0) Background.

Let X be a non-empty subset of a finite directed graph G. A vertex of X is called a *head* of X in G if it is a predecessor of all the other vertices of X. We denote by $H_G(X)$ the set of the heads of X in G. X is called *headed* if $H(X) \neq \phi$ and *totally headed* if all the non-empty subsets of X are headed.

Given a function $f: S \to G$, where S is a topological space, we denote by capital letter V the set of all the f-counterimages of $v \in G$, and, if we want to emphasize the function f, we write $V^f = f^{-1}(v)$.

We call *image-envelope* of a point $x \in S$ by f, and we denote by $\langle f(x)\rangle$, the set of vertices, such that the closure of their f-counterimages includ the point i.e. $v \in \langle f(x)\rangle \Leftrightarrow x \in \overline{V}^f$.

A function $f: S \to G$ is called *o-regular*, if, for all different $v, w \in G$, such that v is not a predecessor of w, it is $V \cap \overline{W} = \emptyset$. We proved that f is o-regular iff:

i) $\langle f(x)\rangle$ is headed, $\forall x \in S$;

ii) $f(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$. (See [5], Proposition 2).

So it is natural to define a more restrictive class of functions by sayi that a function $f: S \to G$ is *completely o-regular* (or simply *c. o-regular*) if

i') $\langle f(x)\rangle$ is totally headed, $\forall x \in S$;

ii') $f(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$.

Afterwards we also consider functions satisfying only condition *i'*, whic we call *completely quasi regular* functions. In [5] we proved that a completely quasi regular function can be replaced by a c. o-regular one by constructing the o-patterns of the function (where an *o-pattern* of a function $f: S \to G$ is a function $g: S \to G$ such that $g(x) \in H(\langle f(x)\rangle)$, $\forall x \in S$).

In the case of pairs of topological spaces S, S' and of pairs of graphs G, G in [5] in order to introduce the o-patterns, we gave the definition of *balanced* function i.e. of a function $f: S, S' \to G, G'$ such that $\langle f(x')\rangle = \langle f'(x')\rangle$, $\forall x' \in S'$. With reference to this we remember that if the subspace S' is open in S, all the functions are balanced.

1) Enlargability of sets in a uniform space.

DEFINITION 1. - *Let $(S, \mathcal{W})$ be a uniform space, where the filter $\mathcal{W}$ is the uniformity of S. Given a vicinity $W \in \mathcal{W}$, we put $W(x) = \{y \in S \ / \ (x,y) \in W\}$, $\forall x \in S$, and* $W(X) = \bigcup_{x \in X} W(x)$, $\forall X \subset S$.

' REMARK. - If (S,d) is a metric space the subsets $W^{\varepsilon} = \{(p,q) \in S \times S \ /$

$p,q) < \varepsilon\}$, $\varepsilon > 0$, constitute a basis of the uniformity induced by the tric d.

DEFINITION 2. - *Let $(S,\mathcal{W})$ be a uniform space and W a vicinity of $\mathcal{W}$. ıen n subsets $X_1,\dots,X_n$ of S are called* W-enlargable *if $W(X_1)\cap\dots\cap$ $W(X_n) = \emptyset$.*

REMARK. If $X_1,\dots,X_n$ are W-enlargable, then all the m-tuples (m > n), ɔtained by adding any n-m subsets of S, are still W-enlargable .

DEFINITION 3. - *Let (S,d) be a metric space and $X_1,\dots,X_n$ subsets of . We call* enlargability *of the n-tuple $X_1,\dots,X_n$, and we denote by* nl$(X_1,\dots,X_n)$ *the non-negative real number r such that:*

$$W^{\varepsilon}(X_1)\cap\dots\cap W^{\varepsilon}(X_n) \begin{cases} = \emptyset, & \forall\varepsilon \leqslant r \\ \neq \emptyset, & \forall\varepsilon > r. \end{cases}$$

REMARK 1. - If $\overline{X}_1\cap\dots\cap\overline{X}_n \neq \emptyset$, we put $enl(X_1,\dots,X_n) = 0$, while if ɔne at least among the X_i is empty, we put $enl(X_1,\dots,X_n)$ = diameter ɔf S.

REMARK 2. - Let $X_1,\dots,X_m$ be a m-tuple of subsets of S, obtained by adding to the n-tuple $X_1,\dots,X_n$ any $m-n$ subsets of S, then $enl(X_1,\dots,X_n) \leqslant enl(X_1,\dots,X_m)$.

REMARK 3. - Let $X_1 \neq \emptyset$, $X_2 \neq \emptyset$. It results $enl(X_1,X_2) \leqslant d(X_1,X_2) \leqslant 2enl(X_1,X_2)$. In fact if we put $d(X_1,X_2) = \eta$, for all ε there exist $x \in X_1$ and $y \in X_2$ such that $d(x,y) < \eta+\varepsilon$. Hence it is $W^{\eta+\varepsilon}(X_1)\cap W^{\eta+\varepsilon}(X_2) \neq \emptyset$, i.e. $enl(X_1,X_2) < \eta+\varepsilon = d(X_1,X_2) + \varepsilon$. Since ε is arbitrary, it follows $enl(X_1,X_2) \leqslant d(X_1,X_2)$.

