1. General results

A mathematical game is a set $\Gamma = (n; X_1, X_2, \ldots, X_n; \varphi_1, \varphi_2, \ldots, \varphi_n)$, where n is a positive integer, $X_1, X_2, \ldots, X_n$ are arbitrary sets and the functions φ_k $(1 \leq k \leq n)$ are such that $\mathcal{D}(\varphi_k) = X_1 \times X_2 \times \ldots \times X_n$, $\mathcal{R}(\varphi_k) \subset R^1$. Here n is called the number of players, the sets X_k are the strategy sets and the functions φ_k are the pay-off functions. Assuming that the first player chooses the strategy $x_1 \in X_1$, the second player chooses the strategy $x_2 \in X_2$, etc., than the value $\varphi_k(x_1, x_2, \ldots, x_n)$ is considered to be the income of player k $(k = 1,2,\ldots,n)$. In the special case of $\sum_{i=1}^{n} \varphi_i = 0$ the game is called a <u>zero sum n-person game</u>.

<u>Definition 1.</u> A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is a Nash-equilibrium point of the game Γ, if

a/ $x_k^* \in X_k$ $(k = 1,2,\ldots,n)$;

b/ for $k = 1,2,\ldots,n$ and arbitrary $x_k \in X_k$,

$$\varphi_k(x_1^*, \ldots, x_k, \ldots x_n^*) \leq \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*). \quad (1)$$

<u>Remark.</u> The equilibrium strategy x_k^* is optimal for the player k assuming that the other players choose the corresponding components of the equiliberium point.

<u>Example 1.</u> Let $n=2$,

$$X_1 = \{1,2,\ldots,m_1\}, \quad X_2 = \{1,2,\ldots,m_2\}.$$

In this special case the game Γ is called a <u>two-person finite game.</u> Let us introduce the following notations:

$$\varphi_1 (i,j) = a_{ij}$$

$$\varphi_2 (i,j) = b_{ij} \qquad (i=1,2,\ldots,m_1;\ j=1,2,\ldots,m_2),$$

$$\underline{\underline{A}} = (a_{ij}), \quad \underline{\underline{B}} = (b_{ij}).$$

Observe that $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are $m_1 \times m_2$ matrices. The inequalities (1) imply that a pair (i_o, j_o) is an eyuilibrium point if and only if

$$b_{i_o j} \leqq b_{i_o j_o} \qquad (j=1,2,\ldots,m_2)$$

$$a_{i j_o} \leqq a_{i_o j_o} \qquad (i=1,2,\ldots,m_1).$$

In other words, the element $a_{i_o j_o}$ is maximal in its column (in matrix $\underline{\underline{A}}$), and the element $b_{i_o j_o}$ is maximal in its row (in matrix $\underline{\underline{B}}$). From this simple observation we can easily verify that the game determined by matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has no equilibrium point; the game with matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

has a unique equilibrium point $(1,1)$; and any pair (i,j) of the game given by matrices

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is an equilibrium point.

The computation of the equilibrium points for finite games is an easy job since a finite number of inequalities has to be checked.

Example 2. Let $n=2$,

$$X_1 = \left\{ \underline{x}_1 \mid \underline{x}_1 \in R^{m_1},\ \underline{x}_1 \geqq \underline{0},\ \underline{1}^T \underline{x}_1 = 1 \right\},$$

$$X_2 = \left\{ \underline{x}_2 \mid \underline{x}_2 \in R^{m_2},\ \underline{x}_2 \geqq \underline{0},\ \underline{1}^T \underline{x}_2 = 1 \right\},$$

$$\varphi_1(\underline{x}_1, \underline{x}_2) = \underline{x}_1^T \underline{\underline{A}}\, \underline{x}_2, \quad \varphi_2(\underline{x}_1, \underline{x}_2) = \underline{x}_1^T \underline{\underline{B}}\, \underline{x}_2,$$

where $\underline{0}$ is the zero vector, the vector $\underline{1}$ has unit components, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are $m_1 \times m_2$ real matrices. The game defined above is called a bimatrix game. In the special case of $\underline{\underline{B}} = -\underline{\underline{A}}$ the game is called a matrix game. It is known that the equilibrium problem of matrix games is equivalent to the solution of linear programming problems and the equilibrium problem of bimatrix games can be solved by the solution of quadratic programming problems. The details will be discussed later. Note that the bimatrix games are generalizations, extensions of finite two-person games, since the strategies of the players are the choices of distributions defined on the sets $\{1,2,\dots,m_1\}$ and $\{1,2,\dots,m_2\}$ instead of the choices of one-one element from each set. The pay-off of the generalized game is the expectation of the pay-off obtained in the finite game with respect to the distribution chosen by each player.

