CHAPTER III. SOME REGULARITY CONDITIONS IN FINITE PLANAR SPACES.

1. INTRODUCTION.

The generalized projective spaces of dimension > 2 (endowed with all their planes) may be defined as the non-trivial planar spaces S satisfying the following condition :

(∗) for every pair of planes Π and Π' intersecting in a line and for every point $x \notin \Pi \cup \Pi'$ such that there is a line through x intersecting Π and Π' in two distinct points, every line through x intersecting Π intersects Π'.

Indeed, let L and L' be two lines intersecting in some point x and let A and A' be two lines not passing through x and intersecting each of the lines L and L'. Suppose that A and A' are disjoint. Since S is a non-trivial planar space, there is a point y outside $\langle L,L'\rangle$. The planes $\Pi = \langle y,L\rangle$ and $\alpha = \langle y,A\rangle$ intersect in a line. The line L' intersects the planes Π and α in two distinct points and contains a third point $x' \in L' \cap A'$. Therefore, by condition (∗), every line passing through x' and intersecting Π intersects α. In particular, the line A' which intersects $L \subset \Pi$ intersects α. Hence A', which is contained in $\langle L,L'\rangle$, intersects $A = \alpha \cap \langle L,L'\rangle$, a contradiction. Hence Pasch's axiom is satisfied.

In particular, the 3-dimensional generalized projective spaces are the non-trivial planar spaces satisfying

(I) for every pair of planes Π and Π' intersecting in a line, every line intersecting Π intersects Π'.

Indeed, any non-trivial planar space S satisfying condition (I) satisfies also condition (∗), and so is a generalized projective space; moreover, since S is necessarily the smallest linear subspace containing two planes intersecting in a line, S is 3-dimensional.

Note that the condition obtained from (I) by deleting the words "intersecting in a line", though apparently stronger than (I) is equivalent to (I).

Two problems arise now in a natural way : is it possible to classify the non-trivial planar spaces which satisfy the condition obtained from (I) by replacing "intersecting in a line" by "intersecting in a point" (resp. by "having an empty intersection") ? This is the subject of the following two theorems, concerning finite planar spaces.

points would allow us to rule out the rather uninteresting cases (c) and (d), and to shorten the proof a little bit.

Proof of the theorem

The proof is divided into a series of lemmas. The planar space PG(4,1) obviously satisfies the hypotheses and we shall always assume in what follows that $S \neq PG(4,1)$

Lemma 9.1. If each of the two planes Π' and Π'' intersects a third plane Π in exactly one point, then $\Pi \cap \Pi' = \Pi \cap \Pi'' = \Pi' \cap \Pi''$.

Proof. Suppose first that $\Pi \cap \Pi' = \{x'\}$ and $\Pi \cap \Pi'' = \{x''\}$ where the points x' and x'' are distinct. By condition (II), every line of Π' passing through x' intersects Π'' in a point, and so $\Pi' \cap \Pi''$ is a line L. Condition (II), applied to the pair of planes $\{\Pi,\Pi'\}$ (resp. $\{\Pi,\Pi''\}$), shows that any line of Π'' (resp. Π') intersecting L passes through x'' (resp. x'), which implies that $\Pi' = L \cup \{x'\}$ and $\Pi'' = L \cup \{x''\}$. If there is a point $x \notin \Pi \cup L$, the line $\langle x,x''\rangle$ must intersect $\Pi' = L \cup \{x'\}$, a contradiction. Therefore $S = \Pi \cup L$.

Let y be a point of L and let A be a line of Π passing through x'' and distinct from $\langle x',x''\rangle$. Since $\Pi \cap \Pi'' = \{x''\}$, the lines A and L are not coplanar and the plane $\alpha = \langle y,A\rangle$ intersects Π' in the point y only. By condition (II), any line of Π intersecting A (hence α) must intersect Π', and so must contain x'; it follows that $\Pi = A \cup \{x'\}$. Similarly, $\Pi = B \cup \{x''\}$ for any line B of Π passing through x' and distinct from $\langle x',x''\rangle$. Therefore Π contains only three points x, x' and x''. If L has at least three points y, y' and y'', then the line $\{x',y''\}$ intersects the plane $\{x'', x', y'\}$ but not the plane $\{x'', x, y\}$, and condition (II) is not satisfied. Therefore L has size 2 and $S = PG(4,1)$ contradicting the initial assumption.

This proves that $x' = x''$. By condition (II), any line of Π' intersecting $\Pi' \cap \Pi''$ must intersect Π, which implies that $\Pi' \cap \Pi'' = \{x'\}$.

A maximal set of planes having the property that any two of them intersect in the point x only will be called a *direction of planes with top* x. It follows from Lemma 9.1. that any plane Π belongs to at most one direction, denoted by dir Π. The top of dir Π will also be called the top of Π and a top in S.

Corollary 9.1. If dir Π contains at least three planes with top x, then all the lines passing through x and belonging to a plane of dir Π have the same size.

Proof. If Π, Π' and Π'' are three distinct planes of dir Π and if L (resp. L') is a line of Π (resp. Π') passing through x, Lemma 9.1. implies that the plane $\langle L,L'\rangle$ intersects Π'' in a line L". By condition (II), any line intersecting L" and L (resp. L" and L') in two distinct points must intersect L' (resp. L), and so L and L' have the same size. The corollary follows easily.

Lemma 9.2. For any point x *of* S, *the residue* S_x *of* x *is one of the following*

(i) *a projective plane (possibly degenerate)*

(ii) *a punctured projective plane*

(iii) *an affine plane with one point at infinity*

(iv) *an affine plane.*

Proof. Two planes of S intersect in x (and in x only) iff the corresponding lines of S_x are disjoint. Therefore Lemma 9.1. implies that if L and L' are two disjoint lines in S_x, any line of S_x intersecting L in one point must also intersect L' in one point. In other words, the linear space S_x is a semi-affine plane. Since S is assumed to be finite, S_x is finite and we know by (I 6) that S_x is either an affine plane, or an affine plane with one point at infinity, or a punctured projective plane, or a (possibly degenerate) projective plane.

The finiteness assumption is essential here : indeed, Dembowski has constructed infinite semi-affine planes which are not of the four types described above [30].

Note that S_x is always an affino-projective plane, except if S_x is a degenerate projective plane. Note also that S_x is a (possibly degenerate) projective plane iff x is not a top in S.

Corollary 9.2. If S_x *is an affino-projective plane of order* k, *then* x *has degree* k *in every plane with top* x.

Proof. It suffices to observe that a plane with top x corresponds to a line of S_x having at least one disjoint line in S_x, that is a line of size k in S_x.

Lemma 9.3. If S *contains a point* x *such that* S_x *is a degenerate projective plane, then* S *is of type* (c).

