Introduction - In the work (1) a general disjoint decomposition of semigroups was given, wich can be applied for the case of regular semigroups. The aim of the present paper is to obtain a characteristic decomposition of regular semigroups based on the decomposition studied in (1). We shall investigate the components of this decomposition and the interrelations between them.

By making use of the work (2) we study the cases of regular semigroups with or without left or right identity element.

Finally we make some special remarks.

Notations : For two sets A, B we write $A \subset B$ if A is a proper subset of B. By magnifying element we mean a left magnifying element.

§ 1.

Let S be a semigroup without nonzero annihilator. This is not a proper restriction because every semigroup can be reduced to this case.

Then S has the following disjoint decomposition:

/1/ $$S = \bigcup_{i=o}^{5} S_i ,$$

where

$$S_o = \{a \in S \mid a\, S \subset S \text{ and } \exists x \in S,\ x \neq o \text{ and } a\, x = 0\},$$

$$S_1 = \{a \in S \mid a\, S = S \text{ and } \exists y \in S,\ y \neq 0 \text{ and } a\, y = 0\},$$

$$S_2 = \{a \in S \setminus (S_o \cup S_1) \mid a\, S \subset S \text{ and } \exists x_1, x_2 \in S,\ x_1 \neq x_2 \text{ and } ax_1 = ax_2\},$$

$$S_3 = \{a \in S \setminus (S_o \cup S_1) \mid a\, S = S \text{ and } \exists y_1, y_2 \in S,\ y_1 \neq y_2 \text{ and } ay_1 = ay_2\},$$

$$S_4 = \{a \in S \setminus (S_o \cup S_1 \cup S_2 \cup S_3) \mid a\, S \subset S\},$$

$$S_5 = \{a \in S \setminus (S_o \cup S_1 \cup S_2 \cup S_3) \mid a\, S = S\}.$$

It is easy to see that the components $S_i (i = 0,1,\ldots,5)$ are semigroups, $S_i \cap S_j = \emptyset$ $(i \neq j)$ and the following relations hold:

$$/2/ \quad \begin{cases} S_5 S_i \subseteq S_i, & S_i S_5 \subseteq S_i, \quad (0 \leq i \leq 5) \\ S_4 S_3 \subseteq S_2, & S_4 S_2 \subseteq S_2, \quad S_4 S_1 \subseteq S_o, \quad S_4 S_o \subseteq S_o, \\ S_2 S_3 \subseteq S_2, & S_o S_1 \subseteq S_o. \end{cases}$$

It is evidente that an analogous

$$/1'/ \qquad S = \bigcup_{i=o}^{5} T_i$$

decomposition exists, where

$T_o = \{a \in S \mid S\,a \subset S$ and $\exists x \in S$, $x \neq 0$ and $x\,a = 0\}$,

$T_1 = \{a \in S \mid S\,a = S$ and $\exists y \in S$, $y \neq 0$, and $y\,a = 0\}$,

etc.

Our theorems concern for the decomposition /1/, but analogous results can be formulated for the decomposition /1'/.

THEOREM 1.1. S_5 is a right group.

PROOF. It is easy to see, that S_5 is right simple and left cancellative, whence the assertion follows.

Let $S_o \cup S_2 = \overline{S}_2$ and $S_1 \cup S_3 = \overline{S}_3$.

THEOREM 1.2. $\overline{S}_2$ is a subsemigroup of S.

PROOF. If $s_o \in S_o$ and $s_2 \in S_2$, then $s_o s_2 \in \overline{S}_2$. There are elements x, $y \in S$, $x \neq y$ such that $s_2 x = s_2 y$. We have $s_o s_2 \notin \overline{S}_3$ and $s_o s_2 \notin S_5$ because $s_o s_2 S = s_o(s_2 S) \subset S$. If $s_o s_2 \neq 0$, then $(s_o s_2)x = (s_o s_2)y$ $(x \neq y)$, whence $s_o s_2 \in S_2 \subseteq \overline{S}_2$. Similarly $s_2 s_o \in \overline{S}_2$. If $s_o \neq 0$ then $s_2 s_o \neq 0$ because $s_2 \in S_2$. Since $s_o \in S_o$, there is an element $z \neq 0$ such that $s_o z = 0$, hence $(s_2 s_o)z = 0$. Therefore $s_2 s_o \in S_o$. Q.e.d.

