to f.

Proof. - By Proposition 28 of [5] and Theorem 16 of [4] there exists a closed neighbourhood U of S' and an extension $k\colon S,U \to G,G'$ which is c.o-regular and such that $k\colon S,S' \to G,G'$ is c.o-homotopic to f. Then we obtain the result by using Proposition 9 for the function $k\colon S,U \to G,G'$.

REMARK. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, we have only to consider the couples of vertices rather than the n-tuples and to determine $\varepsilon_1 = inf(d(A_i^f, A_j^f))$, $\forall$ couple a_i, a_j of non-adjacent vertices of G, $\varepsilon_2 = inf(d(A_r^{f'}, A_s^{f'}))$, $\forall$ couple a_r, a_s of non-adjacent vertices of G'. Then, if we put $r' = inf(\varepsilon_1, \varepsilon_2)$, as in Remark 3 to Theorem 3, we can choose a covering $P = \{X_j\}$, $j \in J$, with mesh $< \frac{r'}{4}$ (see [8], Corollary 8).

6) The third normalization theorem between pairs.

Now we consider pairs of spaces given by a finite cellular complex C and by a subcomplex C' of C; it follows that $|C'|$ is a closed subspace of $|C|$. Since we use completely o-regular functions $f\colon |C|,|C'| \to G,G'$ balanced by the open set $|st(C')|$ (see [5], Definitions 6 and 12), we put:

DEFINITION 12. - *Let C be a finite complex, C' a subcomplex of C, G a finite graph and G' a subgraph of G. A function $f\colon |C|,|C'| \to G,G'$ is called* pre-cellular w.r.t. *C,C' or C,C'*-pre-cellular *if:*

i) $f\colon |C|,|st(C')| \to G,G'$ is completely o-regular.

ii) $f\colon |C| \to G$ is properly C-constant.

iii) $f\colon |C| \to G$ is properly C-constant in C'.

THROREM 11. - (The third normalization theorem between pairs). *Let S be a compact triangulable space, S' a closed triangulable subspace of S, G a finite directed graph, G' a subgraph of G and $f\colon S,S' \to G,G'$ a completely o-regular function. Then for every finite cellular decomposition C,C' of the pair S,S', with suitable mesh, there exists a function $h\colon S,S' \to G,G'$ which is C,C'-pre-cellular and completely o-homotopic to f.*

Proof. - By proceeding as in the proof of Theorem 10, at first we consider an extension $k: S,U \to G,G'$, where U is a closed neighbourhood of S'. Then, by Remark to Proposition 9, we determine a positive real number $r = inf(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}, \varepsilon_3)$, where $\varepsilon_1 = inf(enl(A_1^k,\dots,A_n^k))$, $\forall n$-tuple $a_1,\dots,a_n$ non-headed of G, $\varepsilon_2 = inf(enl(A_1'^{k'},\dots,A_m'^{k'}))$, $\forall m$-tuple $a_1',\dots,a_m'$ non-headed of G', ε_3 is such that $W^{\varepsilon_3}(S') \subset U$. Since we can use $|st(C')|$ as an open neighbourhood of S', now it is not necessary to construct, as in Proposition 9, a closed neighbourhood K of S', included in U, and to consider the interior $\mathring{K}$.

Then, if C,C' is a finite decomposition of S,S' with mesh $< r$, it results $|\overline{st(C')}| \subseteq W^r(S')$, since all the cells have diameter $< r$. Afterwards, we construct the c.quasi-regular function $g: |C|, |st(C')| \to G,G'$ by putting, $\forall \sigma \in C$, (see Proposition 9 and Remark 1 to Theorem 3):

$$g(\sigma) = \begin{cases} \text{a vertex of } H_G(\{k(\bar{\sigma})\}) & \text{if } \sigma \in C - st(C') \\ \text{a vertex of } H_{G'}(\{k(\bar{\sigma})\}) & \text{if } \sigma \in st(C'). \end{cases}$$

To construct a c.o-regular o-pattern h, we must separate the cells of C w.r.t. $st(C')$ as before. Moreover, to obtain h properly quasi-constant, we must separate the cells of C w.r.t. C' in the following way:

a) cells included in $C-C'$: 1) cells τ maximal in C; 2) cells σ non-maximal in C

b) cells included in C': 1) cells τ maximal in C; 2) cells τ' maximal in C' and non-maximal in C; 3) cells σ' non-maximal in C'.

