CHAPTER II. FINITE LINEAR SPACES WITH METRICAL REGULARITIES IN THEIR INCIDENCE GRAPHS.

1. INTRODUCTION.

In a linear space, the classical axiom of Pasch may be reformulated as follows

(*) *for any two disjoint lines* L *and* L', *any point outside* $L \cup L'$ *is on at most one line intersecting both* L *and* L'.

Indeed, suppose that condition (*) is satisfied. Let A and A' be two distinct lines intersecting in a point p and denote by L and L' two lines intersecting A and A' such that neither L nor L' passes through p. If L and L' were disjoint, we would have at least two lines through p intersecting L and L' : a contradiction. Thus L and L' have a point in common and Pasch's axiom is satisfied. The converse is obvious.

It follows that condition (*) characterizes the generalized projective spaces. If "at most one" is replaced by "exactly one" in (*), we get a characterization of the generalized projective spaces of dimension ≤ 3.

Note that the finite affine planes of order n have a similar property : for any two disjoint (hence parallel) lines L and L', any point outside $L \cup L'$ is on exactly n lines intersecting both L and L'.

These examples suggest the problem of classifying the linear spaces which satisfy the following condition :

(D2) *there is a non-negative integer* d_2 *such that for any two disjoint lines* L, L' *and any point* x *outside* $L \cup L'$, *there are exactly* d_2 *lines through* x *intersecting the two lines* L *and* L'.

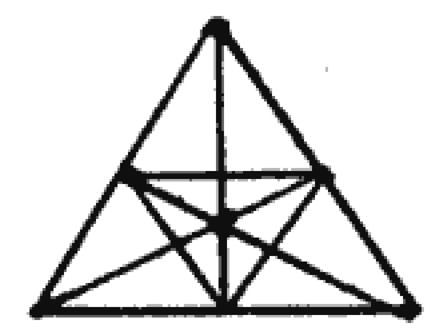
In the finite case, the answer is given by the following result :

Theorem 2. (A. Beutelspacher and A. Delandtsheer [1])
If S *is a finite linear space satisfying condition* (D2), *then one of the following occurs :*

(i) S *is a generalized projective space, and if the dimension of* S *is at least 4, then any line of* S *has exactly two points,*

(ii) S *is an affine plane, an affine plane with one point at infinity, or a punctured projective plane,*

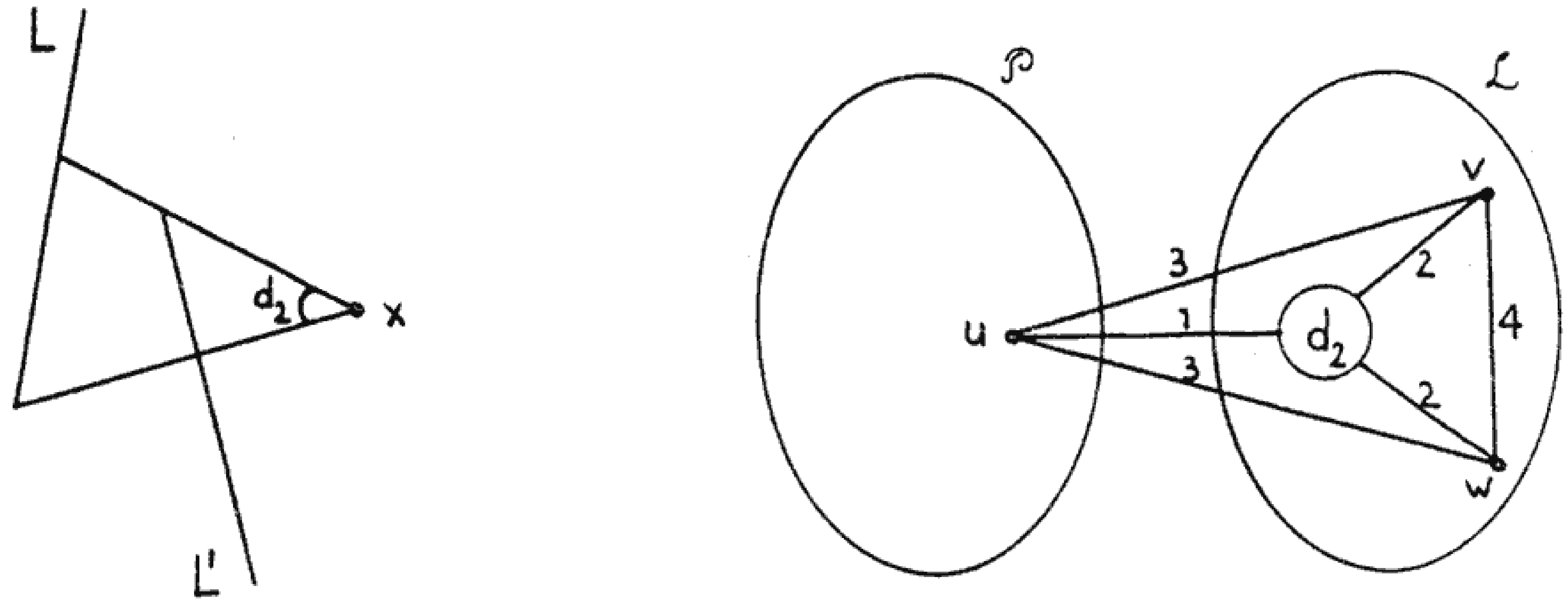
(iii) S *is the Fano quasi-plane, obtained from* PG(2,2) *by "breaking" one of its lines into three lines of size 2.*