THEOREM 1.3 $\overline{S}_3$ contains all the magnifying elements of S and only them.

PROOF. Let $a \in S_1 \cup S_3$. If $a \in S$ and $a S = S$, further there is an $y \neq 0$ so that $a y = 0$, then $S' = S \setminus \{0\} \subset S$ and $aS' = S$, whence a is a magnifying element. If $a \in S_3$ and $a S = S$, further there exist $x,y \in S$ $(x \neq y)$ such that $a x = a y$, then $a(S - \{x\}) = S$ and a is a magnifying element.

Conversely, if $a \in S$ is a magnifying element, then $a \notin S_o \cup S_2 \cup S_4$, and $a M = S$ $(M \subset S)$. Thus there are $m \in M$ and $s \in S \setminus M$ such that $a m = a s$. Hence it follows that $a \in S_1 \cup S_3$, q.e.d.

Remark. Theorem 1.2. and Theorem 1.3. imply

/3/ $$S_oS_2 \subseteq S_o \cup S_2 \quad ; \quad S_2S_o \subseteq S_o \cup S_2 \quad ;$$
$$S_1S_3 \subseteq S_1 \cup S_3 \quad ; \quad S_3S_1 \subseteq S_1 \cup S_3 \quad .$$

In what follows we assume S is a regular semigroup, i. e. to every $a \in S$ there is an $x \in S$ such that $a = a x a$ and $x = x a x$ / x is the inverse of a/. The elements $a x$, $x a$ are idempotent and $aS \supseteq axS \supseteq axaS = aS$ implies $axS = aS$ and similarly $x a S = x S$.

The regular semigroup S can contain a zero element hence the components S_o and S_1 can exist in the decomposition /1/.

THEOREM 1.4. The inverses of elements of $\bar{S}_3$ are in S_4 and the inverses of elements of S_4 are in $\bar{S}_3$.

PROOF. Let $a \in \bar{S}_3$ and $x \in S$ an inverse of a, that is

$a\,x\,a = a$ and $x\,a\,x = x$. First of all, we show that $x\,S \subset S$.
Suppose that $x\,S = S$, then there is a subset $S' \subset S$ such that $a\,S' = S$ because a is a magnifying element. Hence it follows that $x\,a\,S' = x\,S = S$. But we have $(x\,a)S = x\,S = S$ and $x\,a$ is idempotent, that is, $x\,a$ is a left identity of S, therefore $(x\,a)S' = S' \neq S$, which is a contradiction, Thus $x\,S \subset S$, whence x is contained by S_o, S_2 or S_4. If $x \in S_2$, then
$x\,s_1 = x\,s_2$ $(s_1 \neq s_2)$ and $(a\,x)s_1 = (a\,x)s_2$. Since $(a\,x)S = a\,S = S$ and $a\,x$ is idempotent we obtain that $a\,x$ is a left identity of S, i.e. $(a\,x)s_1 = (a\,x)s_2$ implies $s_1 = s_2$, which is a contradiction. It can similarly be proved that $x \notin S_o$. It remains the case $x \in S_4$.