Proof. The hypothesis implies that S is the union of a plane Π and of a line A intersecting Π in x. Let z be a point on A, distinct from x. Since $S = \Pi \cup A$, every line passing through z intersects Π. Therefore the plane Π is isomorphic to S_z and, by Lemma 9.2 , Π is a semi-affine plane.

Suppose that there are two points z and z', distinct from x, on the line A. The plane Π contains two intersecting lines L and L' not passing through x (except if Π is a degenerate projective plane in which all lines through x have size 2, but in this case S is a 3-dimensional generalized projective space and condition (II') is not satisfied). If Π contains either a point $y \notin L \cup L' \cup \{x\}$ or a line L" intersecting L' but not L, then the planes $\langle L,z \rangle$ and $\langle L',z' \rangle$ intersect in the point $L \cap L'$ only, and either the line $\langle y,z' \rangle$ or the line L" intersects $\langle L',z' \rangle$ but not $\langle L,z \rangle$, in contradiction with (II). Therefore the semi-affine plane Π has no such point y and no such line L", and so Π is necessarily a degenerate projective plane with 4 points, in which x is of degree 2. Denote by B the line of size 3 in Π and by x' the point of degree 3 in Π. Then $S = \langle A,B \rangle \cup \{x'\}$ and $S_{x'}$ is isomorphic to $\langle A,B \rangle$. It follows that $\langle A,B \rangle$ is an affine plane of order 2 with the point x at infinity, and so S is of type (c).

Therefore we may assume that A is a line $\{x,z\}$ of size 2. Then $S = \Pi \cup \{z\}$ and all lines through z have size 2. If Π is a (possibly degenerate) projective plane, then S is a 3-dimensional generalized projective space and condition (II') is not satisfied. Therefore the semi-affine plane Π is either a punctured projective plane, or an affine plane with one point at infinity, or an affine plane, and the Lemma is proved.

From now on, we shall always assume that there is no point $x \in S$ such that S_x is a degenerate projective plane.

Lemma 9.4. If S *contains a point* x *such that* S_x *is an affine plane of order* k *with one point at infinity, then* S *is obtained from* PG(3,k) *by deleting an affino-projective plane which is neither projective nor punctured projective.*

Proof. Denote by L_∞ the line of S corresponding to the point at infinity of S_x, by y any point of L_∞ distinct from x, and by Π any plane passing through x and not containing L_∞. S is the union of L_∞ and of all planes of dir Π. Therefore any line through y intersects at least one (hence every) plane of dir Π, and so we define an isomorphism between S_y and Π by mapping any line passing through y onto its point of intersection with Π. Therefore Π is a semi-affine plane (distinct from a degenerate projective plane). By Corollary 9.2 , x has degree k in Π, and so either Π has order k-1 or Π is an affine plane of order k with the point x at infinity.

If Π is a projective plane of order k-1, then all lines of S distinct from L_∞ and passing through x have size k. Let Π_∞ be a plane of S containing L_∞ and let Π_∞^* denote the linear space induced on $\Pi_\infty - (L_\infty - \{x\})$ by the linear structure of Π_∞. Since Π_∞ intersects every plane of dir Π in a line through x and since every line of Π_∞ not passing through x intersects each of the k planes of dir Π in a point, all the lines of the linear space Π_∞^* have size k. The degree of x in Π_∞^* is $k = |\text{dir } \Pi|$, and so Π_∞^* is a projective plane. On the other hand, the lines of Π_∞ passing through y induce pairwise disjoint lines in Π_∞^*, a contradiction.

If the semi-affine plane Π has order k-1 and is not a projective plane, then Π contains a line L of size k-1 not passing through x. Let $\Pi_1 = \langle L,y \rangle$ where $y \neq x$ is a point of L_∞. The intersections of Π_1 with the planes of dir Π form a partition Δ_1 of $\Pi_1 - \{y\}$ into k lines of size $k-1 = |L|$. On the other hand, the lines of Π_1 passing through y define a partition Δ_2 of $\Pi_1 - \{y\}$ into k-1 lines of size k. Let $L' \notin \Delta_1 \cup \Delta_2$ be a line of Π_1. By condition (II), L' intersects each of the lines of Δ_1, which is impossible since $L' \notin \Delta_2$.

Therefore every plane Π containing x but not L_∞ is an affine plane of order k with the point x at infinity. Since any line of S distinct from L_∞ is either contained in some plane of dir Π or intersects every plane of dir Π in a point, the lines of S distinct from L_∞ have size k+1 or k according as they intersect L_∞ or not. Moreover, the planes of S containing x have exactly k^2 points outside L_∞ and the planes not containing x intersect the planes of dir Π in k pairwise disjoint lines of size k. Therefore, in the planar space of k^3 points induced on $S-L_\infty$, all lines have k points and all planes have k^2 points. In other words, $S-L_\infty$ is a planar space of k^3 points in which all planes are affine planes of order k. If k=2, $S-L_\infty$ is the unique Steiner system S(3,4,8), that is the affine space AG(3,2). If k=3, $S-L_\infty$ is the unique Hall triple system of 27 points [36], that is the affine space AG(3,3). If $k \geq 4$, then by (I5) , $S-L_\infty$ is the affine space AG(3,k).

It follows that S is obtained from an affine space AG(3,k) by adding a line at infinity L_∞ to a direction of parallel planes. Using the classical process of completion by points at infinity we conclude easily that S is obtained by deleting from PG(3,k) an affino-projective plane (which is neither projective nor punctured projective since L_∞ contains at least 2 points).

Corollary 9.4. (i) *If* S *contains a point* x *such that* S_x *is an affine plane with one point at infinity, then for any top* y *in* S, S_y *is also an affine plane with one point at infinity.*

(ii) *If* S *contains a point* x *such that* S_x *is an affine plane, then* x *is the only top in* S.

Proof. (i) is an immediate consequence of Lemma 9.4. In order to prove (ii), suppose on the contrary that there is a top $y \neq x$ in S. By (i), S_y is not an affine plane with one point at infinity, and so, by Lemma 9.2., S_y is either an affine plane or a punctured projective plane. In both cases, the line $\langle x,y \rangle$ is contained in a plane Π with top y. On the other hand, there is in S_x a line disjoint from the line Π_x of S_x corresponding to Π, and so Π is a plane with top x. Therefore Π has two distinct tops x and y, in contradiction with Lemma 9.1.

Lemma 9.5. If S *contains a point* x *such that* S_x *is a punctured projective plane of order* k *or an affine plane of order* k, *then every plane* Π *with top* x *is an affine plane of order* k *with the point* x *at infinity.*

Proof. Let $\Pi' \neq \Pi$ be a plane of dir Π and let $y \neq x$ be a point of Π'. By condition (II), all the lines passing through y and disjoint from Π are included in Π'. Therefore if we map each line of S passing through y and intersecting Π onto its point of intersection with Π, we define an isomorphism between Π and the linear space induced by S_y on $S_y - (\Pi'_y - L_y)$ where Π'_y is the line of S_y corresponding to the plane Π' and L_y is the point of S_y corresponding to the line $L = \langle x,y \rangle$. Thanks to Corollary 9.4., we know that S_y is either a projective plane or a punctured projective plane. If $|\text{dir } \Pi| > 2$, then all lines of Π passing through x have the same size by Corollary 9.1. If $|\text{dir } \Pi| = 2$, then S_x must be an affine plane of order 2 and Corollary 9.4. implies that S_y is a projective plane. Therefore, in any case, Π is an affine plane with the point x at infinity and, by Corollary 9.2., the order of Π is k.

