3. The classical oligopoly game

In this section we will discuss a special economic game with sets of strategies

$$X_k = [0, L_k] \quad (L_k > 0, \; k=1,2,\dots,n) \tag{19}$$

and pay-off functions

$$\varphi_k (x_1,\dots,x_n) = x_k \, f\left(\sum_{i=1}^{n} x_i\right) - K_k(x_k), \tag{20}$$

where the functions f and K_k must have the properties:

$$\mathcal{D}(f) = [0, L], \quad \text{where} \quad L = \sum_{i=1}^{n} L_i \; ; \; \mathcal{D}(K_k) = [0, L_k];$$

$\mathcal{R}(f) \subset R^1$ and $\mathcal{R}(K_k) \subset R^1$. The game defined by the sets of strategies (19) and pay-off functions (20) is called the classical oligopoly game.

Before discussing the equilibrium problem of this game we shaw how the game appears in seme applications.

Application 1. Assume that n factories manufacture the same product and they sell it on the same market. Let f be the unit price of the product being a function of the total production level, and let K_k be the cost function of the manufacturer k. Then L_k is the production bound for manufacturer k and $\varphi_k(x_1,\dots,x_n)$ is its netto income assuming that x_i is the production level of the manufacturer i for $i=1,2,\dots,n$.

Application 2. Assume that a multipurpose water supply system has to be designed. Let the water users denoted by k ($k=1,2,\dots,n$) and let the water quantity given to user k be denoted by x_k. If the capacity bounds of the users are denoted by L_k, then obviously $x_k \in [0, L_k]$ for $k=1,2,\dots,n$. Let I be the investment cost being a function of $\sum_{k=1}^{n} x_k$, let $u_k(x_k)$, $v_k(x_k)$ and $w_k(x_k)$ be the production cost, income and the economic loss of the water shortage /penalty e.t.c./ of user k, respectively. Let us assume, that the total investment cost is devided by the users in the rate of the water quantity used by the water users. Thus the total income of user k can be determined by the function

$$-\frac{x_k}{\sum_{i=1}^{n} x_i} I\left(\sum_{i=1}^{n} x_i\right) - u_k(x_k) + v_k(x_k) - w_k(x_k). \qquad (21)$$

By introduceing the notations

$$f\left(\sum_{i=1}^{n} x_i\right) = -\frac{1}{\sum_{i=1}^{n} x_i} I\left(\sum_{i=1}^{n} x_i\right)$$

$$K_k(x_k) = u_k(x_k) - v_k(x_k) + w_k(x_k)$$

function (21) has immediately form (2o).

<u>Application 3.</u> Let us now assume that n factories are on the bank of a river and they send a certain quantity of waste-water to the river. It is also assumed that the total penalty paid by the factories is a function of the total waste-water quantity sent to the river and it is devided among the factories proportionally to the waste-water quantity sent to the river by the different factories. Let L_k be the total waste-

-water quantity produced by factory k, let x_k be the waste-water quantity sent to the river by factory k. Then the total "income" of factory k can be given by the formula

$$\frac{-x_k}{\sum_{i=1}^{n} x_i} P\left(\sum_{i=1}^{n} x_i\right) - C_k\left(L_k - x_k\right), \tag{22}$$

where P is the penalty function, C_k is the cleaning cost of factory k. Let

$$f\left(\sum_{i=1}^{n} x_i\right) = -\frac{1}{\sum_{i=1}^{n} x_i} P\left(\sum_{i=1}^{n} x_i\right), \quad K_k(x_k) = C_k\left(L_k - x_k\right),$$

then the function (22) immidiately has the form of (20).

First we shew that the equilibrium problem of the classical oligopoly game is equivalent to a fixed point problem of a one dimension point-to-set mapping. It will be much more convenient than the application of the fixed point problem of Lemma 3., since the latter is an n-dimensional problem.