Now (see Theorem 6), by induction, we construct the o-pattern h, by putting at the first step:

i) $h(\tau) = g(\tau)$

ii) $h(\sigma) =$ a vertex of $H_\Gamma(g(st^m(\sigma)))$ where $\begin{cases} \Gamma = G & \text{if } \sigma \in C-st(C') \\ \Gamma = G' & \text{if } \sigma \in st(C') \end{cases}$

iii) $h(\tau') =$ a vertex of $H_{G'}(g(st^m(\tau')))$.

If we define, as before, the images of the cells maximal in C', at the second and last step, we put:

$h(\sigma') =$ a vertex of $H_{G'}(h(st^m_{C'}(\sigma')))$.

Hence $h: |C|, |st(C')| \to G,G'$ is the sought function. □

REMARK. - If G is an undirected graph, it is not necessary to construct the extension of the function $f: |C|, |C'| \to G,G'$. In fact, if we determine the upper bound $\frac{r'}{4}$ of the mesh as in Remark to Theorem 10, and, consequently, if we consider the cellular decomposition C,C', we can obtain the strongly regular function $g: S, |st(S')| \to G,G'$, by putting, $\forall \sigma \in C$:

$$g(\sigma) = \begin{cases} \text{a vertex of } f(\bar{\sigma}) & \text{if } \sigma \in C - st(C') \\ \text{a vertex of } f(\bar{\sigma}) \cap \sigma' & \text{if } \sigma \in st(C'). \end{cases}$$

Moreover, in the construction of the o-pattern h, we have only to separate the cells w.r.t. C and C'.

Theorem 8 can be generalized by:

THEOREM 12. - (The third normalization theorem for homotopies of functions between pairs). *Let S be a compact triangulable space, S' a closed triangulable subspace of S, G a finite directed graph, G' a subgraph of G, C,C' and D,D' two finite cellular decompositions of S,S' and e,f: $S,S' \to G,G'$ two functions pre-cellular w.r.t. C,C' and D,D' respectively, which are completely o-homotopic. Then, from any finite cellular decomposition Γ_2, Γ'_2 of the pair $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, $S'\times\left[\frac{1}{3},\frac{2}{3}\right]$ of suitable mesh, which induces on the pairs of bases $S\times\left\{\frac{1}{3}\right\}$ and $S'\times\left\{\frac{2}{3}\right\}$ decompositions $\tilde{C}$, $\tilde{C}'$ and $\tilde{D}$, $\tilde{D}'$ finer than C,C' and D,D', we obtain a finite cellular decomposition Γ,Γ' of the pair $S\times I$, $S'\times I$ and a homotopy between e and f which is a Γ,Γ'-pre-cellular function.*

Proof. - Since $|st(C')|$ and $|st(D')|$ are rispectively balancers (see [5], Definition 12) of e and f in S', the open set $U = |st(C')| \cap |st(D')|$ is a common balancer of e and f. Now let F: $S\times I$, $S'\times I \to G,G'$ be a complete o-homotopy between e and f and, by Proposition 30 of [5] we can construct a closed neighbourhood V of $S'\times I$ and a c. o-regular function $\hat{k}$: $S\times I, V \to G,G'$, which is a homotopy between e and f. Then, the c.o-homotopy $\hat{k}$ can be replaced by the c.o-homotopy M given by:

$$M(x,t) = \begin{cases} e(x) & \forall x \in S, \quad \forall t \in \left[0,\frac{1}{3}\right] \\ F(x,3t-1) & \forall x \in S, \quad \forall t \in \left[\frac{1}{3},\frac{2}{3}\right] \\ f(x) & \forall x \in S, \quad \forall t \in \left[\frac{2}{3},1\right] \end{cases}$$

and, by considering the restriction of M to $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, we determine the real number r, upper bound of the mesh (see the proof of Theorem 11). Moreover, if Γ_2, Γ'_2 is a cellular decomposition, which satisfies the conditions of the theorem and with mesh $< r$, we can construct the cellular decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\Gamma' = \Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$ of the pair of cylinders $S\times I$, $S'\times I$, where $\Gamma_1, \Gamma'_1, \Gamma_2, \Gamma'_2$ are the product decompositions, respectively, of $\tilde{C}\times L_1$, $\tilde{C}'\times L_1$, $\tilde{D}\times L_3$, $\tilde{D}'\times L_3$ (see Theorem 8).