Conversely, let $b \in S_4$, that is, $b\,S = S' \subset S$. Let y be an inverse of b in S. Hence $b\,y\,S = b\,S = S'$. Suppose that $y\,S \subset S$. Let $y\,S = S''$ $(\neq S)$. Hence $b\,S'' = b\,y\,S = b\,S$. Thus there are elements $s \notin S''$, and $s'' \in S''$ such that $b\,s'' = b\,s$.
But every element a of S for which $a\,x_1 = a\,x_2 (x_1 \neq x_2)$ is contained by $S_o \cup S_1$ or $S_2 \cup S_3$, which is a contradiction with $b \in S_4$.
Thus necessarily $y\,S = S$, that is, $y \notin S_o \cup S_2 \cup S_4$. If $y \in S_5$, then $(y\,b)S = y\,S = y(b\,S) = y\,S' = S$ $(S' \neq S)$, i.e., $y \in S_1 \cup S_3$, wich is a contradiction. It remains the only case $y \in S_1 \cup S_3 = \bar{S}_3$, q.e.d.

It is casy to see, that the inverses of $\bar{S}_3$ exhaust S_4 and the inverses of the elements of S_4 also exhaust $\bar{S}_3$.

COROLLARY. 1.5. If a regular semigroup S does not contain magnifying elements ($\bar{S}_3 = \emptyset$) , then $S_4 = \emptyset$, and conversely, $S_4 = \emptyset$ implies $\bar{S}_3 = \emptyset$.

COROLLARY 1.6. If a regular semigroup S does not contain left identity, then $\bar{S}_3 = \emptyset$ and hence $S_4 = \emptyset$.

For if $a \in \bar{S}_3$ and $x \in S_4$ is an inverse of a , then a x is a left identity of S.

THEOREM 1.7. $\bar{S}_2$ is a regular semigroup and the inverses of an element of $\bar{S}_2$ are contained by $\bar{S}_2$.

PROOF. Let $a \in \bar{S}_2$ and x an inverse of a in S. Since $a \in S_o \cup S_2$, we have $a S \subset S$. Assume that $x S = S$. Then $(x a)S = x(a S) = x S = S$, whence x is a magnifying element, i.e., $x \in \bar{S}_o$. But every inverse of $\bar{S}_3$ is /by Theorem 1.4./ in S_4, thus $a \in S_4$, which is a contradiction. Therefore $x S \subset S$. But $x \notin S_4$ because $a \in \bar{S}_2$. We conclude that $x \in S_o \cup S_2 = \bar{S}_2$, q.e.d.

The above results yield the following result.

THEOREM 1.8. A semigroup S is regular if and only if it has a decomposition /1/

$$S = \bigcup_{i=o}^{5} S_i ,$$

where

a/ $\bar{S}_2 = S_o \cup S_2$ is regular;

b/ the inverses of elements of $\bar{S}_3 = S_1 \cup S_3$ are contained by S_4 and conversely;

c/ S_5 is a right group.

PROOF. The necessity follows from Theorems 1.1., 1.4, 1.7. The sufficiency it follows from the fact that a right group is regular.

§. 2.

In this § we shall deepen our knowledge concerning the decomposition /1/ of a regular semigroup S as well as on the components $\bar{S}_2, \bar{S}_3$ and $\bar{S}_4$.

THEOREM 2.1. Let S be a regular semigroup without /left/ magnifying elements, Using the notations $\bar{S}_2 = \bar{S}_2^{\,1}$, $S_5 = S_5^1$ we obtain the following decompositions:

$S = \bar{S}_2^1 \cup S_5^1$ and if $\bar{S}_2^1$ has no magnifying element,

$\bar{S}_2^1 = \bar{S}_2^2 \cup S_5^2$ and if $\bar{S}_2^2$ has no magnifying element,

....

$\bar{S}_2^k = \bar{S}_2^{k+1} \cup S_5^{k+1}$

....

where $\bar{S}_2^k$ are regular semigroups, S_5^k are right groups and the following inclusions hold:

/4/
$$S_5^k S_5^j \subseteq S_5^k \qquad (k \geq j)$$
$$S_5^j S_5^k = S_5^k \qquad (k \geq j)$$
$$S_5^k \bar{S}_2^j = \bar{S}_2^j \qquad (k \leq j)$$
$$\bar{S}_2^j S_5^k \subseteq \bar{S}_2^j \qquad (k \leq j)$$

PROOF. It is enough to give a proof for the following cases:

$$S_5^1 S_5^k , \quad S_5^k S_5^1 , \ S_5^1 \bar{S}_2^j \ , \ \bar{S}_2^j S_5^1$$

because the proof is similar in the semigroups $\bar{S}_2^i$.