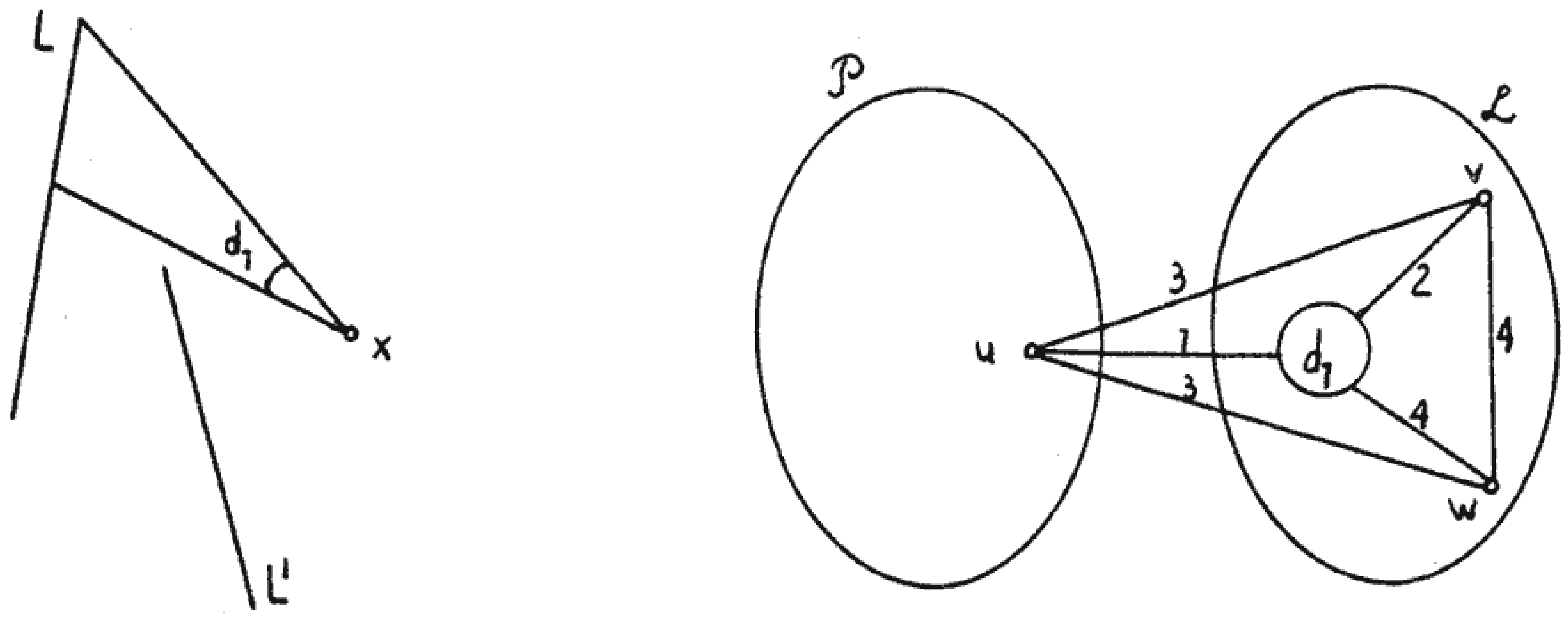
Conversely, each of these finite spaces satisfies (D2)

