronger terminology may be used when the cardinality of the index set is measurable) are a tool in non-standard analysis for the construction of the relevant superstructure.

In [4] it is shown how some fundamental ideas in this field may be introduced through a finitely additive probability measure μ (i.e., a mass) on the index set, extending the concept of ultrapower to that of "μ-power".

To facilitate the exposition, the attention was confined to a structure $\mathcal{E} = \langle E, \mathbb{R}\rangle$, consisting of a non-empty set E and a set $\mathbb{R}$ of relations on E. Given an index set J, let μ be a mass on $\mathcal{P}(J)$, with $\mu(J) = 1$: the superstructure ${}^*\mathcal{E} = \langle {}^*E, {}^*\mathbb{R}\rangle$ consists of the set ${}^*E$ of all functions $f : J \to E$ (modulo μ-null sets) and any relation $R \in {}^*\mathbb{R}$ "is true" (loosely speaking) in the "model" ${}^*\mathcal{E}$ if and only if it is true in $\mathcal{E}$ for almost all $j \in J$ (here each value of one of the "equivalent" functions, with domain J, defining an element of ${}^*E$, is a point of E; ${}^*E$ is a proper extension of E, by virtue of condition (1) : cfr. [4]) .

Let us consider, to be definite, the structure given by the ordered field $\mathbb{R}$. It is clear that, using μ-powers instead of ultrapowers (i.e., arbitrary masses on J instead of ultrafilter ones), ${}^*\mathbb{R}$ is not necessarily a field (and, moreover, it is only partially ordered): take, for example, $A \subset J$ with $0 < \mu(A) < 1$. Its characteristic function χ_A is not equivalent to the null function 0, and so gives rise, in ${}^*\mathcal{E}$, to an element (i.e., an equivalence class modulo μ-null sets) $[\chi_A]$ which is not $^*0 = [0]$; the same is true for χ_{J-A}. But $\chi_A \cdot \chi_{J-A} \equiv 0$, and so in $^*\mathcal{E}$ the product $[\chi_A] \cdot [\chi_{J-A}]$ of this two non-null elements is null (i.e., $^*R$ has zero divisors).

This fact, from the usual point of view adopted in the costruction of non-standard models of the reals, should be considered a "defect", since it is possible (as it is well known) to build up an enlargement $^*R$ which is an ordered field (though, of course, a non-archimedean one). But it is possible to look at the question from different viewpoints, similar (apart from the dropping out of the condition of σ-additivity for the measure on the index set) to those sketched by D . Scott in [12] .

A problem of interest in probability theory is the following: if we take μ to be a measure, it does not exist a denumerable "uniform" (and measurable) partition of the index set $J = \bigcup_{n=1}^{\infty} E_n$, with $\mu(E_n)=0$ for each n, otherwise

$$0 = \sum_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) = 1 \quad \text{(impossible)}.$$

On the other hand, if μ is only a mass and such a partition exists, the latter inequality is consistent with Prop. 1; moreover we show the possibility of looking at it as a sort of "non-standard" countable additivity.

To begin with, we may give a meaning to $\sum_{n=1}^{\infty} {}^*a_n$ (with $^*a_n \in {}^*R$) by choosing suitable representatives a_n of $^*a_n$ ($n = 1,2,...$) such that $\sum_{n=1}^{\infty} a_n(i)$ converges for almost all $i \in J$, and by putting

(7) $$\sum_{n=1}^{\infty} {}^{*}a_n = \left[\left(\sum_{n=1}^{\infty} a_n(i)\right)_{i\in J}\right] \quad .$$

Now, we remark that, since $\chi_{E_n} = 0$ for almost all $j \in J$, we have $\left[\chi_{E_n}\right] = {}^{*}0$; on the other hand, $\sum_{n=1}^{\infty}\chi_{E_n} = \chi_J \equiv 1$, and so $[\chi_J] = {}^{*}1$.

If we apply (7) to $\sum_{n=1}^{\infty} {}^{*}0 = \sum_{n=1}^{\infty}[\chi_{E_n}]$, choosing χ_{E_n} as representative of $[\chi_{E_n}]$, we get

$$\sum_{n=1}^{\infty} {}^{*}0 = \sum_{n=1}^{\infty} [\chi_{E_n}] = [\sum_{n=1}^{\infty}\chi_{E_n}] = [\chi_J] = {}^{*}1 \ .$$

So an uniform probability distribution on a countable set does not conflict, in ${}^{*}\mathbb{R}$, with "countable additivity" of μ .

We point out that this approach differs from the well-know one (see, e.g., [5] , [9]) through $*$-finite sets: is there some hope that such "models"would open new trends in this field?