Let

$$\Psi_k(s, x_k, t_k) = t_k f(s - x_k + t_k) - K_k(t_k)$$

for $k=1,2,..,n$, $s \in [0, L]$, $x_k \in [0, L_k]$ and $t_k \in [0, \gamma_k]$, where $\gamma_k = \min\left\{L_k, L - s + x_k\right\}$. Since $\gamma_k \geqq 0$, the interval for t_k can not be empty. For $k=1,2,..,n$; $s \in [0, L]$; $x_k \in [0, L_k]$ let

$$T_k(s, x_k) = \left\{t_k \mid t_k \in [0, \gamma_k], \Psi_k(s, x_k, t_k) = \max_{0 \leq u_k \leqq \gamma_k} \Psi_k(s, x_k, u_k)\right\}$$

and for $k=1,2,\ldots,n$; $s \in [0, L]$ let

$$X_k(s) = \{x_k \mid x_k \in [0, L_k], \ x_k \in T_k(s, x_k)\} .$$

Lemma 7. A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point of the classical oligopoly game if and only if $x_k^* \in X_k(s^*)$ $(k=1,2,\ldots,n)$, where $s^* = \sum_{k=1}^{n} x_k^*$.

Proof. The definition of the equilibrium point implies that a strategy vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point if and only if

$$x_k^* f(s^* - x_k^* + x_k^*) - K_k(x_k^*) \geq t_k f(s^* - x_k^* + t_k) - K_k(t_k) \quad (23)$$

for $k=1,2,\ldots,n$ and $t_k \in [0, L_k]$. /It is easy to observe that for $s^* = \sum_{i=1}^{n} x_i^*$, $\gamma_k = L_k$/ Inequality (23) is equivalent to the fact that $x_k^* \in T_k(s^*, x_k^*)$, that is $x_k^* \in X_k(s^*)$. ∎

Let us finally introduce the following one dimensional point-to-set mapping:

$$X(s) = \{u \mid u = \sum_{i=1}^{n} x_i , \ x_i \in X_i(s)\} \quad (s \in [0, L]). \qquad (24)$$

Lemma 7. and definition (24) imply the following important result.

Theorem 3. A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point of the classical oligopoly game if and only if for $s^* = \sum_{i=1}^{n} x_i^*$, $s^* \in X(s^*)$ and for $k=1,2,\ldots,n$, $x_k^* \in X_k(s^*)$.

Remark. The solution of the game has two steps:

Step 1: the solution of the one dimensional fixed point problem $s^* \in X(s^*)$;

Step 2: the determination of sets $X_k(s^*)$ and the computation of the vectors $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ such that

$$x_k^* \in X_k(s^*) \quad (k=1,2,\ldots,n) \quad \text{and} \quad s^* = \sum_{k=1}^{n} x_k^* .$$

In the following parts of this section we will assume that the conditions given below are satisfied.

1. There exists a constant $\xi > 0$ such that

a/ $f(s) = 0$ for $s \geqq \xi$;

b/ f is continuous, concave and strictly decreasing in the interval $[0, \xi]$.

2. For $k=1,2,\ldots,n$ function K_k is continuous, convex and strictly increasing in the interval $[0, L_k]$.

Theorem 4. Under the above conditions the game has at least one equilibrium point.

Proof. The proof consists of several steps.

a/ First we prove that if $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point, then $\sum_{k=1}^{n} x_k^* \leqq \xi$. Let us suppose that $\sum_{k=1}^{n} x_k^* > \xi$. Then there are positive x_k^* and x_k such that $0 < x_k < x_k^*$ and $\sum_{i \neq k} x_i^* + x_k > \xi$. This implies

$$\varphi_k\,(x_1^*,\ldots,x_k,\ldots,x_n^*) = x_k\cdot 0 - K_k(x_k) > x_k^*\cdot 0 - K_k(x_k^*) =$$

$\varphi_k\,(x_1^*,\ldots,x_k^*,\ldots,x_n^*)$, which is a contradiction to inequality (1).

b/ Let

$$X = \left\{\underline{x}\;\middle|\;\underline{x} = (x_1,\ldots,x_n),\ \sum_{k=1}^{n} x_k \leq \xi\ ,\ x_k \in [0,\ L_k],\ k=1,2,\ldots,n\right\}.$$