The proof is by induction on k and j. It is trivial that

$$S_5^1 S_5^1 = S_5^1 , \quad s_5^1 \bar{S}_2^1 = \bar{S}_2^1 , \quad s_5^2 \bar{S}_2^1 = \bar{S}_2^1 . \qquad (s_5^k \in S_5^k) .$$

Hence

$$s_5^1 s_5^2 \bar{S}_2^1 = \bar{S}_2^1 ,$$

i.e., $s_5^1 s_5^2 \in S_5^2$ for all $s_5^1 \in S_5^1$ and $s_5^2 \in S_5^2$.
Since $s_5^1 \bar{S}_2^1 = \bar{S}_2^1$, furthermore $s_5^1 S_5^2 \subseteq S_5^2$ and $s_5^1 (s_2^2 \bar{S}_2^1) \subset \bar{S}_2^1$, that is, $s_5^1 s_2^2 \in \bar{S}_2^2$, we conclude $s_5^1 S_5^2 = S_5^2$ and $s_5^1 \bar{S}_2^2 = \bar{S}_2^2$, whence $S_5^1 S_5^2 = S_5^2$, $S_5^1 \bar{S}_2^2 = \bar{S}_2^2$.

Thus we have $S_5^1 S_5^1 = S_5^1$, $S_5^1 \bar{S}_2^1 = \bar{S}_2^1$, $S_5^1 S_5^2 = S_5^2$,

$$S_5^1 \bar{S}_2^2 = \bar{S}_2^2 , \quad S_5^2 S_5^1 \subseteq S_5^2$$

because $s_5^2\ s_5^1\ S_5^2 = s_5^2\ S_5^2 = S_5^2$ and thus $s_5^2\ s_5^1 \in S_5^2$.

The first step of the proof is true.

Now suppose that the following conditions hold:

$$S_5^1\ S_5^k = S_5^k\ ,\quad S_5^k\ S_5^1 \subseteq S_5^k\ ,\quad S_5^1\ \bar{S}_2^j = \bar{S}_2^j\ ,\quad \bar{S}_2^j\ S_5^1 \subseteq \bar{S}_2^j\ .$$

By the definition we have $s_5^{k+1}\ \bar{S}_2^k = \bar{S}_2^k$. Hence

$$(s_5^1\ s_5^{k+1})\ \bar{S}_2^k = s_5^1 \bar{S}_2^k = \bar{S}_2^k,$$

whence $s_5^1\ s_5^{k+1} \in S_5^{k+1}$.

Thus we obtain

$$S_5^{k+1} = (s_5^1\ s_5^{k+1})\ S_5^{k+1} = s_5^1\ S_5^{k+1}\ ,$$

whence $S_5^1\ S_5^{k+1} = S_5^{k+1}$.

It holds $(s_5^{k+1}\ \ s_5^1) S_5^{k+1} = S_5^{k+1}$, furthermore $s_5^{k+1}\ \ s_5^1 \in \bar{S}_2^k$,

thus

$s_5^{k+1}\ \ s_5^1 \in S_5^{k+1}$ implies $S_5^{k+1}\ \ S_5^1 \subseteq S_5^{k+1}$.

We have also $(s_5^1\ s_2^{j+1}) \bar{S}_2^j \subset s_5^1\ \bar{S}_2^j = \bar{S}_2^j$, whence $s_5^1\ s_2^{j+1} \in \bar{S}_2^{j+1}$

and $s_5^1\ \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$ implies $S_5^1 \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$.