Note that condition (D2) can be viewed as a metrical condition on the incidence graph $\mathcal{G}$ of S. Indeed, remember that in the incidence graph of a linear space, any two points are at distance 2, two lines are at distance 2 or 4 according as they intersect or not, and a point and a line are at distance 1 or 3 according as they are incident or not. Therefore, condition (D2) may be translated in the following way :
there is a non-negative integer d_2 such that if u,v,w are any three vertices of $\mathcal{G}$ with distances d(u,v) = d(u,w) = 3 and d(v,w) = 4, then $\mathcal{G}$ contains exactly d_2 vertices t such that d(u,t) = 1, d(v,t) = d(w,t) = 2.
This leads naturally to the more general question : what happens if we choose other values for the distances in this condition ?
Among other things, we shall investigate the finite linear spaces satisfying one of the three conditions (D2), (D1), (D0), which are pictured below, first from a naive point of view, then in terms of the incidence graph.

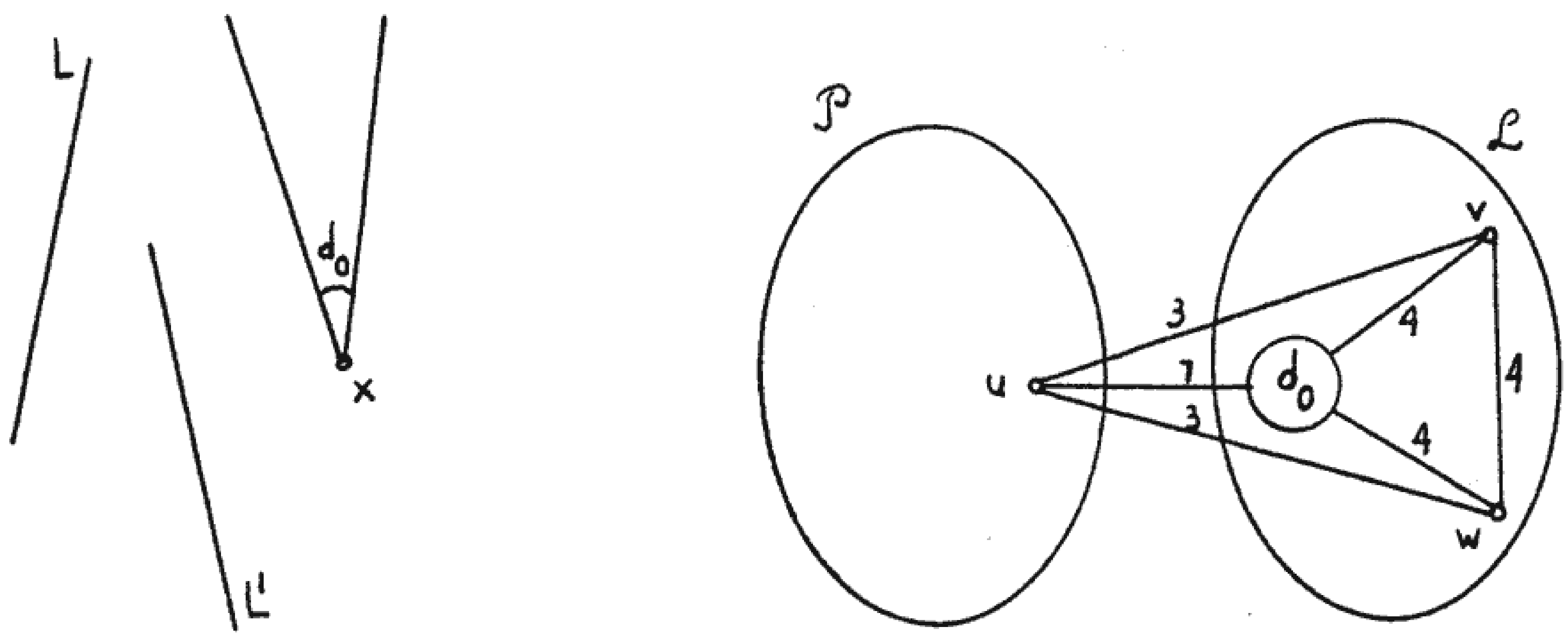
(D2)



(D1)