Lemma 9.6. If S *contains a point* x *such that* S_x *is an affine plane of order* k, *then* S *is obtained from* PG(3,k) *by deleting a punctured projective plane of order* k.

Proof. By Lemma 9.5. and condition (II), the lines of S have k+1 or k points according as they contain x or not, and the planes of S have k^2+1 or k^2 points according as they contain x or not. Therefore $S - \{x\}$ is a planar space of k^3 points in which all lines have k points and all planes have k^2 points. By the same arguments as in the proof of Lemma 9.4., we conclude that $S - \{x\}$ is an affine space AG(3,k) and that S is obtained from the projective space PG(3,k) by deleting a punctured projective plane of order k.

affine planes of order 3 or planes of 13 points consisting of three concurrent lines of size 5 (all the other lines having size 3). We denote by S'_H the linear space obtained from S_H by replacing every line of size 5 by 10 lines of size 2. Let F_9 be the affine plane induced on S'_H by a plane intersecting S'_H in 9 points and let x be a point of $S'_H - F_9$ which is on at least one line of size 3 intersecting F_9. The smallest linear subspace of S'_H containing F_9 and x has exactly 18 points and is a Fischer space, denoted by F_{18}. S_{18} is the planar space induced by PG(3,4) on the set of points of F_{18}.

In order to define S_{36}, let H' be a Hermitian quadric in PG(3,4) having exactly one singular point s (for instance, the quadric of equation $x\bar{y} + \bar{x}y + z\bar{z} = 0$). The planar space induced on H' by PG(3,4) has lines of size 3 or 5, and its planes are either affine planes of order 3 not passing through s or planes of 13 points consisting of 3 lines of size 5 concurrent in s (all the other lines having size 3). S_{36} is the planar space induced by PG(3,4) on $H' - \{s\}$ and F_{36} is obtained from S_{36} by replacing every line of size 4 by 6 lines of size 2.

We still need a notation for five small spaces satisfying (III) and (III'). The space K_7^7 is obtained from PG(2,2) by taking as points the points of PG(2,2), as lines the pairs of points and as planes the lines of PG(2,2) and their complements. The planar space of 6 points in which all lines have size 2 and which contains 0, 1, 2 or 3 planes of 4 points (all the other planes having 3 points) will be denoted by K_6^0, K_6^1, K_6^2 and K_6^3 . It is easy to check that these spaces are uniquely determined by the above properties.

Statement of the theorem.

We shall first prove two fundamental lemmas. In what follows, S denotes always a finite planar space satisfying (III) and (III').

Lemma 10.1. For any plane Π *of* S *and for any point* $x \in S - \Pi$, *there is at most one plane passing through* x *and disjoint from* Π.

Proof. Suppose on the contrary that x is on two distinct planes Π' and Π'' disjoint from Π. Let L be a line passing through x, contained in Π' but not in Π''. Since L intersects Π'', condition (III) implies that L must intersect Π, a contradiction.

A maximal set of pairwise disjoint planes will be called a *direction* of planes, provided there are at least two planes in it. By Lemma 10.1, a plane Π either intersects any other plane or is in exactly one direction, denoted by *dir* Π.

We can now state our main result :

Theorem 10 [25] *If* S *is a finite planar space such that*
(III) *for any two disjoint planes* Π *and* Π', *every line intersecting* Π *intersects* Π'
and (III') *there are at least two disjoint planes*
then one of the following occurs :
(a) S *is obtained from* PG(3,k) *by deleting a line,*
(b) S *is obtained from* PG(3,k) *by deleting an affino-projective (but not affine) plane of order* k,
(c) $S = S_{36}$,
(d) $S = S_{18}$,
(e) S *is a space* S_7 *of 7 points lying on 3 concurrent lines of size 3, all the other lines having size 2, in which the planes either have only 3 points or are unions of two lines of size 3,*
(f) $S = K_7^7, K_6^0, K_6^1, K_6^2$ or K_6^3 ,
(g) S *has only one direction of planes and* $S-S^*$ *contains at least four non-coplanar points.*

We do not know whether there is a finite planar space of type (g).

The proof will be divided into three main parts : we shall first handle some small exceptional spaces (types (e) and (f)), then we shall classify the spaces having at least two directions of planes (types (b) and (c)) and finally we shall examine the spaces having exactly one direction of planes (types (a), (d) and (g)).

3.1. Small exceptional spaces

Lemma 10.3. If S *contains two disjoint planes* Π *and* Π' *such that* $S = \Pi \cup \Pi'$, *then* S *is the union of any two disjoint planes (in particular, every direction has exactly two planes).*

Proof. Suppose on the contrary that Π_1 and Π_1' are two disjoint planes of S such that there is a point $x \notin \Pi_1 \cup \Pi_1'$. We may assume without loss of generality that $x \in \Pi$. Then for any point $y \in \Pi_1 \cap \Pi'$, the line <x,y> has size 2, and so <x,y> is disjoint from Π_1', in contradiction with (III).

Proposition 10.1. If S *contains two disjoint planes* Π *and* Π' *such that* $S = \Pi \cup \Pi'$, *then* S *is the affine space* AG(3,2), $K_7^7, K_6^0, K_6^1, K_6^2$ or K_6^3 .

1.3) Suppose finally that $\Pi_1 \cap \Pi_2$ is a point t and that $\Pi_1' \cap \Pi_2'$ is a point z. Then Π_1, Π_1', Π_2, Π_2' are degenerate projective planes and $\Pi_1 \cap \Pi_2' = X$, $\Pi_1' \cap \Pi_2 = Y$ are lines. Suppose that X has at least three points x,x',x'' and let y,y' be two points of Y. If there is a transversal plane $\Pi = \{x,y,z\}$ not containing t, then the planes $\langle x',y',t\rangle$ and $\langle x'',y',t\rangle$ must intersect Π by Lemma 10.3, and so $\{x',y',z,t\}$ and $\{x'',y',z,t\}$ are distinct planes, a contradiction since they have three non-collinear points in common. Therefore, any transversal plane contains t and $\langle x,y,z\rangle = \langle x,z,t\rangle = \langle x,z,y'\rangle$, a contradiction. This shows that X (and similarly Y) has size 2. Hence S has exactly 6 points and all lines have size 2. Since the union of two disjoint planes contains at least 6 points, S will automatically be the union of any two disjoint planes. It is a trivial exercise to check that there are exactly 4 non-isomorphic planar spaces of 6 points in which all lines have size 2 and which satisfy conditions (III) and (III') (they have respectively 0, 1, 2 or 3 planes of 4 points).

