So we obtain:

THEOREM 8. -(The third normalization theorem for homotopies). *Let S be a compact triangulable space, G a finite directed graph, C,D two finite decompositions of S and e,f: $S \to G$ two functions pre-cellular w.r.t. C and D respectively, which are completely o-homotopic. Then, from any finite cellular decomposition Γ_2 of $S\times\left[\frac{1}{3},\frac{2}{3}\right]$ of suitable mesh which induces on the bases $S\times\left\{\frac{1}{3}\right\}$ and $S\times\left\{\frac{2}{3}\right\}$ decompositions $\tilde{C}$ and $\tilde{D}$ finer than C and D, we obtain a finite cellular decomposition Γ of $S\times I$ and a homotopy between f and g which is a Γ-pre-cellular function.*

Proof. - Let F: $S\times I \to G$ be a complete o-homotopy between e and f. We define the complete o-homotopy M: $S\times I \to G$ between e and f as in the introduction of this paragraph. Then, if we consider the restriction of M to $S\times\left[\frac{1}{3},\frac{2}{3}\right]$, we can determine the real number r, upper bound of the mesh. Now if Γ_2 is a finite cellular decomposition, satisfying the conditions of the theorem and with mesh $< r$, we can consider the cellular decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ of the cylinder $S\times I$, such that:

i) Γ_1 is the product decomposition $\tilde{C}\times L_1$ of $S\times\left[0,\frac{1}{3}\right]$, where $L_1 = \left\{\{0\}, \left]0,\frac{1}{3}\right[, \left\{\frac{1}{3}\right\}\right\}$.

ii) Γ_3 is the product decomposition $\tilde{D}\times L_3$ of $S\times\left[\frac{2}{3},1\right]$, where $L_3 = \left\{\left\{\frac{2}{3}\right\}, \left]\frac{2}{3},1\right[, \{1\}\right\}$.

Then we define the function $\hat{g}$: $S\times I \to G$, given by:

$$\hat{g}(\sigma) = \begin{cases} M(\sigma), & \forall\sigma\in \Gamma-\Gamma_2, \\ \text{a vertex of } H(\{M(\bar{\sigma})\}), & \forall\sigma\in\Gamma_2. \end{cases}$$

Afterwards, by Theorem 6, we construct the o-pattern $\hat{h}$ of $\hat{g}$, by choosing as element of $H(\hat{g}(st^m(\sigma)))$, the value $\hat{g}(\sigma) = M(\sigma)$ if $\sigma\in \Gamma-\Gamma_2$. By construction $\hat{h}$ is a Γ-pre-cellular function. Hence $\hat{h}$ is the sought homotopy since $\hat{h}/_{S\times\{0\}} = e$ and $\hat{h}/_{S\times\{1\}} = f$. □

REMARK. - The finite cellular decomposition Γ induces on the bases $S\times\{0\}$ and $S\times\{1\}$ the decompositions $\tilde{C}$ and $\tilde{D}$.

5) *The second normalization theorem between pairs* .

Given a set A, a non-empty subset A' of A, a finite graph G and a subgraph G' of G, we can generalize Definition 4, by considering functions f: $A,A' \to G,G'$ which are quasi-constant w.r.t. a partition $P = \{X_j\}$, $j\in J$, of A. In this case it follows that the image of every X_j, such that $X_j\cap A' \neq \emptyset$, necessary is a vertex of G'. Moreover, if A is a topological space and A' a subspace of A, we can also generalize the definition of weakly P-constant. So we have:

PROPOSITION 9. - *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, S' a closed subspace of S, U a closed neighbourhood of S', G a finite directed graph, G' a subgraph of G and $f:S,U\to G,G'$ a completely o-regular function. If we choose in $\mathring{U}$ a closed neighbourhood K of S', we can determine a vicinity $W\in\mathcal{W}$ such that, for all the W-partitions $P=\{X_j\}$, $j\in J$, there exists a function $h: S,\mathring{K}\to G,G'$, which is completely o-regular, weakly P-constant and completely o-homotopic to $f: S,S'\to G,G'$.*