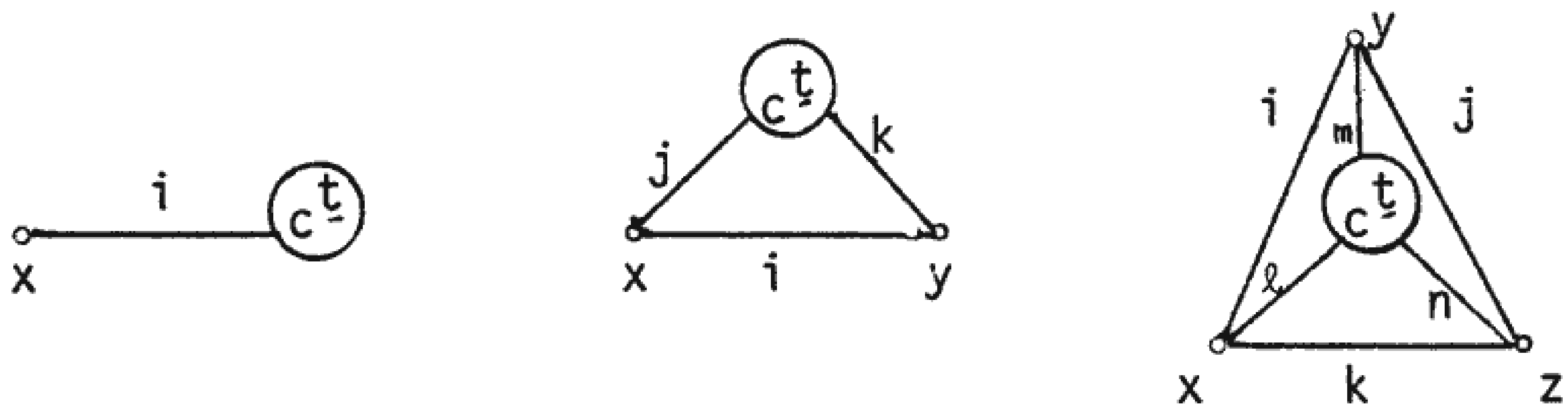
(D0)



The reason for the notations d_0, d_1 and d_2 is clear : the letter d reminds that the two lines L and L' are disjoint and the subscript reminds that we count certain lines intersecting 0, 1 or 2 of the lines L and L'.

2. GRAPH THEORETICAL BACKGROUND.

Actually, the above conditions form a part of the definition of a 3-metrically regular graph. Indeed, a connected graph is called *3-metrically regular* if G is metrically regular and if for any triple (x,y,z) of vertices such that d(x,y) = i, d(y,z) = j, d(z,x) = k, the number of vertices which are at distance ℓ from x, at distance m from y and at distance n from z depends only on the distances i,j,k,ℓ,m,n but not on the choice of triple (x,y,z)



These graphs have been studied quite a lot during the past few years. For instance, Cameron, Goethals and Seidel [16] have proved that if G is a connected 3-metrically regular graph of diameter 2, whose complement $\bar{G}$ is also connected, then G is the pentagon, or G is of pseudo or negative Latin square type, or G or $\bar{G}$ is a Smith graph (for more details, see [21]). On the other hand, Meredith [45] has proved that if G is connected 3-metrically regular graph of girth > 4, then G is a cycle (actually, the hypothesis of Meredith is stronger : he assumes that for any two isometric triples of vertices, there is an automorphism of G mapping the first onto the second; but his proof is essentially combinatorial).

Such metrical conditions are satisfied by point-, line- or incidence graphs of some classical geometries and have been used in certain characterization problems.

For instance, it follows immediately from theorems of Bose [5], Thas and Payne [48] that the point-graph of a generalized quadrangle of order (s,t) with s > 1 and t > 1 is 3-metrically regular if and only if $t = s^2$ if and only if every triad has exactly s+1 centers (i.e. for any triple (x,y,z) of pairwise non-collinear points, there are exactly s+1 points which are collinear with x,y and z). The point-graphs of generalized quadrangles with s = 1 or t = 1 are obviously 3-metrically regular.

Metrical conditions have also been used to characterize some classical generalized hexagons. Let us mention two examples. Thas [54] has proved that if S is a finite generalized hexagon of order (s,t) with $2 \leq t \leq s$, whose point-graph satisfies the following condition :

for any triple of vertices (x,y,z) with d(x,y) = d(y,z) = 3, d(z,x) = 2 (resp. d(z,x) = 3), there is at least one vertex v such that d(y,v) = 1, d(x,v) = d(z,v) = 2,

then t = s, s is a prime power, and S is isomorphic to the classical generalized hexagon H(s) associated with $G_2(s)$. Ronan [50] has characterized, among the finite generalized hexagons satisfying the regulus condition, those which are associated with $G_2(q)$, ${}^3D_4(q)$ and their duals, by means of the number n which counts, given any four vertices x,y,z,u with d(x,y) = d(y,z) = 6, d(x,z) = d(u,y) = 4, d(u,x) = d(u,z) = 2, the number (if it is distinct from t+1) of vertices v such that d(x,v) = d(z,v) = 4, d(y,v) = 2.

