and also gave an iterative algorithm for computing it.

3. If the cost functions K_k are identical and the conditions of O. Opitz are satisfied, then E. Burger [1] proved the existence and uniqueness of the equilibrium point and also gave an algorithm to compute it. We remark that the algorithm of Szidarovszky is a generalization of Burger's method.

4. If the functions f and K_k $(k=1,2,\dots,n)$ are linear, then the existence and uniqueness was proved by M. Maňas, [4], who gave an algorithm which is independent of the method of Szidarovszky. We remark that using the result of Theorem 5. the equilibrium point in this special case can be given in closed form (see pp. 37-39 of [10]).

4. The group equilibrium problem

In this paragraph we will discuss the generalized version of the classical oligopoly game Γ having the srategy sets

$$X_k = [0, L_{k1}] \times [0, L_{k2}] \times \dots \times [0, L_{ki_k}] \quad (1 \le k \le n) \tag{25}$$

and pay-off functions

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \left(\sum_{i=1}^{i_k} x_{ki}\right) f\left(\sum_{\ell=1}^{n}\sum_{j=1}^{i_\ell} x_{\ell j}\right) - K_k(\underline{x}_k), \tag{26}$$

where for $k=1,2,\dots,n$, $\underline{x}_k = (x_{k1},\dots,x_{ki_k}) \in X_k$. This game can occur when the players of the classical oligopoly game form disjoint groups and they tend to the optimal income of the group. If the number of members in group k is equal to i_k, and the capacity limit of member i of group k is given by L_{ki}, then the strategy set of group k is the set X_k and the income of group k is the sum of the individual incomes of its members, given by the function (26).

For $k=1,2,\dots,n$ and $s_k \in \left[0, \sum_{i=1}^{i_k} L_{ki}\right]$ consider the problem

$$
\begin{aligned}
&0 \leqq x_{ki} \leqq L_{ki} \qquad (i=1,2,\dots,i_k)\\
&\sum_{i=1}^{i_k} x_{ki} = s_k \qquad\qquad (27)\\
&\overline{\qquad\qquad\qquad}\\
&K(\underline{x}_k) \longrightarrow \min.
\end{aligned}
$$

If function K is continuous then problem (27) has an optimal solution. Let the optimal objective function value be denoted by $\varphi_k(s_k)$. Some properties of the functions φ_k are given in the following lemma.

<u>Lemma 10.</u> If K is continuous, convex and strictly increasing in the components of $\underline{x}_k$, then φ_k is continuous, convex and strictly increasing in s_k.

<u>Proof.</u> See Lemmas 2,3,4 of the paper [10]. ∎

<u>Remark.</u> Observe that the same properties were assumed in the main theorems of the previous section which are now stated in this lemma.

Let us now consider the classical oligopoly game $\tilde{\Gamma}$ with sets of strategies

$$\tilde{X}_k = \left[0, \sum_{i=1}^{i_k} L_{ki} \right] \quad (k=1,2,\ldots,n) \tag{28}$$

and pay-off functions

$$\tilde{\varphi}_k(s_1,\ldots,s_n) = s_k \, f\left(\sum_{\ell=1}^{n} s_\ell\right) - \varphi_k(s_k). \tag{29}$$

The connection between the generalized game (25), (26) and the classical oligopoly game (28), (29) is shown in the following theorem.

<u>Theorem 6.</u> Assume that K_k is continuous for $k=1,2,\ldots,n$.

a/ Let $\underline{x}^* = (\underline{x}_1^*,\ldots,\underline{x}_n^*)$ $\left(\underline{x}_k^* = (x_{k1}^*,\ldots,x_{ki_k}^*)\right)$ be an equilibrium point of Γ, and let $s_k^* = \sum_{i=1}^{i_k} x_{ki}^*$. Then $(s_1^*,\ldots,s_n^*)$ is an equilibrium point of $\tilde{\Gamma}$ and for $k=1,2,\ldots,n$ $(x_{k1}^*,\ldots,x_{ki_k}^*)$ is an optimal solution of problem (27) with $s_k = s_k^*$.

b/ Let $(s_1^*,\ldots,s_n^*)$ be an equilibrium point of $\tilde{\Gamma}$ and let $\underline{x}_k^* = (x_{k1}^*,\ldots,x_{ki_k}^*)$ be an optimal solution of problem (27) with $s_k = s_k^*$. Then $(\underline{x}_1^*,\ldots,\underline{x}_n^*)$ gives an equilibrium point of game Γ.

<u>Proof.</u> See Lemma 1. of paper [10]. ∎

<u>Remark.</u> The group equilibrium problem is not a real generalization of the classical oligopoly game, since it can be reduced to the classical case.

Finally let as assume that the functions f and K_k are linear. Let

$$f(s) = As + B$$

$$K_k(\underline{x}_k) = \sum_{i=1}^{i_k} a_{ki} x_{ki} + b_k ,$$

then the solution of the optimization problem (27) is a piece-wise linear function φ_k. In this case the reduced game can be solved easily as it is shown in [10], pp. 43-44.

5. Multiproduct oligopoly game

In this paragraph we will consider the game having the sets of strategies

$$X_k = \left[0, L_k^{(1)}\right] x \dots x \left[0, L_k^{(M)}\right] \qquad (30)$$

and pay-off functions

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \sum_{m=1}^{M} x_k^{(m)} f_m\left(\sum_{\ell=1}^{n} x_\ell^{(1)},\dots,\sum_{\ell=1}^{n} x_\ell^{(M)}\right) - K_k(\underline{x}_k), \qquad (31)$$

where $\underline{x}_k = \left(x_k^{(1)},\dots,x_k^{(M)}\right)$, $\mathcal{D}(K_k) = X_k$, $\mathcal{R}(K_k) \subset R^1$,

$\mathcal{D}(f_m) = \left[0, \sum_{\ell=1}^{n} L_\ell^{(1)}\right] x \dots x \left[0, \sum_{\ell=1}^{n} L_\ell^{(M)}\right]$, $\mathcal{R}(f_m) \subset R^1$ for

$k=1,2,\dots,n$ and $m=1,2,\dots,M$. This game can come up if the factories manufacture different products and sell them on the same market. Let M be the number of products, and let $x_k^{(m)}$, $L_k^{(m)}$ be the production level and capacity limit of factory k from product m. If f_m donotes the unit price of product m, than it is assumed that f_m is a function of the total production levels of the different products. The function K_k is the