$\hat{h}$ coincides with $M$ on $S\times\left[0,\frac{1}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$. □

REMARK. - If $G$ is an undirected graph, it is not necessary to construct the extension $\hat{k}$ of the function F. (See Remark to Theorem 11).

*7) Case of n subspaces and n subgraphs.*

The previous results can be easily generalized to the case between $(n+1)$-tuples (see [3], §8b and [5], §11).

Let $S$ be a compact topological space, $G$ a finite directed graph, $S_1,\dots,S_n$ closed subspaces of $S$ and $G_1,\dots,G_n$ subgraphs of $G$, such that $S_j$ is a subspace of $S_i$ and $G_j$ a subgraph of $G_i$, $\forall i,j = 1,\dots,n$, $j>i$. In this case we have to consider functions $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ between $(n+1)$-tuples and their restrictions $f_1\colon S_1\to G_1$, $\dots$, $f_n\colon S_n\to G_n$.

7a) Given a c.o-regular function $f\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$, where $S$ is compact and $S_1,\dots,S_n$ are closed subspaces, by [5], §11.6, we can construct $n$ closed neighbourhoods $U_i$ of $S_i$, $i=1,\dots,n$ and a c.o-regular extension $k\colon S,U_1,\dots,U_n\to G,G_1,\dots,G_n$ such that $k\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$ is c.o-homotopic to $f$. Now, for all the pairs $U_i,S_i$, $i=1,\dots,n$, we determine a closed neighbourhood $K_i$ of $S_i$, included in $\mathring{U}_i$. Then, if the filter $\mathcal{W}$ is the uniformity of $S$, by following the proof of Proposition 9, we can obtain:

*i)* a vicinity $V\in\mathcal{W}$ such that $V(A_1^k)\cap\dots\cap V(A_n^k)\neq\emptyset$, $\forall r$-tuple $a_1,\dots,a_r$ non-headed of $G$;

*ii)* $\forall i=1,\dots,n$ a vicinity $Z_i$ of the trace-filter $\mathcal{W}_i$ of $\mathcal{W}$ on $U_i\times U_i$, such that $Z_i(A_1^{k_i})\cap\dots\cap Z_i(A_s^{k_i})=\emptyset$, $\forall s$-tuple $a_1,\dots,a_s$ non-headed of $G_i$, and, consequently, we obtain a vicinity $V_i\in\mathcal{W}$ / $Z_i = V_i\cap(U_i\times U_i)$. At least, we choose a symmetric vicinity $W$, such that $W\circ W\subset V\cap V_1\cap\dots\cap V_n$ and $W(K_i)\subseteq U_i$, $i=1,\dots,n$.

Given, now, a $W$-partition $P=\{X_j\}$, $j\in J$, of the space $S$, we define a relation $g\colon S,\mathring{K}_1,\dots,\mathring{K}_n\to G,G_1,\dots,G_n$ by putting, $\forall X_j$, $j\in J$, the constant value:

$$g(X_j)=\begin{cases}\text{a vertex of } H_G(\{f(X_j)\}) & \text{if } X_j\cap K_1=\emptyset\\ \text{a vertex of } H_{G_1}(\{f_1(X_j)\}) & \text{if } X_j\cap K_1\neq\emptyset \text{ and } X_j\cap K_2=\emptyset\\ \dots\dots\dots & \\ \text{a vertex of } H_{G_n}(\{f_n(X_j)\}) & \text{if } X_j\cap K_n\neq\emptyset.\end{cases}$$

Similarly to Proposition 9, we verify that $g$ is c. quasi-regular and that every o-pattern $h$ of $g$ is c.o-homotopic to $f$. Hence we can give:

THEOREM 13. - (The second normalization theorem between $n$-tuples). *Let $S,S_1,\ldots,S_n$ be a (n+1)-tuple of topological spaces, where $S$ is compact and $S_1,\ldots,S_n$ are closed subspaces of $S$, $G,G_1,\ldots,G_n$ a (n+1)-tuple of finite directed graphs and $f\colon S,S_1,\ldots,S_n \to G,G_1,\ldots,G_n$ a completely o-regular function. Then, if the filter $\mathcal{W}$ is the uniformity of $S$, we can determine $n$ closed neighbourhoods $K_i$ of $S_i$, $i=1,\ldots,n$ and a vicinity $W \in \mathcal{W}$ such that, for all the $W$-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h\colon S, \mathring{K}_1,\ldots,\mathring{K}_n \to G,G_1,\ldots,G_n$ which is completely o-regular, weakly $P$-constant and completely o-homotopic to $f$.* □

REMARK 1. - If $S$ is a compact metric space, we can determine a positive real number $r$ and consider partitions with mesh $< r$.