3. TERMINOLOGY AND NOTATIONS FOR LINEAR SPACES.

Let S be a finite linear space and $\mathcal{G}$ its incidence graph. We shall say that a triple (u,v,w) of vertices of $\mathcal{G}$ is of *type* ($\mathcal{S}$,i,j,k) (where $\mathcal{S}$ denotes either the point-set $\mathcal{P}$ or the line-set $\mathcal{L}$) if $u \in \mathcal{S}$, d(u,v) = i, d(v,w) = j and d(w,u) = k. For a given type ($\mathcal{S}$,i,j,k) and a given triple of positive integers (ℓ,m,n), the *problem* ($\mathcal{S}$,i,j,k;ℓ,m,n) consists in classifying the finite non-trivial linear spaces which satisfy the following condition : there is a constant c such that for any triple (u,v,w) of vertices of type ($\mathcal{S}$,i,j,k) in $\mathcal{G}$, the number of vertices t which are at distance ℓ from u, m from v and n from w is exactly c.

Obviously, certain choices of $\mathcal{S}$,i,j,k,ℓ,m,n are absurd. An easy but rather tedious enumeration leads to 102 problems ($\mathcal{S}$,i,j,k;ℓ,m,n) which have a sense (i.e. such that there exists a linear space whose incidence graph contains at least one 4-tuple (u,v,w,t) of the desired type). In the following sections, we shall investigate the most interesting of these problems namely ($\mathcal{P}$,3,4,3; 1,2,2), ($\mathcal{P}$,3,4,3;1,2,4), ($\mathcal{P}$,3,4,3;1,4,4), ($\mathcal{P}$,1,2,1;3,4,4), ($\mathcal{P}$,3,2,3;1,4,4), ($\mathcal{P}$,3,2,3;1,4,2) and also a problem which is trivially equivalent to ($\mathcal{P}$,3,2,3; 1,2,2).

Most of the remaining problems are easily solved and the answers are often the Steiner systems S(2,k,v), the projective planes or some "very small" linear spaces. However a few problems are still unsolved. For example, we have no other characterization of the finite linear spaces satisfying condition ($\mathcal{P}$,3,4,1;3,4,4) than saying that they are the finite linear spaces in which for any two disjoint lines L and L', the number of lines disjoint from L ∪ L' is a constant independent from L and L'. Note also that some of these problems may seem rather artificial : this is due to the fact that the distances between three vertices of $\mathcal{G}$ are not always sufficient to describe completely the corresponding geometrical configuration in S (for instance, three lines which are pairwise at distance 2 in $\mathcal{G}$ may be concurrent or form a triangle in S : this explains why the only solution of problem ($\mathcal{L}$,2,2,2;1,1,1) is the most trivial of all non-trivial linear spaces, namely the triangle S(2,2,3)).

Before starting the proofs of the main results, we briefly define some notations used in this chapter :

S will always denote a finite linear space of v points, with point-set $\mathcal{P}$ and line-set $\mathcal{L}$.

K is the set of line sizes in S

r_x is the degree of the point $x \in S$ (also denoted by r if all points of S have the same degree)

$r_L = \sum_{x \in L} r_x$ is the sum of the degrees of the points of L.
Usually, the size of a line A,B,C, ... will be denoted by a,b,c, ... respectively. A line G of S will be called *projective* if G intersects all the other lines of S. The total number of projective lines in S will be denoted by π and the number of projective lines containing a point x by ρ_x (or ρ if this number is independent of x).

A *bisecant* of two lines L and L' will be a line distinct from L and L' and intersecting $L \cup L'$ in exactly two points. A *trisecant* of three lines L, L', L" will be a line distinct from L, L', L" and intersecting $L \cup L' \cup L''$ in exactly three points.

For any triple (x,L,L') where L and L' are two disjoint lines and x is a point outside $L \cup L'$, we denote by $d_2(x,L,L')$ the number of lines through x which intersect both L and L', by $d_1(x,L,L')$ the number of lines through x which intersect L but not L' and by $d_0(x,L,L')$ the number of lines through x which are disjoint from L and L'. Then, the conditions (D2), (D1) and (D0) express that $d_2(x,L,L')$, $d_1(x,L,L')$ and $d_0(x,L,L')$ respectively are independent of the triple (x,L,L').