2) In order to complete the proof, it remains to show that the case where S has only one pair of disjoint planes Π and Π' such that $S = \Pi \cup \Pi'$ leads to a contradiction.

Suppose first that Π and Π' are two projective planes (possibly degenerate). If there is a plane intersecting Π in a line A and Π' in a line A', let $a \in \Pi - A$ and $a' \in \Pi' - A'$. The planes $\langle A,a'\rangle = A \cup \{a'\}$ and $\langle A',a\rangle = A' \cup \{a\}$ are disjoint, a contradiction. Hence there is no plane intersecting both Π and Π' in a line. Let L (resp. L') be a line of Π (resp. Π') and let $x \in \Pi - L$, $x' \in \Pi' - L'$. The planes $\langle L,x'\rangle = L \cup \{x'\}$ and $\langle L',x\rangle = L' \cup \{x\}$ are disjoint, a contradiction.

Therefore we may assume that Π contains two disjoint lines A and B. If there is a plane α containing A and intersecting Π' in only one point x', then every plane $\beta \neq \Pi$ containing B must intersect α, and so must contain x', a contradiction because two such planes β_1 and β_2 would have three non-collinear points in common. Therefore the planes $\alpha_1,\ldots,\alpha_n \neq \Pi$ containing A intersect Π' in lines $A_1', \ldots, A_n'$ partitioning Π'. Similarly, the planes $\beta_1, \ldots, \beta_m \neq \Pi$ containing B intersect Π' in lines $B_1', \ldots, B_m'$ partitioning Π'. Since Π and Π' are the only two disjoint planes of S, any line A_i' intersects each line B_j', and so there is no line in Π' which is coplanar with A and also coplanar with B. For the same reason, there is no line in Π which is coplanar with A_1' and also coplanar with A_2', a contradiction since A is coplanar with A_1' and with A_2'.

From now on, we shall assume that S is not the union of two disjoint planes.

Proposition 10.2. If there is a plane $\Pi \notin \operatorname{dir} \Pi_1$ *intersecting* Π_1 *in only one point* x*, then* $S = S_7$.

Proof. Let $\Pi_1' \neq \Pi_1$ be any other plane of dir Π_1. By Lemma 10.2, Π is contained in $\Pi_1 \cup \Pi_1'$ (which implies that dir $\Pi_1 = \{\Pi_1, \Pi_1'\}$), and so Π is a degenerate projective plane. Let y be a point outside $\Pi_1 \cup \Pi_1'$ (hence outside Π). By condition (III), the line $\langle x,y\rangle$ intersects Π_1' in a point $x' \notin L' = \Pi \cap \Pi_1'$, and, by Lemma 10.2, the plane $\langle L',y\rangle$ intersects Π_1 in a line L not containing x. Thus the plane $\Pi' = \langle x',L\rangle$ is disjoint from Π and is isomorphic to Π by Corollary 10.2.1. If there is a line L'' disjoint from $\Pi \cup \Pi'$, then Lemma 10.2 implies that the planes through L'' which are not disjoint from Π intersect Π in disjoint lines, which is impossible since Π is a degenerate projective plane. Therefore all lines passing through y intersect Π or Π', and so, by (III), intersect Π and Π'.

If L contains at least three points u, v, w, let $u' = L' \cap \langle u,y\rangle$, $v' = L' \cap \langle v,y\rangle$, $w' = L' \cap \langle w,y\rangle$. The planes $\alpha = \langle x,u,w'\rangle$ and $\alpha' = \langle x',u',v\rangle$ are two disjoint planes of 3 points and the line $\langle v,v'\rangle$ intersects α' but not α, so that condition (III) is not satisfied. Therefore the lines L and L' have size 2 and y is on exactly 3 lines.

If the line $\langle x,y\rangle$ contains a fourth point x'', then the planes $\langle L',x''\rangle$ and $\langle L',x\rangle = \Pi$ have a line in common and are both disjoint from $\Pi' = \langle L,x'\rangle$, contradicting Lemma 10.1. The same argument shows that the 3 lines passing through y have size 3, and so $S = S_7$.

From now on, we shall assume that $S \neq S_7$ so that, by Proposition 10.2, any plane not belonging to a direction dir Π intersects all planes of dir Π in a line.

3.2. Suppose that there are at least two directions of planes dir Π_1 and dir Π_2.

By Lemma 10.2 and Corollary 10.2.2, the set of lines $\Pi_1^i \cap \Pi_2^j$, where $\Pi_1^i \in \operatorname{dir} \Pi_1$ and $\Pi_2^j \in \operatorname{dir} \Pi_2$, is a partition of S^* and will be denoted by $\delta(\Pi_1 \cap \Pi_2)$.

Lemma 10.4. If a plane Π *intersects a line of* $\delta(\Pi_1 \cap \Pi_2)$ *in a single point, then* Π *intersects every line of* $\delta(\Pi_1 \cap \Pi_2)$ *in a single point and* $\Pi^* = \Pi \cap S^*$ *is an affine plane of order* $k = |\operatorname{dir} \Pi_1| = |\operatorname{dir} \Pi_2|$.

Proof. The intersections of the planes of dir Π_1 (resp. dir Π_2) with Π^* define a partition δ_1 (resp. δ_2) of Π^* into lines of S. Note that $\delta_1 \neq \delta_2$, otherwise

Π^* would not intersect a line of $\delta(\Pi_1 \cap \Pi_2)$ in a single point. If L^* is any line of Π^* not in δ_1 (resp. not in δ_2), condition (III) implies that L^* intersects every line of δ_1 (resp. δ_2). There is a line of δ_1 which is not in δ_2; since this line must intersect every line of δ_2, δ_1 and δ_2 have no line in common. Therefore, if L_i is any line of δ_i $(i = 1,2)$ and if L^* is any line of Π^* not in $\delta_1 \cup \delta_2$, we have

$$|L_1| = |\delta_2| = |L^*| = |\delta_1| = |L_2|$$

and so all lines of Π^* have the same size

$$k = |\delta_1| = |\mathrm{dir}\ \Pi_1| = |\delta_2| = |\mathrm{dir}\ \Pi_2|$$

Moreover,

$$|\Pi^*| = |\delta_1| \ . \ |L_1| = k^2$$

and so Π^* is an affine plane of order k.

Since every plane of dir Π_1 is partitioned into $k = |\mathrm{dir}\ \Pi_2|$ lines of $\delta(\Pi_1 \cap \Pi_2)$, since Π^* intersects such a plane in a line of size k and since Π^* contains no line of $\delta(\Pi_1 \cap \Pi_2)$ (because $\delta_1 \cap \delta_2 = \emptyset$), we conclude that Π^* intersects every line of $\delta(\Pi_1 \cap \Pi_2)$ in a single point.