Then we define the function $\hat{g}$: $S\times I$, $S'\times I \to G,G'$ by putting:

$$\hat{g}(\sigma) = \begin{cases} M(\sigma), & \forall \sigma \in \Gamma - \Gamma_2 \\ \text{a vertex of } H_G(\{M(\tilde{\sigma})\}) & \text{if } \sigma \in \Gamma_2 - st_{\Gamma_2}(\Gamma'_2) \\ \text{a vertex of } H_{G'}(\{M(\tilde{\sigma})\}) & \text{if } \sigma \in st_{\Gamma_2}(\Gamma'_2). \end{cases}$$

Hence, by Theorem 11, we construct the o-pattern $\hat{h}$ of $\hat{g}$, by choosing, if $\sigma \in \Gamma - \Gamma_2$, as value of $\hat{h}(\sigma)$, the value $\hat{g}(\sigma) = M(\sigma)$. In this way

$\hat{h}$ coincides with M on $S\times\left[0,\frac{1}{3}\right]$ and $S\times\left[\frac{2}{3},1\right]$. □

REMARK. - If G is an undirected graph, it is not necessary to construct the extension $\hat{k}$ of the function F. (See Remark to Theorem 11).

7) *Case of n subspaces and n subgraphs.*

The previous results can be easily generalized to the case between $(n+1)$-tuples (see [3], §8b and [5], §11).

Let S be a compact topological space, G a finite directed graph, $S_1,\dots,S_n$ closed subspaces of S and $G_1,\dots,G_n$ subgraphs of G, such that S_j is a subspace of S_i and G_j a subgraph of G_i, $\forall i,j = 1,\dots,n$, $j>i$. In this case we have to consider functions $f\colon S,S_1,\dots,S_n \to G,G_1,\dots,G_n$ between $(n+1)$-tuples and their restrictions $f_1\colon S_1\to G_1$, $\dots$, $f_n\colon S_n\to G_n$.

7a) Given a c.o-regular function $f\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$, where S is compact and $S_1,\dots,S_n$ are closed subspaces, by [5], §11.6, we can construct n closed neighbourhoods U_i of S_i, $i=1,\dots,n$ and a c.o-regular extension $k\colon S,U_1,\dots,U_n\to G,G_1,\dots,G_n$ such that $k\colon S,S_1,\dots,S_n\to G,G_1,\dots,G_n$ is c.o-homotopic to f. Now, for all the pairs U_i,S_i, $i=1,\dots,n$, we determine a closed neighbourhood K_i of S_i, included in $\mathring{U}_i$. Then, if the filter $\mathcal{W}$ is the uniformity of S, by following the proof of Proposition 9, we can obtain:

i) a vicinity $V\in\mathcal{W}$ such that $V(A_1^k)\cap\dots\cap V(A_n^k)\neq\emptyset$, $\forall r$-tuple $a_1,\dots,a_r$ non-headed of G;

ii) $\forall i=1,\dots,n$ a vicinity Z_i of the trace-filter $\mathcal{W}_i$ of $\mathcal{W}$ on $U_i\times U_i$, such that $Z_i(A_1^{k_i})\cap\dots\cap Z_i(A_s^{k_i})=\emptyset$, $\forall s$-tuple $a_1,\dots,a_s$ non-headed of G_i, and, consequently, we obtain a vicinity $V_i\in\mathcal{W}\ /\ Z_i = V_i\cap(U_i\times U_i)$. At least, we choose a symmetric vicinity W, such that $W\circ W\subset V\cap V_1\cap\dots\cap V_n$ and $W(K_i)\subseteq U_i$, $i=1,\dots,n$.

Given, now, a W-partition $P=\{X_j\}$, $j\in J$, of the space S, we define a relation $g\colon S,\mathring{K}_1,\dots,\mathring{K}_n\to G,G_1,\dots,G_n$ by putting, $\forall X_j$, $j\in J$, the constant value:

$$g(X_j)=\begin{cases}\text{a vertex of } H_G(\{f(X_j)\}) & \text{if } X_j\cap K_1=\emptyset\\ \text{a vertex of } H_{G_1}(\{f_1(X_j)\}) & \text{if } X_j\cap K_1\neq\emptyset \text{ and } X_j\cap K_2=\emptyset\\ \dots\dots\dots & \\ \text{a vertex of } H_{G_n}(\{f_n(X_j)\}) & \text{if } X_j\cap K_n\neq\emptyset.\end{cases}$$