Finally we have $s_2^{j+1}\ \ s_5^1 \in \bar{S}_2^j$ and $s_2^{j+1} s_5^1 \bar{S}_2^j = s_2^{j+1}\ \ S_2^j \subset S_2^j$,

whence it follows that $s_2^{j+1}\ s_5^1 \in \bar{S}_2^{j+1}$ and $\bar{S}_2^{j+1} S_5^1 \subseteq \bar{S}_2^{j+1}$.

Q.e.d.

COROLLARY 2.2. If S and $\bar{S}_2^k$ $(k \geqslant 1)$ are regular semigroups without magnifying elements, then S has one of the following four types of decompositions:

a/ $S = ((((\ldots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$,

with infinite number of components;

b/ $S = \bar{S}_2^1 \cup ((((\ldots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$,

where $\bar{S}_2^1$ is a semigroup of type $\bar{S}_2$ and there are infinitely many components;

c/ $S = (((S_5^n \cup \ldots) \cup S_5^3) \cup S_5^2) \cup S_5^1$,

where the number of components is n;

d/ $S = (((\bar{S}_2^m \cup S_5^m) \cup \ldots) \cup S_5^3) \cup S_5^2) \cup S_5^1$,

where the number of components is $m + 1$.

We shall treat some properties of the semigroups $\bar{S}_3$ and S_4.

THEOREM 2.3. Let $a,b \in \bar{S}_3$, an inverse of a is x. an inverse of b is y $(x,y \in S_4)$. Then $x\,y$ is an inverse of $b\,a$.

PROOF. Since $a\,x$ and $b\,y$ are left identities of S, we have

$$ba\,xy\,ba = b\,(a\,x\,y\,)\,ba = by\,ba = ba,$$

$$xy\,ba\,xy = x\,y\,b\,(a\,x\,y) = x\,y\,b\,y = x\,y\,, \qquad \text{q.e.d.}$$

THEOREM 2.4. If $a,b \in S_4$ and x is an inverse of a, y is an inverse of b, then yx and ab are inverses of each other.

PROOF. By theorem 2.3. $(y\,b\,y)(x\,a\,x)$ is an inverse of $a\,b$, i.e., $a\,b = a\,b\,(y\,b\,y)(x\,a\,x)\,a\,b =$

$$= a\,(b\,y\,b)\,y\,x\,(a\,x\,a)\,b = a\,b\,y\,x\,a\,b\,,$$

$$y\,x\,a\,b\,y\,x = y\,b\,y\,x = y\,x$$

using that $x\,a$, $y\,b$ are left identities of S. Q.e.d.

By Theorem 1.4. $\bar{S}_3 \cup S_4$ is a regular subset of S, but it fails to be a subsemigroup, because, e.g., $S_4 S_3 \subseteq S_2$ /cf.(2)/ .

Let $X_1 = \{x \in S_4 \mid x$ is an inverse of some $a \in S_1\}$

$X_3 = \{y \in S_4 \mid y$ is an inverse of some $b \in S_3\}$

Then $S_4 = X_1 \cup X_3$.

COROLLARY 2.5. X_1 and X_3 are subsemigroups of S_4 . In general, if $A \subseteq \bar{S}_3$ is a subsemigroup, then the inverses of elements of A is a subsemigroup of S_4 .

PROOF. This is an easy consequence of Theorem 2.3.

COROLLARY 2.6. $\bar{S}_3$ and S_4 have no idempotent elements.

PROOF. Every element of $\bar{S}_3$ is a magnifying one, thus $a \neq a^2$ $(a \in \bar{S}_3)$. Assume that $e \in S_4$ is idempotent. Since e is an inverse of e , $e \in \bar{S}_3$ /by Theorem 1.4/, which is a contradiction. Q.e.d.

THEOREM 2.7. Every element of $\bar{S}_3$ and S_4 generates an infinite cyclic semigroup.

PROOF. In opposite case $\bar{S}_3$ or S_4 contains an idempotent element which is a contradiction by 2.6.