The planes of S (or S^*) intersecting every line of $\delta(\Pi_1 \cap \Pi_2)$ will be called *transversal* and those containing a line of $\delta(\Pi_1 \cap \Pi_2)$ will be called *non-transversal*. Note that any plane of S is either transversal or non-transversal. For any triple of non-coplanar lines L, L', L" $\in \delta(\Pi_1 \cap \Pi_2)$, the product $|L|.|L'|.|L''|$ counts the total number of transversal planes in S. It follows that all lines of $\delta(\Pi_1 \cap \Pi_2)$ have the same size ℓ. Since Π_1 (resp. Π_2) is partitioned into k lines of size ℓ by its intersections with the planes of dir Π_2 (resp. dir Π_1), we have

$$|\Pi_1| = |\Pi_2| = k\ell \qquad (1)$$

and $|S^*| = |\mathrm{dir}\ \Pi_1| \ . \ |\Pi_1| = k^2\ell$

Lemma 10.5. S - S^* *is a linear subspace of* S, *and any transversal plane* Π *of* S *has at most* k-1 *points outside* S^*.

Proof. By condition (III), any line intersecting a plane of dir Π_1 intersects every plane of dir Π_1. It follows that, for any point $x \in \Pi - \Pi^*$, the set of all lines passing through x and intersecting Π^* determines a partition of Π^* into lines, i.e. a parallel class in the affine plane Π^*. Therefore, if x and

y are any two points of $S-S^*$, the line $<x,y>$ must be disjoint from S^*. This proves that $S-S^*$ is a linear subspace of S. Since there are k+1 parallel classes in Π^* and since at least two (induced by dir Π_1 and dir Π_2) are also classes of pairwise disjoint lines in Π, there are at most k-1 points in $\Pi-\Pi^*$.

Note that the planes of S are not necessarily the smallest linear subspaces containing three non-collinear points. On the contrary, if x, y, z are non-collinear points in $S-S^*$, it follows from Lemma 10.2 that the plane $<x,y,z>$ of S intersects S^* in a plane of S^*.

Proposition 10.3. If $\ell=k$, *then* S *is obtained from a 3-dimensional projective space* PG(3,k) *by deleting an affino-projective (but not affine) plane of order* k.

Proof. If $\ell=k$, then S^* is a planar space of k^3 points in which all lines have k points and all planes have k^2 points, hence S^* is the 3-dimensional affine space AG(3,k). Indeed, by (I 5) if k = 2, S^* is the unique Steiner system S(3,4,8), that is the affine space AG(3,2); if k = 3, S^* is the unique Hall triple system of 27 points, that is the affine space AG(3,3); if $k \geq 4$, S^* is the affine space AG(3,k).

On the other hand, if the linear subspace $S - S^*$ contains three non-collinear points x, y, z, then the planes containing $<x,y>$ and those containing $<x,z>$ induce two distinct partitions of the affine space S^* into classes of parallel planes, but these partitions have the plane $<x,y,z> \cap S^*$ in common, a contradiction. Therefore $S - S^*$ is either empty, or a point, or a line of size at most k-1, and the lemma is proved.

From now on, we shall assume $\ell \neq k$, so that any transversal plane Π intersects all the other planes of S (otherwise Π would belong to a direction dir Π of planes of S with $|\text{dir } \Pi| = \ell$ and, by applying to dir Π and dir Π_1 the arguments used in the proof of Lemma 10.4, we would get $\ell = k$).

Lemma 10.6. For every transversal plane Π, *the number of planes of* S *whose intersection with* Π *is disjoint from* S^* *is a constant* c *independent from* Π, *and*

$$c = (\ell-1)[\ell^2 + \ell + 1 - (k^2 + k) - k^2(\ell-k)]$$

$$= (\ell-1)[b' + v'(\ell+1-k) - \sum_{x\in\Pi-\Pi^*} r'_x]$$

where v' *(resp.* b'*) denotes the number of points (resp. the number of lines) of the linear subspace* $\Pi-\Pi^*$ *and* r'_x *denotes the degree of* x *in* $\Pi-\Pi^*$.

Proof. Since every non-transversal plane intersects Π^* in a line, any plane of S whose intersection with Π is disjoint from S^* is necessarily a transversal plane. The total number of transversal planes distinct from Π is ℓ^3-1. The number of transversal planes intersecting Π^* in one line (resp. one point) is $(k^2+k)(\ell-1)$ (resp. $k^2(\ell-1)(\ell-k)$), since any line (resp. any point) of Π^* is in exactly $\ell-1$ (resp. ℓ^2-1) transversal planes distinct from Π. Hence the number of planes of S intersecting Π outside S^* is equal to

$$c = (\ell-1)[(\ell^2 + \ell + 1) - (k^2 + k) - k^2(\ell-k)] \quad ,$$

which is clearly independent of the choice of the transversal plane Π.

Let L be a line of $\Pi-\Pi^*$ (if there is one). Since the planes containing L intersect Π_1 in pairwise disjoint lines, the number of planes intersecting Π in L is $\ell-1$. Therefore the number of planes intersecting Π in a line outside Π^* is equal to $(\ell-1)b'$.

Now let x be a point of $\Pi-\Pi^*$ (if there is one). Any plane of S disjoint from Π^* intersects Π_1 in a line disjoint from $\Pi \cap \Pi_1$ and, for any line A of Π_1 disjoint from $\Pi \cap \Pi_1$, the plane $\langle A,x\rangle$ is disjoint from Π^* (otherwise it would intersect Π in a line intersecting $\Pi \cap \Pi_1$, a contradiction). Hence the number of planes through x which are disjoint from Π^* is equal to the number of lines of Π_1 which are disjoint from $\Pi \cap \Pi_1$, that is $(\ell-1)(\ell+1-k)$. Therefore the number of planes whose intersection with Π is the point x is equal to $(\ell-1)(\ell+1-k-r'_x)$. It follows that

$$c = (\ell-1)(b' + v'(\ell+1-k) - \sum_{x\in\Pi-\Pi^*} r'_x)$$

Corollary 10.6. If some transversal plane Π *contains at least one line of* $S - S^*$*, then every transversal plane of* S *contains at least one line of* $S - S^*$.

Proof. Let $x, y \in \Pi-\Pi^*$. We have seen that the number of planes of S passing through x (resp. y) and disjoint from Π^* is equal to $(\ell-1)(\ell+1-k)$, and that the number of planes of S intersecting Π in the line $\langle x,y\rangle$ is $\ell-1$. Therefore

$$c \geq 2(\ell-1)(\ell+1-k) - (\ell-1) > (\ell+1)(\ell+1-k)$$

since $\ell > k$. The existence of a transversal plane of S contained in S^* or having a single point outside S^* would imply $c = 0$ or $c = (\ell+1)(\ell+1-k)$, a contradiction.