Proof. - At first there exists a closed neighbourhood K of S', included in U, since S is normal. Then, by following the proof of Theorem 3, we determine a vicinity $V\in\mathcal{W}$ such that $V(A_1^f)\cap\ldots\cap V(A_n^f) = \emptyset$, $\forall n$-tuple $a_1,\ldots,a_n$ non-headed of G. Moreover, if $\mathcal{W}'$ is the trace filter of $\mathcal{W}$ on $U\times U$, we obtain, as before, a vicinity $Z'\in\mathcal{W}'$ such that $Z'(A'^{f'}_1)\cap\ldots\cap Z'(A'^{f'}_m) = \emptyset$, $\forall m$-tuple $a'_1,\ldots,a'_m$ non-headed of G'. Since $Z'\in\mathcal{W}'$, necessarily it is $Z' = V_1\cap(U\times U)$, where $V_1\in\mathcal{W}$. Then we choose a symmetric vicinity $W\in\mathcal{W}$ such that $W\circ W\subseteq V\cap V_1$ and $W(K)\subset U$. Now, given a W-partition $P=\{X_j\}$, $j\in J$, of S, we define a relation $g: S,\mathring{K}\to G,G'$, by putting, for every X_j, $j\in J$, the constant value:

$$g(X_j) = \begin{cases} \text{a vertex of } H_G(\{f(X_j)\}) & \text{if } X_j\cap K=\emptyset,\\ \text{a vertex of } H_{G'}(\{f'(X_j)\}) & \text{if } X_j\cap K\neq\emptyset.\end{cases}$$

We verify that g satisfies the following conditions:

i) g is a function. In fact it results:

a) $\forall X_j \,/\, X_j\cap K=\emptyset$, the set $\{f(X_j)\}$ is headed in G. For proving this we go on as in i) of the proof of Theorem 3.

b) $\forall X_j \,/\, X_j\cap K\neq\emptyset$, the set $\{f'(X_j)\}$ is headed in G'. At first we prove that $X_j\subseteq U$. Let $z\in X_j\cap K$, $\forall y\in X_j$ it is $(z,y)\in X_j\times X_j\subseteq W$, i.e: $X_j\subseteq W(z)\subseteq W(K)\subseteq U$. Then, if we go on as in i) of Theorem 3, we obtain that $\{f'(X_j)\}$ is headed in G'. Moreover, we remark that the vertex $g(x)$, chosen in $H_{G'}(\{f'(X_j)\})$, is also an element of $H_G(\{f(X_j\})$, since $f(X_j) = f'(X_j)$.

From a) and b) it follows that there exists $g(x)$, for every $x\in S$; hence g is a function.

ii) and *iii) The function $g:S,\mathring{K}\to G,G'$ and the homotopy $F:S\times I,\mathring{K}\times I\to G,G$ between f and g given by:*

$$F(x,t) = \begin{cases} f(x) & \forall x\in S, \quad \forall t\in\left[0,\tfrac{1}{2}\right[\\ g(x) & \forall x\in S, \quad \forall t\in\left[\tfrac{1}{2},1\right]\end{cases}$$

are completely quasi-regular functions.

a) $g: S \to G$ and $F: S \times I \to G$ are c. quasi-regular functions. We obtain this result as in ii) and iii) of Theorem 3.

b) The restrictions $g': \mathring{K} \to G'$ and $F': \mathring{K} \times I \to G'$ are c. quasi-regular. At first we observe that, by the definition of g, it is $g(K) \subset G'$ and then $F(K \times I) \subset G'$. Secondly we go on as in ii) and iii) of Theorem 3, by choosing, $\forall x' \in \mathring{K}$, the neighbourhood $W(x') \cap \mathring{K}$, rather than $W(x')$, and by using the vicinity Z' rather than V. Then, for example, if we suppose that the m-tuple $a'_1, \ldots, a'_m \in \langle g'(x') \rangle$ is non-headed, we obtain the contradiction $x' \in Z'(A'^{f'}_1) \cap \ldots \cap Z'(A'^{f'}_m)$.

From a) and b) it follows ii) and iii).

Now if we consider any o-pattern h of g, we obtain the sought function. In fact we have:

i') $h: S, \mathring{K} \to G, G'$ is completely o-regular (see [5], Proposition 15).

ii') h is weakly P-constant by the definition of o-pattern of a quasi-constant function.

iii') h is completely o-homotopic to $f: S, S' \to G, G'$. Since the homotopy $F: S, \mathring{K} \to G, G'$ is c.quasi-regular by iii) and $\mathring{K}$ is open, there exists an o-pattern E (which is c.o-regular by [5], Proposition 15) of F. We can choose E such that $E(x,0) = f(x)$ and $E(x,1) = h(x)$, $\forall x \in S$, for f and g are c.o-regular i.e.:

a) $f(x) \in H_G(\langle f(x) \rangle) = H_G(\langle F(x,0) \rangle)$ and $h(x) \in H_G(\langle g(x) \rangle) = H_G(\langle F(x,1) \rangle)$, $\forall x \in S$.

b) $f'(x) \in H_{G'}(\langle f'(x) \rangle) = H_{G'}(\langle F'(x,0) \rangle)$ and $h'(x) \in H_{G'}(\langle g(x) \rangle) = H_{G'}(\langle F'(x,1) \rangle)$, $\forall x \in \mathring{K}$.