Next we prove that any equilibrium point $\underline{x}^*$ of the generalized game $\Gamma = (n;\ X_1,\ldots,X_n,\ X;\ \varphi_1,\ldots,\varphi_n)$ gives an equilibrium point for the classical oligopoly game. Let $x_k \in [0,\ L_k]$. If $(x_1^*,\ldots,x_{k-1}^*,\ x_k,\ x_{k+1}^*,\ldots,x_n^*) \in X$, then the equilibrium property for game Γ gives

$$\varphi_k\,(x_1^*,\ldots,x_k^*,\ldots,x_n^*) \geq \varphi_k\,(x_1^*,\ldots,x_k,\ldots,x_n^*),$$

and if $(x_1^*,\ldots,x_k,\ldots,x_n^*) \notin X$, then

$$\varphi_k\,(x_1^*,\ldots,x_k,\ldots,x_n^*) = x_k\cdot 0 - K_k(x_k) < -\,K_k(0) =$$

$$= 0.f\left(\sum_{i\neq k} x_i^*\right) - K_k(0) = \varphi_k\,(x_1^*,\ldots,0,\ldots,x_n^*) \leq \varphi_k(x_1^*,\ldots,x_k^*,\ldots,x_n^*)$$

since $(x_1^*,\ldots,0,\ldots,x_n^*) \in X$.

c/ Next we prove that if function h is continuous, concave and strictly decreasing in a nonnegative interval $[A,\ B]$, then the function $xh(x)$ is concave in the same interval.

Let us first assume that h is twice continuously differentiable.

Then

$$\{xh(x)\}' = xh'(x) + h(x),$$

$$\{xh(x)\}'' = 2h'(x) + xh''(x) < 0,$$

which implies the assertion.

If h is continuous, then let h_m $(m=1,2,..)$ be twice continuously differenciable, concave, strictly decreasing functions such that $\lim_{m\to\infty} h_m = h$.

Let $A \leq x < y \leq B$; $\alpha, \beta \geq 0$; $\alpha + \beta = 1$, then for $m=1,2...$

$$(\alpha x + \beta y)h_m(\alpha x + \beta y) \geqq \alpha x h_m(x) + \beta y h_m(x).$$

By the limit relation $m \to \infty$ we obtain

$$(\alpha x + \beta y)h(\alpha x + \beta y) \geqq \alpha x h(x) + \beta y h(y),$$

thus $xh(x)$ is concave.

d/ The parts a/ and b/ imply that the classical oligopoly game and the generalized game $\Gamma = (n; X_1, \ldots, X_n, X; \varphi_1, \ldots, \varphi_n)$ have the same equilibrium points. Under the assumptions of the theorem X is a convex, closed, bounded subset of R^n, φ_k is continuous and part c/ implies that φ_k is concave in x_k. Thus the conditions of the Nikaido-Isoda theorem are satisfied, consequently the game has at least one equilibrium point. ∎

Remark. The uniqueness of the equilibrium point is not assured in general as the following example shows.

Example 4. Let $n=2$; $L_1 = L_2 = 1,2$;

$$f(s) = \begin{cases} 1,75 - 0,5s , & \text{if } 0 \leqq s \leqq 1,5 \\ 2,5 - s, & \text{if } 1,5 \leqq s \leqq 2,5 \\ 0 , & \text{if } s > 2,5 ; \end{cases}$$

$$K_1(x) = K_2(x) = 0,5x \quad (x \geqq 0).$$

We will prove that an arbitrary point of the set

$$X^* = \left\{(x_1, x_2) \middle| 0,5 \leqq x_1 \leqq 1, 0,5 \leqq x_2 \leqq 1, x_1 + x_2 = 1,5\right\}$$

gives an equilibrium point of the game.

Lét $x^* \in [0,5 ; 1]$ be fixed, and let

$$\Psi(x) = xf(1,5 - x + x^*) - K_k(x) \quad (k=1,2,).$$

It is easy to verify that

$$\Psi'(x^* - 0) = x^*(-0,5) + 1 - 0,5 = 0,5(1 - x^*) \geqq 0 ,$$

and

$$\Psi'(x^* + 0) = x^*(-1) + 1 - 0,5 = 0,5 - x^* \leqq 0.$$

Part c/ implies that function Ψ is concave in x, consequently from the inequalities $\Psi'(x^* - 0) \geqq 0$ and $\Psi'(x^* + 0)$ we can conclude that x^* is a maximumpoint of the function Ψ . Thus arbitrary $x^* \in X^*$ is an equilibrium point.

Next we discuss a numerical algorithm for findig the equilibrium points of the classical oligopoly game. Under the assumptions of Theorem 4. the following statements are true.