Lemma 10.7. Any non-transversal plane Π_i^* *of* S *belongs to a unique partition* dir Π_i^* *of* S^* *into non-transversal planes. The set of all planes of* S *whose intersections with* S^* *are the planes of* dir Π_i^* *will be called the pseudo-direction* $\mathrm{dir}^* \Pi_i$.

Proof. Let Π^* be a transversal plane of S^*. If $\Pi_i^* \in \mathrm{dir}\ \Pi_1$, the lemma is obvious. If $\Pi_i^* \notin \mathrm{dir}\ \Pi_1$, Π_i^* is partitioned by its intersections with the planes of dir Π_1 into k lines belonging to $\delta(\Pi_1 \cap \Pi_2)$. Since all these lines intersect Π^*, $\Pi_i^* \cap \Pi^*$ is a line of Π^*. On the other hand, every line of Π^* is in a unique plane containing a line of $\delta(\Pi_1 \cap \Pi_2)$, that is in a unique non-transversal plane. Therefore Π_i^* belongs to a partition of S^* into k pairwise disjoint non-transversal planes, each of which contains one of the k parallels to $\Pi_i^* \cap \Pi^*$ in the affine plane Π^*. Such a partition is clearly unique.

Proposition 10.1. If $\ell \neq k$, *then* $S = S^* = S_{36}$.

Proof. Suppose on the contrary that $S - S^*$ is non-empty. A point $x \in S - S^*$ cannot belong to two non-transversal planes Π_i and Π_j whose pseudo-directions are distinct, because $\Pi_i \cap \Pi_j$ is a line of $\delta(\Pi_1 \cap \Pi_2)$ included in S^* . On the other hand, any line through x intersecting S^* is contained in a unique non-transversal plane. Therefore x belongs to the planes of exactly one pseudo-direction $\mathrm{dir}^* \Pi_i$. We shall say that x and $\mathrm{dir}^* \Pi_i$ are *associated*. Obviously, all the points of $S - S^*$ associated with a given pseudo-direction are collinear.

Suppose first that all points of $S - S^*$ are associated with the same pseudo-direction $\mathrm{dir}^* \Pi_i$. Then the points of $S - S^*$ are collinear (this includes the case where $|S - S^*| = 1$), and so there is a transversal plane Π having exactly one point outside S^*. Hence, by Lemma 10.6, $c = (\ell-1)(\ell+1-k)$. Since $k \neq \ell+1$ (because $c \neq 0$), there is a line A of Π_1 disjoint from $\Pi \cap \Pi_1$. The number of transversal planes through A is $|\Pi_1|/k = \ell$ and the number of transversal planes through A intersecting Π in a line is k-1. Therefore, since any two transversal planes have a non-empty intersection, the number of transversal planes through A intersecting Π in exactly one point is $\ell-(k-1)$. By counting in two ways the number of pairs (y,α) where y is a point of Π and $\alpha = \langle y,A \rangle$ we get

$$k^2 + 1 = k + (k-1)k + \ell - k + 1$$

which implies $\ell = k$, a contradiction.

This proves that $S - S^*$ contains two points associated with distinct pseudo-directions. If all the points of a line L_i of $S - S^*$ are associated with a pseudo-direction $\mathrm{dir}^* \Pi_i$, let x be a point of $S - S^*$ associated with another pseudo-direction $dir^* \Pi_j$. Then L_i is the intersection of the plane $<x,L_i>$ with each plane of $dir^* \Pi_i$. Since the planes of $dir^* \Pi_i$ induce a partition of S^*, the plane $<x,L_i>$ must be disjoint from S^* , a contradiction. Therefore any pseudo-direction of planes is associated with at most one point outside S^*. Since there are k+1 pseudo-directions and since $dir^* \Pi_1$ and $dir\ \Pi_2$ are not associated with any point of $S - S^*$, we have $|S - S^*| \leq k-1$. Since $S - S^*$ contains at least one line (all of whose points are associated with distinct pseudo-directions), there is a transversal plane having at least one line outside S^*. Therefore, by Corollary 10.6, every transversal plane has at least one line outside S^*. It follows that the number n of pairs (Π,L), where Π is a transversal plane and L a line of $\Pi-\Pi^*$, is not less than the number ℓ^3 of transversal planes and is equal to ℓ times the number of lines in $S - S^*$, that is

$$\ell^3 \leq n \leq \ell(k-1)(k-2)/2 \qquad (3)$$

On the other hand, the degree $\ell+1$ of a point in Π_1 cannot be less than the size k of some of the lines of Π_1, and so $\ell^2 \geq (k-1)^2$, contradicting (3).

We have proved that $S = S^*$. Therefore c = 0 and, using Lemma 10.6, we get

$$\ell = (k^2 - 1 \pm (k-1)\sqrt{k^2 - 2k - 3})/2$$

Obviously, k = 2 is excluded and, for k > 3

$$(k-2)^2 < k^2 - 2k - 3 < (k-1)^2$$

shows that $k^2 - 2k - 3$ is not a perfect square. Therefore k = 3 and $\ell = 4$.

Thus every transversal plane is an affine plane of order 3 and every non-transversal plane consists of 3 pairwise disjoint lines of size 4, all the other lines having size 3. If we replace each line of size 4 by 6 lines of size 2, we get a linear space F of 36 points consisting of lines of sizes 2 and 3 and in which the smallest linear subspaces containing three non-collinear points are degenerate projective planes of 3 points, punctured projective planes of order 2 or affine planes of order 3, and so F is a Fischer space of 36 points. Let $dir\ \Pi_1 = \{\Pi_1, \Pi_1', \Pi_1''\}$ and let F_6 be a linear subspace of 6 points of F contained in the plane Π_1 of S. If $x \in \Pi_1'$, x is joined to every point of F_6 by a line of size 3, and the smallest linear subspace of F containing x and F_6 has obviously at least 6 points in each of the planes Π_1, Π_1' and Π_1''. Buekenhout [10] has proved that a Fischer space having at least 18 points and generated by a plane α isomorphic to F_6 and by a point joined by a line of size 3 to at least one point of α is necessarily either F_{18} or F_{36}. Since x is joined to every point of F_6 by a line of size 3, a situation which does not occur in F_{18}, F is isomorphic to F_{36}. There is a unique way to construct 9 mutually disjoint lines of size 4 from the lines of size 2 of F_{36} and to provide this new linear space with planes isomorphic to those of S. The planar space S_{36} constructed in this way from F_{36} has the required properties.

3.3. Suppose that S contains only one direction Δ of planes

The planes of Δ will be called *Δ-planes*, the lines contained in a Δ-plane will be called *Δ-lines*, and the lines intersecting all Δ-planes will be called *transversal lines* (by condition (III), a line intersecting a Δ-plane must intersect all Δ-planes).