Example 3. Let us consider the following n-person game:

$$X_k = \{\underline{x}_k \mid \underline{\underline{A}}_k \underline{x}_k \leq \underline{b}_k\} \qquad (k=1,2,\dots,n)$$

$$\varphi_k(\underline{x}_1,\dots,\underline{x}_n) = \sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} a^{(k)}_{i_1 i_2 \dots i_n} x^{(1)}_{i_1} x^{(2)}_{i_2} \dots x^{(n)}_{i_n} ,$$

where $\underline{\underline{A}}_k$ is an $\ell_k \times m_k$ real matrix, $\underline{b}_k \in R^{\ell_k}$ is a real vector, the numbers $a^{(k)}_{i_1 \dots i_n}$ are given real parameters, and for $k=1,2,\dots,n$, $\underline{x}_k = (x_1^{(k)},\dots, x^{(k)}_{m_k})$. This game is called a generalized polyhedral game. To simplify our notations let

$$a_i^{(k)}(\underline{x}) = \sum_{i_1=1}^{m_1} \dots \sum_{i_{k-1}=1}^{m_{k-1}} \sum_{i_{k+1}=1}^{m_{k+1}} \dots \sum_{i_n=1}^{m_n} a^{(k)}_{i_1 \dots i_{k-1} i i_{k+1} \dots i_n} \cdot x^{(1)}_{i_1} \dots$$

$$\dots x^{(k-1)}_{i_{k-1}} x^{(k+1)}_{i_{k+1}} \dots x^{(n)}_{i_n} ,$$

and

$$\underline{a}_k(\underline{x}) = \left(a_1^{(k)}(\underline{x}),\dots, a_{m_k}^{(k)}(\underline{x})\right)^T ,$$

then

$$\varphi_k(\underline{x}) = \underline{a}_k(\underline{x})^T \underline{x}_k ,$$

where $\underline{a}_k(\underline{x})$ is independent of $\underline{x}_k$.

In the special case of

$$\underline{\underline{A}}_k = \begin{pmatrix} -\underline{\underline{I}} \\ \underline{1}^T \\ -\underline{1}^T \end{pmatrix} , \qquad \underline{b}_k = \begin{pmatrix} \underline{0} \\ 1 \\ -1 \end{pmatrix}$$

(where $\underline{\underline{I}}$ is the m_k dimensional unit matrix, $\underline{1}, \underline{0} \in R^{m_k}$, the vector $\underline{0}$ is the zero vector, the vector $\underline{1}$ has unit components) we have

$$X_k = \left\{ \underline{x}_k \;\middle|\; \underline{x}_k \in R^{m_k} ,\ \underline{x}_k \geqq \underline{0} ,\ \underline{1}^T \underline{x}_k = 1 \right\} ,$$

and the game is called the <u>mixed extension</u> of finite n-person games. Observe that for n=2 we have the bimatrix games with

$$\underline{\underline{A}} = \left(a^{(1)}_{i_1 i_2} \right) \quad \text{and} \quad \underline{\underline{B}} = \left(a^{(2)}_{i_1 i_2} \right) ,$$

since

$$\varphi_1(\underline{x}_1, \underline{x}_2) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} a^{(1)}_{i_1 i_2} x^{(1)}_{i_1} x^{(2)}_{i_2} = \underline{x}_1^T \underline{\underline{A}} \underline{x}_2$$

and

$$\varphi_2(\underline{x}_1, \underline{x}_2) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} a^{(2)}_{i_1 i_2} x^{(1)}_{i_1} x^{(2)}_{i_2} = \underline{x}_1^T \underline{\underline{B}} \underline{x}_2 .$$

First we will show the connection between certain mathematical programming problems and two-person zero sum games.

Let us consider the mathematical programming problem

$$\begin{array}{c} \underline{x} \in X \\ \underline{g}(\underline{x}) \geqq \underline{0} \\ \hline f(\underline{x}) \longrightarrow \max , \end{array} \qquad (2)$$

where X is an arbitrary subset of R^n /it may be discrete/, $\mathcal{D}(\underline{g}) \subset R^n$, $\mathcal{R}(\underline{g}) \subset R^m$, $\mathcal{D}(f) \subset R^n$, $\mathcal{R}(f) \subset R^1$. Let

$$R^m_+ = \left\{ \underline{u} \;\middle|\; \underline{u} \in R^m ,\ \underline{u} \geqq \underline{0} \right\} ,$$

and let us consider the two-person zero sum game

$$\Gamma = (2 ; X , R_m^+ ; F, -F), \qquad (3)$$

where

$$F(\underline{x}, \underline{u}) = f(\underline{x}) + \underline{u}^T g(\underline{x}).$$

Lemma 1. If $(\underline{x}^*, \underline{u}^*)$ is an equilibrium point of the game Γ, than $\underline{x}^*$ is an optimal solution to the programming problem (2).