For any triple (x,L,L') where L and L' are two intersecting lines and x is a point outside $L \cup L'$, we denote by $i_2(x,L,L')$ the number of bisecants of L and L' through x, by $i_1(x,L,L')$ the number of lines through x which intersect L but not L' and by $i_0(x,L,L')$ the number of lines through x which are disjoint from L and L'. The conditions (I2), (I1) and (I0) express that $i_2(x,L,L')$, $i_1(x,L,L')$ and $i_0(x,L,L')$ respectively are independent of the triple (x,L,L').

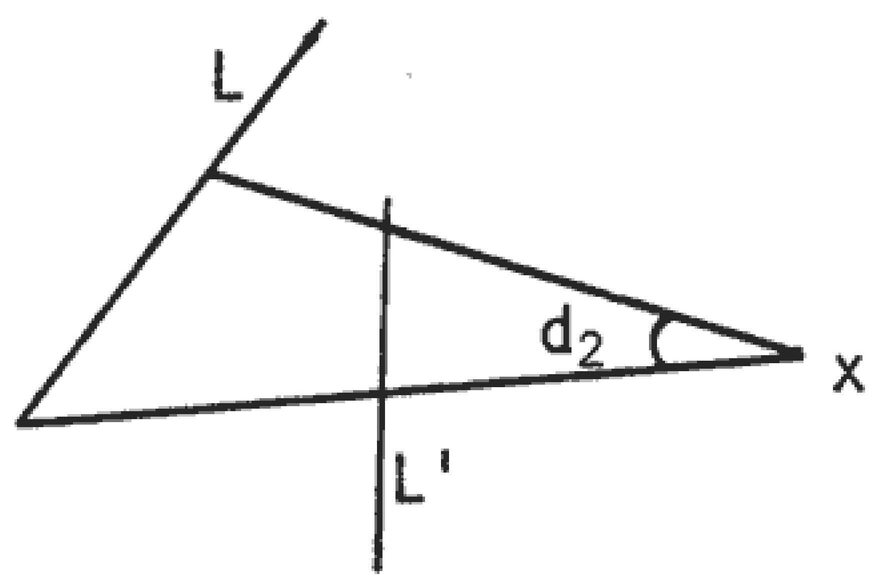
Note that in a Steiner system S(2,k,v), the three conditions (I0), (I1) and (I2) are equivalent. Indeed, for any two intersecting lines L and L', the degree of any point x outside L and L' is

$$\begin{aligned} r &= k + i_1(x,L,L') + i_0(x,L,L') \\ &= 2k - i_2(x,L,L') + i_0(x,L,L'), \end{aligned}$$

so that the constancy of one of the i_j's implies the constancy of the other two. This remark will be useful in the study of conditions (I0) and (I1). A similar argument shows that the conditions (D0), (D1) and (D2) are equivalent in a Steiner system S(2,k,v).

4. LINEAR SPACES SATISFYING CONDITION (D2)

We now prove Theorem 2 stated on pages II 1-2. Throughout this proof, S denotes a finite linear space satisfying condition (D2).



The proof is divided into two main parts : we first investigate the case where some additional regularity conditions are satisfied, then we handle the case in which S contains a projective line.

4.1. Some Additional Regularity Conditions.

Proposition 2.1. If $d_2 \leqslant 1$, *then* S *is a generalized projective space of dimension* d. *Moreover if* $d \geqslant 4$, $S = PG(d,1)$.

Proof. If $d_2 \leqslant 1$, condition (∗) (hence also Pasch's axiom) is satisfied, so that S is a generalized projective space of dimension d.

Suppose that $d \geqslant 4$ and that there is a line L containing at least three distinct points p_1, q_1, q_2. There exist two disjoint lines L_1 and L_2 through q_1 and q_2, respectively. This implies $d_2 = 1$. On the other hand, since $d \geqslant 4$, there is a line L' disjoint from L and there is a point p outside the 3-dimensional subspace generated by L and L'. Clearly, there is no line through p intersecting L and L'. Hence $d_2 = 0$, a contradiction.

Thanks to Proposition 2.1, we may now assume that $d_2 \geqslant 2$, and also that there are two disjoint lines in S and that for any two disjoint lines there is a point outside their union (otherwise S would be a generalized projective space of dimension $\leqslant 3$).

First we consider the situation in which all lines of S have the same size.

Proposition 2.2. If all lines of S *have size* n, *then* S *is an affine plane of order* n.