Hence the o-pattern $h(x) = E(x,1)$ is c.o-homotopic to f by E. □

REMARK. - If S is a compact metric space, we can determine a positive real number r and choose partitions P with mesh $< r$. In fact, we put $\varepsilon_1 = inf(enl(A^f_1, \ldots, A^f_n))$, $\forall n$-tuple $a_1, \ldots, a_n$ non-headed of G and $\varepsilon_2 = inf(enl(A'^{f'}_1, \ldots, A'^{f'}_m))$, $\forall m$-tuple $a'_1, \ldots, a'_m$ non-headed of G' and we choose ε_3 such that $W^{\varepsilon_3}(K) \subset U$. Then the real number r is given by $inf(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}, \varepsilon_3)$.

THEOREM 10. -(The second normalization theorem between pairs). *Let S be a compact space, the filter $\mathcal{W}$ the uniformity of S, S' a closed subspace of S, G a finite directed graph, G' a subgraph of G and $f: S, S' \to G, G'$ a completely o-regular function. Then we can determine a closed neighbourhood K of S' and a vicinity $W \in \mathcal{W}$ such that, for all the W-partitions $P = \{X_j\}$, $j \in J$, there exists a function $h: S, \mathring{K} \to G, G'$, which is completely o-regular, weakly P-constant and completely o-homotopic* *to f.*

Proof. - By Proposition 28 of [5] and Theorem 16 of [4] there exists a closed neighbourhood U of S' and an extension $k: S,U \to G,G'$ which is c.o-regular and such that $k: S,S' \to G,G'$ is c.o-homotopic to f. Then we obtain the result by using Proposition 9 for the function $k: S,U \to G,G'$.

REMARK. - If G is an undirected graph, the function g can be choosen quasi-constant. Moreover if S is a compact metric space, we have only to consider the couples of vertices rather than the n-tuples and to determine $\varepsilon_1 = inf(d(A_i^f, A_j^f))$, $\forall$ couple a_i, a_j of non-adjacent vertices of G, $\varepsilon_2 = inf(d(A_r^{f'}, A_s^{f'}))$, $\forall$ couple a_r, a_s of non-adjacent vertices of G'. Then, if we put $r' = inf(\varepsilon_1, \varepsilon_2)$, as in Remark 3 to Theorem 3, we can choose a covering $P = \{X_j\}$, $j \in J$, with mesh $< \frac{r'}{4}$ (see [8], Corollary 8).

6) *The third normalization theorem between pairs.*

Now we consider pairs of spaces given by a finite cellular complex C and by a subcomplex C' of C; it follows that $|C'|$ is a closed subspace of $|C|$. Since we use completely o-regular functions $f: |C|, |C'| \to G,G'$ balanced by the open set $|st(C')|$ (see [5], Definitions 6 and 12), we put:

DEFINITION 12. - *Let C be a finite complex, C' a subcomplex of C, G a finite graph and G' a subgraph of G. A function $f: |C|, |C'| \to G,G'$ is called* pre-cellular w.r.t. *C,C' or C,C'*-pre-cellular *if:*

i) $f: |C|, |st(C')| \to G,G'$ is completely o-regular.

ii) $f: |C| \to G$ is properly C-constant.

iii) $f: |C| \to G$ is properly C-constant in C'.

THROREM 11. - (The third normalization theorem between pairs). *Let S be a compact triangulable space, S' a closed triangulable subspace of S, G a finite directed graph, G' a subgraph of G and $f: S,S' \to G,G'$ a completely o-regular function. Then for every finite cellular decomposition C,C' of the pair S,S', with suitable mesh, there exists a function $h: S,S' \to G,G'$ which is C,C'-pre-cellular and completely o-homotopic to f.*

Proof. - By proceeding as in the proof of Theorem 10, at first we