Proof. The inequalities (1) imply that if $(\underline{x}^*, \underline{u}^*)$ is equilibrium point, than

$$f(\underline{x}^*) + \underline{u}^{*T} g(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^{*T} g(\underline{x}) \qquad (\forall\, \underline{x} \in X) \qquad (4)$$

$$f(\underline{x}^*) + \underline{u}^{*T} g(\underline{x}^*) \leqq f(\underline{x}^*) + \underline{u}^{T} g(\underline{x}^*) \qquad (\forall\, \underline{u} \in R^m_+). \qquad (5)$$

First we observe that $g(\underline{x}^*) \geqq \underline{0}$. Let us assume that a component $g_i(\underline{x}^*) < 0$, then taking the ith component of $\underline{u}$ sufficiently large, the inequality (5) will not hold. Let $\underline{u} = \underline{0}$, then inequality (5) implies $\underline{u}^{*T} g(\underline{x}^*) \leqq 0$. But $\underline{u}^* \geqq \underline{0}$, $g(\underline{x}^*) \geqq \underline{0}$, consequently $\underline{u}^{*T} g(\underline{x}^*) \geqq 0$. Thus $\underline{u}^{*T} g(\underline{x}^*) = 0$.

Since $\underline{x}^* \in X$, $g(\underline{x}^*) \geqq \underline{0}$, the vector $\underline{x}^*$ is a feasible solution of the problem (2). We can easily prove that $\underline{x}^*$ is an optimal solution. Let $\underline{x}$ be any feasible solution of the problem (2). Then inequality (4) implies

$$f(\underline{x}^*) = f(\underline{x}^*) + \underline{u}^{*T} g(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^{*T} g(\underline{x}) \geqq f(\underline{x}),$$

thus $\underline{x}^*$ is an optimal solution. ∎

Remark. The opposite statement is not true in general. The Kuhn-Tucker theory gives sufficent conditions, for an arbitrary optimal solution of the problem (2) to be obtainable from an equilibrium point of the game Γ.

Newt we will prove that the equilibrium problem of n-person general games is equivalent to the fixed-point problem of a certain point-to-set mapping. Let us consider the n-person game in a more generalized form: $\Gamma = (n\ ;\ X_1, X_2, \ldots, X_n, X\ ;\ \varphi_1, \varphi_2, \ldots, \varphi_n)$, where n is the number of players; $X_1, X_2, \ldots, X_n$ are the strategy sets of the players, $X \subset X_1 \times X_2 \times \ldots \times X_n$ is the simultaneous strategy set, the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the pay-off functions such that $\mathcal{D}(\varphi_k) = X$, $\mathcal{R}(\varphi_k) \subset R^1$ $(k=1,2,\ldots,n)$. Here we assume that the players can not choose their strategies independently of each other because of circumstances independent of the players /for instance in production games it is inpossible all players to have maximal production because of the bounded quantity of row materials/, and in the concrete realizations of the game only the elements of X can appear as strategy vectors.

Definition 2. A vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point of the game Γ if

a/ $\underline{x}^* \in X$;

b/ for $k=1,2,\ldots,n$ for arbitrary $(x_1^*, \ldots, x_k, \ldots, x_n^*) \in X$,

$$\varphi_k(x_1^*, \ldots, x_k, \ldots, x_n^*) \leqq \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*). \quad (6)$$

Let us consider the following function,

$$\Phi(\underline{x}, \underline{y}) = \sum_{k=1}^{n} \varphi_k(x_1, \ldots, y_k, \ldots, x_n),$$

where for $k=1,2,\ldots,n$, $(x_1, \ldots, y_k, \ldots, x_n) \in X$. Let us say that the pair $(\underline{x}, \underline{y})$ is feasible if $\underline{x} \in X$ and for $k=1,2,\ldots,n$, $(x_1, \ldots, y_k, \ldots, x_n) \in X$. Then function Φ is defined for arbitrary feasible pairs $(\underline{x}, \underline{y})$.

Lemma 2. The vector $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is an equilibrium point of the game Γ if and only if for arbitrary feasible pairs $(\underline{x}^*, \underline{y})$, $\Phi(\underline{x}^*, \underline{x}^*) \geq \Phi(\underline{x}^*, \underline{y})$.