Lemma 10.8. Every Δ-line L *contained in a Δ-plane* Π *belongs to at least one partition of* Π *into lines which are coplanar with the same line of a Δ-plane* $\Pi' \neq \Pi$.

Proof. Let $\Pi' \neq \Pi$ be a Δ-plane and let L' be a line of Π' coplanar with L. The set of intersections of Π with the planes passing through L' (and distinct from Π') is clearly a partition of Π into lines, and L belongs to this partition.

Since we have assumed that S is not the union of two disjoint planes, all Δ-planes are isomorphic by Corollary 10.2.1. Let v' denote the number of points of any Δ-plane.

Lemma 10.9. If $S = S^*$, *then*

(i) *all transversal lines have size* $\ell = |\Delta| \geq 3$

(ii) *any two coplanar Δ-lines contained in two distinct Δ-planes have the same size*

(iii) *the number* p_L *of planes containing a Δ-line* L *is* $1 + v'/|L|$.

Proof. $S = S^*$ is partitioned by the Δ-planes. Moreover, $|\Delta| \geq 3$ because S is not the union of two disjoint planes. This proves (i).

(ii) is a consequence of (i) and of Lemma 10.2.

Let Π be a Δ-plane not containing L. The planes not belonging to Δ and containing L intersect Π in lines of size $|L|$ by (ii). This proves (iii).

Lemma 10.10. If $S = S^*$, *then any two disjoint Δ-lines contained in the same Δ-plane* Π *have the same size.*

Proof. Suppose on the contrary that Π contains two disjoint lines A of size a and B of size b with a > b. Let α be a plane intersecting Π in A. Any line C of α distinct from A and coplanar with B is disjoint from A, hence C is a Δ-line and, by Lemma 10.9, C has size a (because C is coplanar with A) and C has size b (because C is coplanar with B), a contradiction. Therefore A is the only line of α coplanar with B, and so any plane containing B and distinct from Π intersects α in exactly one point. This, together with Lemma 10.9, implies that

$$p_B - 1 = v'/b = a(\ell-1) \tag{4}$$

By Lemmas 10.8 and 10.9, Π contains at least one line B' disjoint from B and having size b. Let $\beta \neq \Pi$ be a plane containing B and let n be the number of lines of β which are distinct from B and coplanar with B' (such a line being necessarily a Δ-line, $0 \leqslant n \leqslant \ell-1$). Moreover, since a plane containing B' must intersect β in a line or in a point, we have

$$p_{B'} - 1 = v'/b = n + b(\ell-1-n) \tag{5}$$

(4) and (5) yield

$$a(\ell-1) + n(b-1) = b(\ell-1) \quad \text{where } n \geqslant 0 ,$$

contradicting the assumption a > b.

Proposition 10.5. If $S = S^*$ *contains two* Δ*-lines of different sizes, then* $S = S_{18}$.

Proof. Let a > b be two sizes of Δ-lines and let Π be a Δ-plane. Since all Δ-planes are isomorphic, Π contains a line of size a and a line of size b. By Lemmas 10.8 and 10.9, there is a partition of Π into lines of size a and a partition of Π into lines of size b. Since a > b, any line L of Π which does not belong to any of these two partitions is necessarily disjoint from at least one line of size b in the second partition, and so L has size b by Lemma 10.10. Moreover, by Lemma 10.10 again, in the plane Π, every line of size b must intersect every line of size a. Therefore Π contains exactly v' = ab points, and every point of Π is on exactly one Δ-line of size a, on exactly a Δ-lines of size b and has degree r' = a+1 in Π. It follows that the Δ-lines of size a partition S and are pairwise coplanar.

Let A be a line of size a in Π. By Lemma 10.9, the number p_A of planes containing A is given by

$$p_A = 1 + v'/a = 1+b$$

Let L be a transversal line disjoint from A. Each of the ℓ points of L is on a unique Δ-line of size a and the union of these ℓ lines is a plane λ. Since Δ is the only direction of planes in S, every plane containing A intersects the plane λ and this intersection is necessarily a Δ-line of size a. Thus every plane containing A intersects the line L, and so

$$p_A = 1 + v'/a = 1+b = \ell \qquad (6)$$

Let x be a point of Π and let α (resp. β) be a plane containing x and intersecting Π in a line A (resp. B) of size a (resp. b). We shall count the number $n(x,\alpha)$ (resp. $n(x,\beta)$) of planes intersecting α (resp. β) in the point x only. Let $\Pi' \neq \Pi$ be a Δ-plane and let $A' = \Pi' \cap \alpha$, $B' = \Pi' \cap \beta$. Since any plane intersecting α in the point x only intersects Π' in a line disjoint from A' and since all lines of Π' which are disjoint from A' have size a and are coplanar with A, we have

$$n(x,\alpha) = 0 \qquad (7)$$

On the other hand, the number of planes (distinct from Π) containing B is $v'/b = a$, the number of planes intersecting β in a transversal line passing through x is $|B'|(r'-1) = ba$, and the total number planes (distinct from Π) passing through x is equal to the number a^2+b of lines in Π'. Therefore

$$n(x,\beta) = a^2 + b - ba - a \qquad (8)$$

Since any plane of S belongs to Δ or is α (resp. β) or intersects α (resp. β) in a Δ-line or intersects α (resp. β) in a transversal line or intersects α (resp. β) in a single point, the total number p of planes of S is given respectively by

$$\begin{aligned} p &= |\Delta| + 1 + |\Delta|(v'/a-1) + a^2(r'-1) + |\alpha| n(x,\alpha) \\ &= |\Delta| + 1 + |\Delta|(v'/b-1) + b^2(r'-1) + |\beta| n(x,\beta) \end{aligned}$$

from which it follows, by (7) and (8), that

$$\ell(b-1) + a^3 = \ell(a-1) + b^2a + b\ell(a^2 + b - ab - a)$$

Using (6), we get, after simplification by $a-b \neq 0$ and $a-1 \neq 0$,

$$b^2 = a+1 \qquad (9)$$

Let B" be a line of Π' disjoint from B'. The number of planes containing B" is

$$1 + v'/b = 1+a = m + b(\ell-m) \qquad (10)$$

where $1 \leqslant m \leqslant \ell$ denotes the number of lines of β which are coplanar with B"
By (6), (9) and (10), we get

$$m(b-1) = b$$

and so $m = b = 2$, $a = \ell = 3$ and $|S| = 18$.
Therefore every transversal line has size 3, every Δ-plane is the union of two disjoint lines of size 3 and the planes not belonging to Δ are punctured projective planes of order 2 or affine planes of order 3 according as their Δ-lines have size 2 or 3. This implies that the linear space S is a Fischer space of 18 points. Moreover, it is easy to check that the smallest linear subspace of S containing a punctured projective plane π of order 2 and a point $x \notin \pi$ joined to a point of π by a line of size 3 is S itself. Buekenhout [10] has proved that a Fischer space of 18 points having this property is necessarily isomorphic to F_{18}. Moreover, there is a unique way to provide F_{18} with planes isomorphic to those of S. The planar space S_{18} constructed in this way from F_{18} has the required properties.

