and 12, we also obtain:

THEOREM 15. - (The third normalization theorem for homotopies of functions between $(n+1)$-tuples). *Let $S,S_1,\dots,S_n$ be a $(n+1)$-tuple of topological spaces, where S is a compact triangulable space, $S_1,\dots,S_n$ are closed triangulable subspaces, $G,G_1,\dots,G_n$ a $(n+1)$-tuple of finite directed graphs, $C,C_1,\dots,C_n$ and $D,D_1,\dots,D_n$ two finite cellular decompositions of $S,S_1,\dots,S_n$ and $e,f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ two functions pre-cellular w.r.t. $C,C_1,\dots,C_n$ and $D,D_1,\dots,D_n$ respectively, which are o-homotopic. Then, from any finite cellular decomposition of the $(n+1)$-tuple $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, $S_1\times\left[\frac{1}{3},\frac{2}{3}\right],\dots,$ $S_n\times\left[\frac{1}{3},\frac{2}{3}\right]$, of suitable mesh, which induces on the bases decompositions finer than $C,C_1,\dots,C_n$ and $D,D_1,\dots,D_n$, we obtain a finite cellular decomposition $\Gamma,\Gamma_1,\dots,\Gamma_n$ of the $(n+1)$-tuple $S\times I,S_1\times I,\dots,S_n\times I$, and a homotopy between e and f, which is a pre-cellular function w.r.t. $\Gamma,\Gamma_1,\dots,\Gamma_n$.* $\square$

8) Case of homotopy groups.

Since the n-cube I^n is a triangulable compact manifold, we can apply the results of the previous paragraphs to the case of absolute and relative n-dimensional groups of regular homotopy. So we can choose, as representative of any homotopy class, a loop which is pre-cellular w.r.t. a suitable cellular decomposition of I^n. Now, the cellular decompositions of I^n which are relevant for applications, are the triangulations and the subdivisions into cubes (the latter are determinated by a partition into k parts of equal size of every edge of I^n). To construct the absolute groups $Q_n(G,v)$ we consider *o-regular loops* i.e. o-regular functions $f\colon I^n,\dot{I}^n \to G,v$ where $\dot{I}^n$ is the boundary of I^n and v a vertex of G, whereas, in the case of relative groups $Q_n(G,G',v)$ we use the *o-regular relative loops*, i.e. o-regular functions $f\colon I^n,\dot{I}^n,J^{n-1} \to G,G',v$ where J^{n-1} is the union of the $(n-1)$-faces of I^n, different from the face $x_n = 0$. Since the subspaces $\dot{I}_n$, J^{n-1} are an union of faces of I^n, they are closed subspaces, which can be triangulated and subdivided into cubes. So, by applying the third normalization theorem (see Theorems 11 and 14), directly we obtain:

THEOREM 16. - *On the previous assumptions, in every o-homotopy class of the group $Q_n(G,v)$ (resp. $Q_n(G,G',v)$) there exists a loop which is pre-cellular w.r.t. a suitable triangulation (subdivision into cubes) of I^n.*

Proof. - Let α be an o-homotopy class and $f \in \alpha$ a loop. By [4], Theorem 15 and its generalization, we can replace f by a c.o-regular function $g \in \alpha$. Moreover, by Theorems 11 and 14, we can replace g by a function $h \in \alpha$ which satisfies the sought conditions, since there always exist triangulations and subdivisions into cubes with mesh $< r$, where r is a predeterminate real number. □

REMARK. - If G is a finite undirected graph, we obtain Property 13 of [8] again. Nevertheless, we remark that the meaning of properly quasi-constant function of Definition 10 is weaker than that one given there. In fact, now, the constant value of a cell σ is equal to the value of a maximal cell $\tau \in st(\sigma)$, whileas, before, the value of σ must also correspond to that one of a cell of properly upper dimension.

To obtain the third normalization theorem for homotopies, we recall that the cellular decompositions Γ_1 and Γ_2 are product decompositions. Consequently, we have:
i) To obtain a triangulation of $I^n \times I$, first we must triangulate every prism of the product. To this aim, we remark that it can be done by retaining the same triangulations $\widetilde{C}$ and $\widetilde{D}$ on the respective bases.
ii) Whileas, to obtain a subdivision of $I^n \times I$ into k^{n+1} cubes (where k is a multiple of 3), we must complete the subdivision of $I^n \times \left[\frac{1}{3}, \frac{2}{3}\right]$ into $\frac{1}{3}k^{n+1}$ cubes, by giving a subdivision into cubes of the parallelepipeda of the product cellular decompositions Γ_1 and Γ_2.
Then we have:

THEOREM 17. - *On the previous assumptions, let f, g be two o-homotopic loops which are pre-cellular w.r.t. the triangulations T and T' (subdivisions into cubes Q and Q') of I^n. Then, between f and g there exists a homotopy which is pre-cellular w.r.t. a suitable triangulation (subdivision into cubes), which induces on $I^n \times \{0\}$ and $I^n \times \{1\}$ triangulations (subdivisions into cubes) finer than T and T' (than Q and Q').* □

REMARK 1. - If G is a undirected graph we obtain Property 14 of [8] again. Moreover now we can avoid the extension k of the c.o-regular function, by choosing as image of a cell σ, whose closure intersects the basis $S \times \{0\}$ $(S \times \{1\})$, the value of any maximal cell of $\bar{\sigma} \cap (S \times \{0\})$ $(\bar{\sigma} \cap (S \times \{1\}))$.

REMARK 2. - The subdivision into cubes is useful to obtain the regular homotopy groups by blocks of vertices of G. (See [10]).