REMARK 2. - If $G$ is an undirected graph, the function $g$ can be choosen $P$-constant. Moreover, since it is not necessary to replace $f$ with an extension $k$, we have only to consider a symmetric vicinity $W$ / $W \circ W \subset V \cap V_1 \cap \ldots \cap V_n$.

7b) Now let $C,C_1,\ldots,C_n$ be a $(n+1)$-tuple of spaces which consists of a finite cellular complex $C$ and of $n$ subcomplexes $C_1,\ldots,C_n$. A function $f\colon |C|,|C_1|,\ldots,|C_n| \to G,G_1,\ldots,G_n$ is called *pre-cellular w.r.t.* $C,C_1,\ldots,C_n$ if:

*i)* $f\colon |C|, |st(C_1)|,\ldots,|st(C_n)| \to G,G_1,\ldots,G_n$ is c.o-regular;
*ii)* $f\colon |C| \to G$ is properly $C$-constant;
*iii)* $f\colon |C| \to G$ is properly $C$-constant in $C_1$ (in $C_2,\ldots,C_n$).

Now, if $f\colon S,S_1,\ldots,S_n \to G,G_1,\ldots,G_n$ is a c.o-regular function, where $S$ is a compact triangulable space, $S_1,\ldots,S_n$ closed triangulable subspaces of $S$, we can consider c.o-regular extension $k\colon S,U_1,\ldots,U_n \to G,G_1,\ldots,G_n$ and determine the positive real number $r = inf(\frac{\varepsilon}{2},\frac{\varepsilon_1}{2},\ldots,\frac{\varepsilon_n}{2},\eta_1,\eta_2,\ldots,\eta_n)$ where $\varepsilon = inf(enl(A_1^k,\ldots,A_r^k))$, $\forall r$-tuple $a_1,\ldots,a_r$ non-headed of $G$, $\varepsilon_i = inf(enl(A_1^{k_i},\ldots,A_{s_i}^{k_i}))$, $\forall\, s_i$-tuple $a_1,\ldots,a_{s_i}$ non-headed of $G_i$, and $\eta_i$ are such that $W^{\eta_i}(S_i) \subset U_i$, $i=1,\ldots,n$.
Given then a finite cellular decomposition $C,C_1,\ldots,C_n$ of $S,S_1,\ldots,S_n$ with mesh $< r$, we consider the following partition of $C$: $D_0 = C - st(C_1)$, $D_1 = st(C_1) - st(C_2),\ldots,$ $D_n = st(C_n)$ and we construct the c.quasi-regular function $g\colon |C|, |st(C_1)|,\ldots,|st(C_n)| \to G,G_1,\ldots,G_n$, by putting, $\forall D_i$,

$\forall \sigma \in D_i$, $g(\sigma)$ = a vertex of $H_{G_i}(\{k(\bar{\sigma})\})$, where $G_0 = G$. We separate the cells of $C$, besides using the subsets $D_i$, also in the following way:

i) cells included in $C-C_1$ $\begin{cases} 1) \text{ cells } \tau \text{ maximal in } C \\ 2) \text{ cells } \sigma \text{ non-maximal in } C \end{cases}$

ii) cells included in $C_1-C_2$ $\begin{cases} 1) \text{ cells } \tau \text{ maximal in } C \\ 2) \text{ cells } \tau_1 \text{ maximal in } C_1 \text{ and non-maximal in } C \\ 3) \text{ cells } \sigma_1 \text{ non-maximal in } C_1. \end{cases}$

. . . . . . . . . . . .

$n$+1) cells included in $C_n$ $\begin{cases} 1) \text{ cells } \tau \text{ maximal in } C \\ 2) \text{ cells } \tau_1 \text{ maximal in } C_1 \text{ and non-maximal in } C \\ \ldots\ldots\ldots \\ n+1) \text{ cells } \tau_n \text{ maximal in } C_n \\ n+2) \text{ cells } \sigma_n \text{ non-maximal in } C_n. \end{cases}$

At least, we can construct, by induction, the o-pattern $h$ in $n$+1 steps by putting:

in the first step: $\begin{cases} h(\tau) = g(\tau) \\ h(\sigma) = \text{a vertex of } H_\Gamma(g(st^m(\sigma))) \\ h(\tau_1) = \text{a vertex of } H_\Gamma(g(st^m(\tau_1))) \end{cases}$

in the second step: $\begin{cases} h(\sigma_1) = \text{a vertex of } H_\Gamma(h(st^m_{C_1}(\sigma_1))) \\ h(\tau_2) = \text{a vertex of } H_\Gamma(h(st^m_{C_1}(\tau_2))) \end{cases}$

in the third step: $\begin{cases} h(\sigma_2) = \text{a vertex of } H_\Gamma(h(st^m_{C_2}(\sigma_2))) \\ h(\tau_3) = \text{a vertex of } H_\Gamma(h(st^m_{C_2}(\tau_3))) \end{cases}$

. . . . . . .

in the ($n$+1) step: $h(\sigma_n)$ = a vertex of $H_\Gamma(h(st^m_{C_n}(\sigma_n)))$,
where, $\forall \lambda \in C$, we put $\Gamma = G_i$ if $\lambda \in D_i$.