THEOREM 2.8. 1./ $\bar{S}_3$ has no/proper/right magnifying element. 2./ S_4 has no left magnifying element. 3./ If $1 \in S$ /i.e. S is a monoid/, then $S_o \cup S_2 \cup S_5$ has no left or right magnifying element. 4./ S_5 has no left magnifying element.

PROOF. /1/ is a consequence of [4] , Chap.III. 5.6. (β)

Since in the product s_4S $(s_4 \in S_4)$ the representation of each element is uniquely, thus the same holds for s_4S_4 , and /2/ is true.

/3/ if follows from [4] , Chap.III. 5.6. / γ / , because the union $S_o \cup S_2 \cup S_5$ does not contain left or right magnifying elements of S.

Finally, S_5 is a right group, and hence it has no left magnifying elements /cf. [4] , Chap. III. 5.3. (γ)/ .

§ 3.

In this § results of the work [2] will be applied to the decomposition /1/ of regular semigroups.

For a regular semigroup S we shall investigate the following cases based on Theorem 4 in [2] :

1/ S has neither left nor right identity element;

2/ S has identity element;

3/ S has either left or right identity element.

In the case 3/ we may assume that S has only left identity element. In the opposite case we need study the decomposition /1'/ instead of /1/.

As it is well known an idempotent element e is $\mathcal{D}$-primitive if it is minimal among the idempotents D_e, where D_e is the $\mathcal{D}$-class of e / $\mathcal{D}$ is a Green's relation/.

In the case 1/ S has no left magnifying elements /cf.corollary 1.6/, that is, $S_1 \cup S_3 = \emptyset$ and $S_4 = \emptyset$, furthermore $S_5 = \emptyset$, because in the contrary case S would have a left identity element. Hence $S = S_o \cup S_2 = \bar{S}_2$.

In the case 2/ suppose that $1 \in S$ is the identity element. Then: a/ if 1 is $\mathcal{D}$ -primitive we have $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ /e.g. $1 \in S_5$/. In this subcase we arrive

$$S = S_o \cup S_2 \cup S_5 .$$

b/ If 1 is not Ω-primitive, then there are magnifying elements, that is $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, S_5 is equal to the subsemigroup of all invertable elements and thus it is nonempty.

Since $S_4S_3 \subseteq S_2$ and $S_4S_1 \subseteq S_o$, at least one of the subsemigroups S_o, S_2 is nonempty. Hence we obtain

$$S = \bar{S}_2 \cup \bar{S}_3 \cup S_4 \cup S_5$$

where all the components are nonvoid.

In the case 3/ suppose that e is a left identity element of S. Then

a/ if e is Ω-primitive, then $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ / for example, $e \in S_5$/. Therefore

$$S = S_o \cup S_2 \cup S_5 .$$

b/ If e fails to be Ω-primitive, then there are magnifying elements, that is $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$ and similarly to the case 2/b, we have $S_o \cup S_2 \neq \emptyset$.

Hence

$$S = \bar{S}_2 \cup \bar{S}_3 \cup S_4 \cup S_5 ,$$

where all the components are nonempty.

Summing up the above statements:

THEOREM 3.1. Let S be a regular semigroup. Then

1/ if S has no left identity element:

$$S_1 \cup S_3 = \emptyset , \quad S_4 = \emptyset , \quad S_5 = \emptyset .$$

2/ If S has identity element:

a/ if 1 is ω-primitive,

$$S_1 \cup S_3 = \emptyset , \quad S_4 = \emptyset, \quad S_5 \neq \emptyset.$$

b/ If 1 is not ω-primitive, we have

$$S_1 \cup S_3 \neq \emptyset , \quad S_4 \neq \emptyset , \quad S_5 \neq \emptyset, \quad S_o \cup S_2 \neq \emptyset .$$

3./ If e is a left identity of S:

a/ if e is ω-primitive, we get

$$S_1 \cup S_3 = \emptyset , \quad S_4 = \emptyset , \quad S_5 \neq \emptyset .$$

b/ if e is not ω-primitive, we have

$$S_1 \cup S_3 \neq \emptyset , \quad S_4 \neq \emptyset , \quad S_5 \neq \emptyset , \quad S_o \cup S_2 \neq \emptyset .$$

Finally we make some remarks concerning the decomposition /1/.