Lemma 8.

a/ For $s \in [0, L]$, $X_k(s)$ is not empty and is a closed interval $[A_k(s), B_k(s)]$, $(k=1,2,\dots,n)$;

b/ for $0 \leq s < s' \leq L$ the inequality $B_k(s') \leqq A_k(s)$ holds for $k=1,2,\dots,n$;

c/ if f is differentiable at the point s, then

$A_k(s) = B_k(s)$;

d/ if f is differentiable in the interval $[0, L]$, then

$A_k(s)$ is a continuous function of s.

Proof. Parts a/ and b/ can be proven by simple modifications of parts C/a/ and C/b/ of the proof of Theorem 1. in paper [10]. The statements c/ and d/ are proven in the C/a,b,c part of the proof of Theorem 1. in paper [10]. ∎

Lemma 9. If $\underline{x}^* = (x_1^*,\dots,x_n^*)$ and $\underline{x}^{**} = (x_1^{**},\dots,x_n^{**})$ are equilibrium points of the classical oligopoly game having the properties given in Theorem 4., then

$$\sum_{k=1}^{n} x_k^* = \sum_{k=1}^{n} x_k^{**} .$$

Proof. Assume that $s^* = \sum_{k=1}^{n} x_k^* < s^{**} = \sum_{k=1}^{n} x_k^{**}$. Then

$$s^* = \sum_{k=1}^{n} x_k^* \geqq \sum_{k=1}^{n} A_k(s^*) \geqq \sum_{k=1}^{n} B_k(s^{**}) \geqq \sum_{k=1}^{n} x_k^{**} = s^{**} ,$$

which is a contradiction. ∎

Corollary. The point-to-set mapping $X(s)$ has exactly one fixed point, which can be computed by the usual bisection method (see F. Szidarovszky, S.Yakowitz [12]).

Theorem 5. Assume that the conditions of Theorem 4. are satisfied. Let s^* be the unique fixed point of the mapping $X(s)$. Then all equilibrium points of the classical oligopoly game can be obtained by the solution of the system of linear equations and inequalities:

$$A_k(s^*) \leqq x_k \leqq B_k(s^*) \qquad (k=1,2,\dots,n)$$

$$\sum_{k=1}^{n} x_k = s^*$$

Proof. The statement is a consequence of Lemma 8. and Lemma 9. ∎

Corollary. If in addition to the conditions of Theorem 4. function f is differentiable on the interval $[0, L]$, then the equilibrium point is unique.

Remark 1. It is interesting to observe that the game is not linear but the set of equilibrium points is a simplex.

Remark 2. The uniqueness of the equilibrium point depends on the differentiability of a function and not on strict concavity as it is usual in the theory of nonlinear programming.

Special cases.

1. In case of f and K_k $(1 \leqq k \leqq n)$ being twice differentiable the uniqueness was proved by O. Opitz [7] without giving any algorithm for finding it.

2. Under the assumptions of O. Opitz, F. Szidarovszky [9] proved the existence and uniqueness of the equilibrium point

and also gave an iterative algorithm for computing it.

3. If the cost functions K_k are identical and the conditions of O. Opitz are satisfied, then E. Burger [1] proved the existence and uniqueness of the equilibrium point and also gave an algorithm to compute it. We remark that the algorithm of Szidarovszky is a generalization of Burger's method.

4. If the functions f and K_k $(k=1,2,\dots,n)$ are linear, then the existence and uniqueness was proved by M. Maňas, [4], who gave an algorithm which is independent of the method of Szidarovszky. We remark that using the result of Theorem 5. the equilibrium point in this special case can be given in closed form (see pp. 37-39 of [10]).

4. The group equilibrium problem

In this paragraph we will discuss the generalized version of the classical oligopoly game Γ having the srategy sets

$$X_k = [0, L_{k1}] \times [0, L_{k2}] \times \dots \times [0, L_{ki_k}] \quad (1 \le k \le n) \tag{25}$$

and pay-off functions

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \left(\sum_{i=1}^{i_k} x_{ki}\right) f\left(\sum_{\ell=1}^{n} \sum_{j=1}^{i_\ell} x_{\ell j}\right) - K_k(\underline{x}_k), \tag{26}$$

where for $k=1,2,\dots,n$, $\underline{x}_k = (x_{k1},\dots,x_{ki_k}) \in X_k$. This game can occur when the players of the classical oligopoly game form disjoint groups and they tend to the optimal income of the group. If the number of members in group k is equal to i_k,