difficult to see that such subset must necessarily be $A \cap B \cap E_o$). So the mass

$$\beta_1(E) = \begin{cases} 0 & \text{if} \quad E \notin \mathcal{U}_1 \\ \alpha_1 & \text{if} \quad E \in \mathcal{U}_1 \end{cases}$$

is atomic. Put $\mu_1 = \mu - \beta_1$; if the mass μ_1 is non-continuous, then

$$\alpha_2 = \inf_{p \in \mathfrak{D}} \mu_1(E^{(p)}) > 0,$$

and so it is possible to go on in the same fashion.

After n steps, we get

$$\mu_n = \mu - \sum_{k=1}^{n} \beta_k$$

and, if μ_n is continuous, eq.(4) holds with $\mu_o = \mu_n$ and with each β_k null for $k > n$. If μ_n is non-continuous for any n, we get a sequence (β_n) such that the corresponding series $\sum_{n=1}^{\infty} \beta_n(E)$ converges for every $E \in \mathcal{Q}$ (since $\mu(E) < +\infty$). Then $\lim_{k\to\infty} \alpha_k = 0$, and it follows that $\mu_o = \lim_{n\to\infty} \mu_n$ is continuous. ■

3. Non atomic masses.

In the classical case of a measure, non-atomicity is equivalent to continuity. This can be seen, for example, as an easy consequence of the following

Theorem 2 (Saks) - Let μ be a measure on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Given any $\varepsilon > 0$, there exists $p \in \mathfrak{D}$ such that each $E_k \in p$ is either an atom or $\mu(E_k) < \varepsilon$.

Proof : see [2], p. 308 . ∎

This equivalence cannot be carried over to the general case of a mass: in [1] it is shown (cfr. also the following Theorem 5) that, if $\mu=\nu+\lambda$, where ν is atomic and λ is continuous, with $\nu<<\lambda$ (i.e. ν is absolutely continuous with respect to λ), then μ is both non-atomic and non-continuous.

In other words, while atomic masses are necessarily non-continuous (see Proposition 2), non-atomic ones can be either continuous or not.

For continuous masses, the following result has been established in [13]:

Theorem 3 - Let μ be a continuous mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (F_n) of mutually disjoint measurable sets, with $\mu(F_n) > 0$, such that $\sum_{n=1}^{\infty}\mu(F_n)$ equals any preassigned α , with $0 < \alpha < \mu(\Omega)$, and

$$(5) \qquad \alpha = \sum_{n=1}^{\infty}\mu(F_n) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) .$$

So to say, the mass μ "behaves" like a measure on each collection (F_n) : then it would seem interesting a deeper investigation of the family (or of some suitable subfamily) of all such sets obtained when α ranges in the open interval from 0 to $\mu(\Omega)$.

A simple corollary of Theorem 3 is the following: the range of a continuous μ is the whole interval $[0,\mu(\Omega)]$. The latter statement is no langer true if μ is non-continuous: a counterexample is given in [1]; moreover, there it is shown that the range of μ need not even be a closed subset of $\mathbb{R}$, contrary to the classical case of a measure.

As far as continuous masses are concerned, let us quote also a recent result obtained through non-standard methods: a necessary and sufficient condition for the existence of a continuous mass, which is invariant for a transformation of Ω into itself, is given in [16].

We want now to extend Theorem 3 to the more general case of a non-atomic μ: the previous remarks show that we can hope, at most, in countable additivity on a suitable sequence of sets (and not also, as in eq. (5), in a beforehand given value of α).

Theorem 4 - Let μ be a non-atomic mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, with $\mu(A_n) > 0$, such that

$$(6) \qquad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \ .$$

Proof - Let $B \in \mathcal{A}$, with $0 < \mu(B) < \mu(\Omega)$: then also $B' = \Omega - B$ satisfies $0 < \mu(B') < \mu(\Omega)$. At least one of them (call it B_1) is such that $\mu(B_1) \leq \frac{1}{2}\mu(\Omega)$. Now, let $B_2 \in \mathcal{A}$, $B_2 \subset B_1$, be such that $0<\mu(B_2)<\mu(B_1)$ and $\mu(B_2) \leq \frac{1}{2}\mu(B_1)$. In general, we define $B_n \subset B_{n-1}$, with $0<\mu(B_n)<\mu(B_{n-1})$ and $\mu(B_n) \leq \frac{1}{2}\mu(B_{n-1}) \leq \frac{1}{2^n}\mu(\Omega)$.

Put $A_n = B_n - B_{n+1}$: we have $\mu(A_n) = \mu(B_n) - \mu(B_{n+1}) > 0$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Moreover

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{k} \mu(A_n) + \mu(\bigcup_{n=k+1}^{\infty} A_n) =$$

$$= \sum_{n=1}^{k} \mu(A_n) + \mu(B_{k+1}) \leq \sum_{n=1}^{k} \mu(A_n) + \frac{1}{2^{k+1}} \mu(\Omega) .$$

As $k \to \infty$, we get

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) ,$$

which, taking into account Proposition 1, gives (6). ■

The next theorem will enable us to extend (in Section 4) the previous result also to a particular class of atomic masses.

Theorem 5 - Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra, ν an atomic mass on $\mathcal{A}$, and λ a continuous mass on $\mathcal{A}$ such that $\lambda(A) > 0$ for any atom A of ν (e.g., such that $\nu \ll \lambda$). Then $\mu = \nu + \lambda$ is non atomic and non-continuous.

Proof : see [1]. ■

Remark - Put $\nu = \sum_n \beta_n$ in Theorem 1, eq.(4): ν need not be atomic (an example is given in [10] , p. 47), and so we may have masses which are at the same time non-atomic and non-continuous, but not of the form given by Theorem 5.

Corollary 1 - Let β be an ultrafilter mass and λ a continuous mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, with $\beta << \lambda$. Then $\mu = \beta + \lambda$ is non-atomic and non-continuous.

Corollary 2 - Let β be an ultrafilter mass on $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ such that $\beta << \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then β cannot be a measure on $\mathcal{A}$.

Remark - It is interesting also to look at Theorem 5 as another counterexample to known results for measures: in [7] it is shown that, given two measures λ and ν, with $\nu << \lambda$ and λ non atomic (i.e. continuous), then ν also is non-atomic. Actually, this need not be true if ν is only a mass (and not a measure), for example if it is an ultrafilter mass β, as that of Corollary 2. The existence of such a mass (given λ) can be proved (cfr. [1]) taking an $\mathcal{A}$-ultrafilter containing the filter

$$\mathfrak{F} = \{E \in \mathcal{A} : \lambda(E) = \lambda(\Omega)\} .$$

4. Atomic masses and measurable cardinals.

Since the mass μ occurring in Theorem 5 is non-atomic (and non-continuous), Theorem 4 can be suitably applied to it, giving easily a countably additive sequence of sets also for the atomic mass ν.

Theorem 6 - Let ν be an atomic mass on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, such that $\nu << \lambda$, where λ is a continuous measure on $\mathcal{A}$. Then there exists a sequence (A_n) of mutually disjoint measurable sets, such that

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) .$$