Moreover let $r = enl(X_1,X_2)$. For all $\varepsilon>0$ it is $W^{r+\varepsilon}(X_1)\cap W^{r+\varepsilon}(X_2) \neq \emptyset$. Then there exist $z \in W^{r+\varepsilon}(X_1)\cap W^{r+\varepsilon}(X_2)$, $x_1 \in X_1$ and $x_2 \in X_2$ such that $d(X_1,X_2) \leqslant d(x_1,x_2) \leqslant d(x_1,z)+d(x_2,z) \leqslant 2r + 2\varepsilon = 2enl(X_1,X_2) + 2\varepsilon$. Since ε is arbitrary, it follows $d(X_1,X_2) \leqslant 2enl(X_1,X_2)$. We remark that it may be $d(X_1,X_2) < 2enl(X_1,X_2)$. In fact if $S = \{x_1,x_2\}$ is the discrete metric space, where $d(x_1,x_2) = 1$, it is $enl(\{x_1\},\{x_2\}) = 1$.

PROPOSITION 1. - *Let S be a compact space and the filter $\mathcal{W}$ the uniformity of S* (*). *If, for n subsets $X_1,\dots,X_n$ of S, it results $\overline{X}_1\cap\dots\cap\overline{X}_n$*

(*) We remark that in a compact space there exists only one uniformity compatible with the topology (see [2] , Cap. 2, §4, n° 1).

$= \emptyset$, *then there exists a vicinity* $W \in \mathcal{W}$ *such that* $X_1, \ldots, X_n$ *are* W-*enlargable.*

Proof. - We suppose all the sets X_i are non-empty, otherwise the proposition is trivial. Since S is compact, $\forall i = 1, \ldots, n$, the family $\{W(\overline{X}_i)\}$, $\forall W \in \mathcal{W}$, constitute a basis of the neighbourhoods filter of $\overline{X}_i$ (see [2], Cap. 2, §4, n° 3); moreover, since S is normal, the neighbourhoods filter of $\overline{X}_i$ is closed. Consequently, $\{W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)\}$ $\forall W \in \mathcal{W}$ is the basis of a closed filter $\mathfrak{F}$. Now, if $\mathfrak{F}$ is the null filter, there exists $W \in \mathfrak{F}$ such that $W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n) = \emptyset = W(X_1) \cap \ldots \cap W(X_n)$, i.e. $X_1, \ldots, X_n$ are W-enlargable. Otherwise, since S is compact, there exists a point x adherent to $\mathfrak{F}$, and since $\mathfrak{F}$ is a closed filter, $x \in W(\overline{X}_1) \cap \ldots \cap W(\overline{X}_n)$, $\forall W \in \mathcal{W}$. Then it is $x \in W(\overline{X}_i)$, $\forall W \in \mathcal{W}$, $i = 1, \ldots, n$. As the sets $W(\overline{X}_i)$ constitute a basis of the neighbourhoods filter of $\overline{X}_i$, it follows $x \in \overline{X}_i$, $i = 1, \ldots, n$, i.e. $x \in \overline{X}_1 \cap \ldots \cap \overline{X}_n$. Contradiction

COROLLARY 2. - *Let* S *be a compact metric space and* $X_1, \ldots, X_n$ *subsets of* S *such that* $\overline{X}_1 \cap \ldots \cap \overline{X}_n = \emptyset$, *then it is* $enl(X_1, \ldots, X_n) > 0$. □

2) *The second normalization theorem.*

DEFINITION 4. - *Let* A *be a non-empty set,* G *a finite graph and* $P = \{X_j\}$, $j \in J$, *a partition of* A. *A function* $f: A \to G$ *is called* quasi constant with respect to P (w.r.t. P) *or* P-constant *if the restrictions of* f *to each* X_j *are constant functions. Moreover , if* A *is a topological space ,* $f: A \to G$ *is called* weakly quasi-constant w.r.t. P *or* **weakly** P-costant *if the restrictions of* f *to the interior of every* X_j *are constant.*

REMARK. - If $P' = \{X'_k\}$, $k \in K$, is a partition of A finer than P, i.e. if all the $X_i \in P$ are the union of elements $X'_k \in P'$, then the function f is obviously quasi-contant also w.r.t. P'.

DEFINITION 5. - *Let* $(S, \mathcal{W})$ *be a uniform space and* W *a vicinity of* $\mathcal{W}$. *A subset* X *of* S *is called* small of order W *or a* W-subset *if* $X \times X \subseteq W$. *Moreover a family* $\mathcal{X} = \{X_j\}$, $j \in J$, *is called* small of order W *or a* W-family *if* $X_j \times X_j \subseteq W$, $\forall j \in J$.

REMARK 1. - If W is closed and $\{X_j\}$, $j \in J$, is a W-family, $\{\overline{X}_j\}$, $j \in J$, is a W-family .