Proof. Denote by L and L' two disjoint lines of S. Counting in two ways the number of flags (p,A) with $p \notin L \cup L'$ and $L \cap A \neq \emptyset \neq L' \cap A$, we get

$$(v - 2n)d_2 = n^2(n-2) ,$$

that is

$$(v-n)d_2 = n(n^2 - 2n + d_2) \qquad (1)$$

On the other hand, all points of S have the same degree r, with

$$v-1 = r(n-1) ,$$

or

$$v-n = (r-1)(n-1) . \qquad (2)$$

Equations (1) and (2) together imply that n-1 is a divisor of $n(n^2 - 2n + d_2)$, and so n-1 divides d_2-1. Using $2 \leq d_2 \leq n$, we conclude that $d_2 = n$. Therefore $v = n^2$ and S is an affine plane of order n.

The following Lemma is crucial for our purpose :

Lemma 2.1. Any two lines L_1 *and* L_2 *disjoint from a given line* L *have the same size.*

Proof. We count in two ways the number of trisecants of L, L_1 and L_2. If L_1 and L_2 are disjoint, we get

$$|L_1|d_2 = |L_2|d_2 .$$

If L_1 and L_2 have a point in common, we get

$$(|L_1| - 1)d_2 = (|L_2| - 1)d_2 .$$

In both cases $|L_1| = |L_2|$.

Proposition 2.3. If S *contains two disjoint lines of different sizes, then* S *is the Fano quasi-plane.*

Proof. Let X and Y be two disjoint lines of different sizes x and y, respectively. We suppose $x < y$. Thanks to Lemma 2.1, any line of S intersects X or Y. Therefore, through any point p outside X and Y, there are d_2 bisecants of X and Y, $y-d_2$ lines of size y disjoint from X and $x-d_2$ lines of size x disjoint from Y. In particular, any point outside X and Y has degree $x+y-d_2$. Since $y > x \geq d_2$, we have $y > d_2$.

Step 1. If all lines disjoint from X have a point q in common, then S is the Fano quasi-plane.

Indeed, through any point not on X or Y there are exactly $y-d_2$ lines disjoint from X; since all these lines pass through $q \in Y$, it follows that $y = d_2+1$. This means conversely that any line through q which is incident with a point $p \neq q$ outside X, is disjoint from X. In other words, any line through q intersecting X is a line of size 2. But for any line X' of size 2, there exists a line Y' disjoint from X' and a point p' outside X' and Y'. Therefore $d_2 \leq 2$, hence $d_2 = 2$, $y = 3$, $x = 2$.

Since there are at least two lines through q disjoint from X, any point $p \neq q$ outside X has degree $x+y-d_2 = 3$. So, every line has at most three points. This implies $v \leq 7$. The assertion follows easily.

Now, let us assume that the lines disjoint from X have no point in common. We shall get a contradiction in three steps.

Step 2. There is a positive integer z such that any line intersecting X has size x or z with

$$z = 1 + (v-x)/y \qquad (3)$$

and

$$y = z + (d_2-1)/(y-d_2) \; . \qquad (4)$$

Indeed, since any line disjoint from X has size y, in our present situation, any point outside X has degree $x+y-d_2$. So, if we denote by d_X the number of lines disjoint from X, we have

$$d_X = (v-x)(y-d_2)/y \; .$$

Let Z be a line intersecting X. If Z is not a line of size x, then - in view of Lemma 2.1 - any line disjoint from X must intersect Z. Hence

$$(|Z| - 1)(y-d_2) = d_X, \text{ or } |Z| = 1 + (v-x)/y \; . \qquad (5)$$

Using Lemma 2.1 again, we see that any two lines disjoint from X intersect. Counting in two ways the number of flags (p,L), where $L \neq Y$ is a line disjoint from X and $p \in Y$, we get by (5) :

$$y(y - d_2 - 1) = d_X - 1 = (z-1)(y-d_2) - 1.$$

Step 3. $d_2 = x$.

Indeed, assume on the contrary $d_2 < x$. Then any point outside X is on at least one line disjoint from Y.

We claim that the lines disjoint from Y have no point in common. (Assume that the lines disjoint from Y intersect in a point q. Since any point outside X and Y is on exactly $x-d_2$ lines disjoint from Y, we have $x = d_2+1$. This implies that any line through q and a point of Y is a line of size 2. Also, any such line X' is disjoint from at least one line disjoint from X. Using Lemma 2.1, we get $2 = |X'| = |X| = x$, and so $d_2 = x-1 = 1$, a contradiction).