Hence we obtain:

THEOREM 14. - (The third normalization theorem between ($n$+1)-tuples). *Let $S,S_1,\ldots,S_n$ be a (n+1)-tuple of topological spaces, where $S$ is a compact triangulable space, $S_1,\ldots,S_n$ are closed triangulable subspaces, $G,G_1,\ldots,G_n$ a (n+1)-tuple of finite directed graphs and $f\colon S,S_1,\ldots,S_n \to G,G_1,\ldots,G_n$ a completely o-regular function. Then, for every finite cellular decomposition $C,C_1,\ldots,C_n$ of $S,S_1,\ldots,S_n$ with suitable mesh, there exists a function $h\colon S,S_1,\ldots,S_n \to G,G_1,\ldots,G_n$ pre-cellular w.r.t. $C,C_1,\ldots,C_n$ and completely o-homotopic to $f$.* □

By a procedure similar to that one used in the proofs of Theorems 8 and 12, we also obtain:

THEOREM 15. - (The third normalization theorem for homotopies of functions between $(n+1)$-tuples). *Let $S,S_1,\ldots,S_n$ be a $(n+1)$-tuple of topological spaces, where $S$ is a compact triangulable space, $S_1,\ldots,S_n$ are closed triangulable subspaces, $G,G_1,\ldots,G_n$ a $(n+1)$-tuple of finite directed graphs, $C,C_1,\ldots,C_n$ and $D,D_1,\ldots,D_n$ two finite cellular decompositions of $S,S_1,\ldots,S_n$ and $e,f\colon S,S_1,\ldots,S_n \to G,G_1,\ldots,G_n$ two functions pre-cellular w.r.t. $C,C_1,\ldots,C_n$ and $D,D_1,\ldots,D_n$ respectively, which are o-homotopic. Then, from any finite cellular decomposition of the $(n+1)$-tuple $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, $S_1\times\left[\frac{1}{3},\frac{2}{3}\right],\ldots,$ $S_n\times\left[\frac{1}{3},\frac{2}{3}\right]$, of suitable mesh, which induces on the bases decompositions finer than $C,C_1,\ldots,C_n$ and $D,D_1,\ldots,D_n$, we obtain a finite cellular decomposition $\Gamma,\Gamma_1,\ldots,\Gamma_n$ of the $(n+1)$-tuple $S\times I, S_1\times I,\ldots,S_n\times I$, and a homotopy between $e$ and $f$, which is a pre-cellular function w.r.t. $\Gamma,\Gamma_1,\ldots,\Gamma_n$.* □

*8) Case of homotopy groups.*

Since the $n$-cube $I^n$ is a triangulable compact manifold, we can apply the results of the previous paragraphs to the case of absolute and relative $n$-dimensional groups of regular homotopy. So we can choose, as representative of any homotopy class, a loop which is pre-cellular w.r.t. a suitable cellular decomposition of $I^n$. Now, the cellular decompositions of $I^n$ which are relevant for applications, are the triangulations and the subdivisions into cubes (the latter are determinated by a partition into $k$ parts of equal size of every edge of $I^n$). To construct the absolute groups $Q_n(G,v)$ we consider *o-regular loops* i.e. o-regular functions $f\colon I^n,\dot{I}^n \to G,v$ where $\dot{I}^n$ is the boundary of $I^n$ and $v$ a vertex of $G$, whereas, in the case of relative groups $Q_n(G,G',v)$ we use the *o-regular relative loops*, i.e. o-regular functions $f\colon I^n,\dot{I}^n,J^{n-1} \to G,G',v$ where $J^{n-1}$ is the union of the $(n-1)$-faces of $I^n$, different from the face $x_n = 0$. Since the subspaces $\dot{I}_n$, $J^{n-1}$ are an union of faces of $I^n$, they are closed subspaces, which can be triangulated and subdivided into cubes. So, by applying the third normalization theorem (see Theorems 11 and 14), directly we obtain:

THEOREM 16. - *On the previous assumptions, in every o-homotopy class of the group $Q_n(G,v)$ (resp. $Q_n(G,G',v)$) there exists a loop which is pre-cellular w.r.t. a suitable triangulation (subdivision into cubes) of $I^n$.*