Proposition 10.6. If $S = S^*$ *and if all* Δ*-lines have the same size* a, *then* S *is obtained from* PG(3,a) *by deleting a line.*

Proof. Any Δ-plane π is a Steiner system S(2,a,v'). Thus, if we denote by b' the number of lines of π and by r' the degree of any point in π, we have

$$v' = r'(a-1) + 1 \qquad (11)$$

and

$$b' = v'r'/a \qquad (12)$$

Let α be a plane not belonging to Δ and let $x \in \alpha$. Counting in the same way as in Lemma 10.14 the number $n(x,\alpha)$ of planes of S intersecting α in the point x only, we have

$$n(x,\alpha) = b' - a(r'-1) - v'/a \qquad (13)$$

and so the total number p of planes of S is

$$p = \ell + 1 + \ell(v'/a - 1) + a^2(r'-1) + a\ell n(x,\alpha) \qquad (14)$$

On the other hand, every plane not belonging to Δ intersects π in a line and every line of π is contained in exactly v'/a planes not belonging to Δ, so that

$$p = \ell + b'v'/a \qquad (15)$$

Let A' be a line of Π disjoint from $A = \Pi \cap \alpha$. The number of planes containing A' is given by

$$1 + v'/a = n + a(\ell-n) \qquad (16)$$

where $1 \leqslant n \leqslant \ell$ is the number of lines of α which are coplanar with A'.
Using (11), (16) becomes

$$r' = (\ell a^2 - a - 1)/(a-1) - na \qquad (17)$$

which implies $a-1 | \ell-2 > 0$
and so $a-1 \leqslant \ell-2$.
On the other hand, the degree $a+1$ of a point in α cannot be less than the size ℓ of a transversal line, and so

$$a+1 \geqslant \ell$$

These two inequalities imply that

$$\ell = a+1 \qquad (18)$$

and (17) becomes

$$r' = a^2 - (n-2)a+1 \qquad (19)$$

From (13), (14), (15), (11), (12) and (18), we deduce, after some straightforward computation,

$$(r'-1)(r'-a-1)(r' - a^2 - a) = 0$$

and so
$r' = 1$, $a+1$ or a^2+a
$r' = 1$ is clearly impossible and $r' = a^2+a$ contradicts (19). Therefore $r' = a+1$ and the Δ-planes are affine planes of order a. The planes not belonging to Δ have exactly $\ell a = a(a+1)$ points and are punctured projective planes of order a. It is now a simple matter to deduce that S is obtained from $PG(3,a)$ by deleting one line.

Proposition 10.7. If $S \neq S^*$, *then* $S - S^*$ *contains at least four non-coplanar points.*

Proof. Suppose on the contrary that $S - S^*$ is contained in a plane α.

Consider first the case where $|\Delta| = 2$. Let Π and Π' be the two Δ-planes. Since any plane which is not in Δ intersects both Π and Π' in a line, $A = \Pi \cap \alpha$ and $A' = \Pi' \cap \alpha$ are two Δ-lines. The planes (distinct from Π) containing A determine a partition of Π' into lines. If this partition contains two lines A'' and A''' distinct from A', then $A \cup A''$ and $A \cup A'''$ are two planes of S because every point of $S - S^*$ is in the plane α. Let $x \in \Pi$-A. The plane $\langle x,A'''\rangle$ must intersect the plane $A \cup A''$, but this is impossible since A and A''' are coplanar. Therefore $\Pi' = A' \cup B'$, where B' is a line of Π' disjoint from A' and $A \cup B'$ is a plane of S. Similarly $\Pi = A \cup B$, where B is a line of Π disjoint from A and $A' \cup B$ is a plane of S. The planes $A \cup B'$ and $A' \cup B$ are disjoint, contradicting the assumption that Π and Π' are the only disjoint planes in S.

Suppose now that $|\Delta| = \ell \geq 3$. Let x_1, x_2 be any two points of a Δ-plane Π. If Π' and Π'' are two Δ-planes distinct from Π and if x_1'' is a point of Π'', the lines passing through x_1'' and intersecting Π determine, by Corollary 10.2.1, an isomorphism φ_1 from Π onto Π'. Let $x_1' = \varphi_1(x_1)$ and let x_2'' be the point of intersection of the line $\langle x_2,x_1'\rangle$ with the plane Π''. The lines passing through x_2'' and intersecting Π' determine an isomorphism φ_2 from Π' onto Π. Since $\varphi_2 \circ \varphi_1$ is an automorphism of Π mapping x_1 on x_2, all points of Π have the same degree r'.

Let $A = \Pi \cap \alpha$ and let β be a plane containing A, distinct from Π and α. Lemma 10.2 implies that any two coplanar Δ-lines contained in two distinct Δ-planes have the same size. Therefore, for any point $x \in \alpha^* = \alpha \cap S^*$ and for any point $y \in \beta$,

$$n(x,\alpha) = b' - a(r'-1) - v'/a = n(y,\beta) \qquad (20)$$

where $a = |A|$ and v' (resp. b') is the number of points (resp. of lines) in a Δ-plane. Counting in two ways the number p of planes in S, we get

$$\begin{aligned} p &= \ell+1 + \ell(v'/a-1) + a^2(r'-1) + |\beta|n(y,\beta) \\ &= \ell+1 + \ell(v'/a-1) + a^2(r'-1) + |\alpha^*|n(x,\alpha) + \sum_{z\in\alpha-\alpha^*} n(z,\alpha) + \sum_{L\subset\alpha-\alpha^*} (p_L-1) \end{aligned}$$

where p_L denotes the number of planes containing the line L. Using (20) and the fact that $|\alpha^*| = |\beta| = \ell a$, this implies

$$\sum_{z\subset\alpha-\alpha^*} n(z,\alpha) + \sum_{L\subset\alpha-\alpha^*} (p_L-1) = 0$$

Since $n(z,\alpha) \geq 0$ and $p_L - 1 \geq 1$ for every line $L \subset \alpha-\alpha^*$, we conclude that $n(z,\alpha) = 0$ for every point $z \in \alpha-\alpha^*$ and that there is no line contained in $\alpha-\alpha^*$. On the other hand, by Lemma 10.8, there is a line B of Π disjoint from A. If $z \in \alpha-\alpha^*$, the plane $\langle B,z\rangle$ is disjoint from A, thus also from α^*. Therefore, either $\langle B,z\rangle$ intersects α in the point z only and $n(z,\alpha) \neq 0$, or $\langle B,z\rangle$ intersects α in a line contained in $\alpha-\alpha^*$. In both cases, we have a contradiction.