If $x \in S_4$, let $B_x = \{a \in S \mid a \text{ is an inverse of } x\}$.

If $a \in \bar{S}_3$, let $C_a = \{x \in S \mid x \text{ is an inverse of } a\}$.

If $x \in S_4$ and $a \in B_x (a \in \bar{S}_3)$, then ax is a left identity of S, that is, $a x = e$ and $a y = e'$ $(a \in \bar{S}_3, y \in C_a)$ are left identities of S.

THEOREM 4.1. /i/ If $x \in S_4$ then B_x fails to be a subsemigroup. /ii/ If $a \in \bar{S}_3$, then C_a fails to be a subsemigroup.

<u>PROOF.</u> Suppose that B_x is a semigroup and $a,b \in B_x$. Then $a\,x\,a = a$, $b\,x\,b = b$ and $b\,a \in B_x$. Hence $b\,a\,x\,b\,a = b\,a$. Since $a\,x$ is a left identity element, hence $b(b\,a) = b\,a$. On the other hand, $b\,a \in \bar{S}_3$, thus $b\,a\,S = S$, whence $b\,s = s$ for all $s \in S$, which is a contradiction /b is a left magnifying element!/

Let $x,y \in C_a$. If C_a is a semigroup, then $a(x\,y)\,a = (a\,x)\,y\,a = y\,a$. But $y\,a \neq a$, because $y\,a$ is idempotent, while the element $a \in \bar{S}_3$ is not. Thus $x\,y \notin C_a$. Q.e.d.

Let $M \subset S$ be a subset of S such that $a\,M = S$. Then the set M is <u>left increasable</u> by a. Such left increasable set of a is not determined uniquely.

THEOREM 4.2. If $a \in \bar{S}_3$ then $a(S_o \cup S_2 \cup S_4) = S$.

PROOF. Let $a \in \bar{S}_3$ and $x \in S_4$ an inverse of a. Then we have $a\,x\,S = a\,S = S$ and $x\,S \subset S$. On the other hand $x\,S \subseteq S_4 S$, furthermore, by making use the relations /2/ we get

$$S_4 S = S_4(S_o \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) \subseteq S_o \cup S_2 \cup S_4.$$

Hence $x\,S \subseteq S_o \cup S_2 \cup S_4$ and thus

$$a(S_o \cup S_2 \cup S_4) = S. \qquad \text{Q.e.d.}$$

Theorem 4.2. implies the existence $y_a \in S_o \cup S_2 \cup S_4$ to every $a \in \bar{S}_3$ such that $a\, y_a = a$.

THEOREM 4.3. a/ If $a \in S_3$, then $y_a \notin S_o$.

b/ The elements $a \in \bar{S}_3$ for which $y_a \in S_4$ $(a\, y_a = a)$ have a two-sided identity element in S.

PROOF. a/ If $y_a \in S_o$, then there is $x \neq 0$ such that $y_a x = 0$. Thus $a\, x = (a\, y_a)x = a(y_a\, x) = a\, 0 = 0$, whence $a \in S_o \cup S_1$ which is a contradiction.

b/ If $y_a \in S_4$, then there exists $b \in \bar{S}_3$, such that $b\, y_a b = b$ and $y_a b\, y_a = y_a$. Then $a\, y_a b = a\, b$, $a\, y_a b\, y_a = a\, b\, y_a$, that is $a\, y_a = a\, b\, y_a$ whence it follows that $a = a(b\, y_a)$. On the other hand $b\, y_a$ is a left identity element of S, whence $b\, y_a\, a = = a = a\, b\, y_a$. Q.e.d.