Proof. a/ Let us assume that $\underline{x}^*$ is an equilibrium point. Then for arbitrary k and $(x_1^*, \ldots, y_k, \ldots, x_n^*) \in X$ the inequality holds

$$\varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*) \geq \varphi_k(x_1^*, \ldots, y_k, \ldots, x_n^*).$$

Let us add these inequalities for $k=1,2,\ldots,n$ and let $\underline{y} = (y_1, \ldots, y_n)$, then we have

$$\sum_{k=1}^{n} \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*) = \Phi(\underline{x}^*, \underline{x}^*) \geq \sum_{k=1}^{n} \varphi_k(x_1^*, \ldots, y_k, \ldots, x_n^*)$$

$$= \Phi(\underline{x}^*, \underline{y}).$$

b/ Let us now assume that $\underline{x}^* \in X$ and for an arbitrary feasible pair $(\underline{x}^*, \underline{y})$, $\Phi(\underline{x}^*, \underline{x}^*) \geq \Phi(\underline{x}^*, \underline{y})$. Let k be fixed and let $\underline{y} = (x_1^*, \ldots, x_k, \ldots, x_n^*) \in X$. Then obviously the pair $(\underline{x}^*, \underline{y})$ is feasible and

$$\Phi(\underline{x}^*, \underline{x}^*) \geqq \Phi(\underline{x}^*, \underline{y}). \tag{7}$$

Since

$$\Phi(\underline{x}^*, \underline{x}^*) = \sum_{i \neq k} \varphi_i(\underline{x}^*) + \varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*)$$

and

$$\Phi(\underline{x}^*, \underline{y}) = \sum_{i \neq k} \varphi_i(\underline{x}^*) + \varphi_k(x_1^*, \ldots, x_k, \ldots, x_n^*)$$

the inequality (7) implies that

$$\varphi_k(x_1^*, \ldots, x_k^*, \ldots, x_n^*) \geqq \varphi_k(x_1^*, \ldots, x_k, \ldots, x_n^*),$$

thus $\underline{x}^*$ is an equilibrium point. ∎

By using the above notations let us introduce the following point-to-set mapping

$$\phi(\underline{x}) = \left\{ \underline{t} \mid (\underline{x}, \underline{t}) \text{ is feasible and } \Phi(\underline{x}, \underline{t}) = \max \left\{ (\underline{x}, \underline{y}) ; (\underline{x}, \underline{y}) \text{ is feasible} \right\} \right\}.$$

As a simple consequence of Lemma 2. we have the following important result.

<u>Lemma 3.</u> A vector $\underline{x}^*$ is an equilibrium point of the game if and only if $\underline{x}^* \in \phi(\underline{x}^*)$ /i.e. $\underline{x}^*$ is a fixed point of the mapping ϕ /.

The most important existence theorem for n-person games can be proven by using the Kakutani fixed point theorem for showing that the mapping ϕ has at least one fixed point. This theorem is called Nikaido-Isoda theorem and it is the following:

<u>Theorem 1.</u> Assume that

a/ X is a bounded, closed and convex subset of a finite dimension Eucledian space;

b/ for $k=1,2,\ldots,n$ the functions φ_k are continuous and for fixed $x_1,\ldots,x_{k-1}, x_{k+1},\ldots,x_n$ they are concave in x_k. Under these assumptions the game has at least one equilibrium point.

Proof. See J.B. Rosen [8] . ■

Remark. If we assume that the functions φ_k are strictly concave in x_k, then the uniqueness of the equilibrium point in general is not true /see Example 4./. For the uniqueness of the equilibrium point of n-person games J.B. Rosen [8] gave sufficient conditions, but the assumptions of the next paragraphs are independent of the conditions introduced by J.B. Rosen.

2. The solution of a special class of concave games

Let us assume that for $k=1,2,\ldots,n$

$$X_k = \left\{ \underline{x}_k \mid \underline{x}_k \in R^{m_k}, \ \underline{h}_k(\underline{x}_k) \geqq \underline{0} \right\},$$

where

a/ $\mathcal{D}(\underline{h}_k) = R^{m_k}$, $\mathcal{R}(\underline{h}_k) \subset R^{\ell_k}$, the components of $\underline{h}_k$ are concave, continuously differentiable functions;

b/ X_k is bounded, and in each point of X_k the Kuhn-Tucker regularity condition holds /see G. Hadley [3]/ ;

c/ φ_k is continuous, concave in $\underline{x}_k$ for fixed $\underline{x}_1,\ldots,\ldots, \underline{x}_{k-1}, \underline{x}_{k+1},\ldots,\underline{x}_n$ and continuously differentiable with respect to $\underline{x}_k$.

Lemma 4. The game $\Gamma = (n; X_1,\ldots,X_n ; \varphi_1,\ldots, \varphi_n)$ has at least one equilibrium point.