Like in step 2 we see that any line intersecting Y has size y or z' with

$$z' = 1 + (v-y)/x \qquad (6)$$

and

$$x = z' + (d_2-1)/(x-d_2) > z'.$$

The set K of line sizes of S is $\{x,y,z\} = \{x,y,z'\}$ with $z' < x < y$. Therefore $z = z'$, which yields, together with (3) and (6),

$$(v-x)/y = (v-y)/x ,$$

or

$$(y-x)(v-y-x) = 0 .$$

Since $x \neq y$, we have $v = x+y$, contradicting the fact that S contains a point belonging neither to X nor to Y.

Step 4. $d_2 \neq x$.

Indeed, assume on the contrary $d_2 = x$. This means that for any line L disjoint from X and any point p outside X and L, any line through p intersecting X has a point in common with L. In particular, any line Z intersecting X intersects any line disjoint from X as well. So, Z has size z by the argument of step 2. Therefore, the lines distinct from X have size z or y, according as they intersect X or not. Since any line $Z \neq X$ which intersects X has size z, we get

$$(v-x-y)x = (v-x-y)d_2 = x\,y\,(z-2) \qquad (7)$$

Moreover, for any point p on X,

$$(r_p - 1)(z-1) = v-x \qquad (8)$$

Equations (7) and (8) together imply $r_p = y+1$.

On the other hand, the degree of any point q not on X is $r_q = y$ since there is no line disjoint from both X and Y and since $d_2 = x$.

Next, we claim that there exist two disjoint lines intersecting X. (Assume that there were a line, say Z, which intersects X and all lines intersecting X. Counting the number of flags (p,L) with $p \notin X \cup Y$, $p \in Z$ and $L \cap X \neq \emptyset \neq L \cap Y$, we would get

$$(z-2)(d_2-1) = (x-1)(y-1) ,$$

that is $y = z-1 < z$, contradicting (4)).

Denote by Z and Z' two disjoint lines intersecting X. Lemma 2.1 states that any line L disjoint from $Z \cup Z'$ has size z. By (4), $z \neq y$, so L intersects X and therefore L meets Y. Counting in two ways the number of lines disjoint from $Z \cup Z'$, we get

$$(x-2)(y+1-2z+x) = (y-2)(y-2z+x)$$

hence

$$(y-x)(x+y-2z) = x-2 ,$$

therefore

$$y-x \mid x-2 .$$

But (4) implies that

$$y-x \mid x-1$$

so

$$y = x+1$$

and (4) yields $z = 2$.
Since any line intersecting X has size 2, no line through a point p outside $X \cup Y$ can intersect both X and Y, a contradiction.
By steps 1, 3 and 4, Proposition 2.3 is proved.

4.2. The case of projective lines.

In view of Propositions 2.2 and 2.3, we may suppose from now on that S contains lines of different sizes, and that any two disjoint lines have the same size.

Lemma 2.2. There is at least one projective line in S.

Proof. We assume that for any line L of S there is a line disjoint from L.

Let M be the maximal and m the minimal size of a line in S. Denote by X, X' (resp. Y, Y') two disjoint lines of size M (resp. m). Some obvious counting yields

$$M^2(m-2) \leq (v-2M)d_2 \qquad (9)$$

and

$$(v-2m)d_2 \leq m^2(M-2) .$$

Together

$$M^2(m-2) + 2\, d_2(M-n) \leq m^2(M-2) ,$$

or

$$Mm(M-m) + 2d_2(M-m) \leq 2(M^2-m^2) = 2(M+m)(M-m).$$

Dividing by $M-m > 0$ gives

$$Mm + 2\, d_2 \leq 2(M+m) ,$$

therefore

$$0 \leq (M-2)(m-2) = Mm - 2(M+m) + 4 \leq Mm - 2(M+m) + 2d_2 \leq 0.$$

Hence $m = d_2 = 2$, and all the above inequalities are in fact equalities. In particular, equality holds in (9), so $v = 2M$, a contradiction.

With the following proposition, Theorem 2 is proved.

Proposition 2.4. S is a punctured projective plane or an affine plane with one point at infinity.

Proof. By Lemma 2.2, there exists a projective line G of size g. Let L and L' denote two disjoint lines, necessarily of the same size ℓ. Counting in two ways the number of flags (p,X) with $p \notin L \cup L'$, $p \in G$, where $X \neq G$ is a line intersecting L and L', we get

$$(g-2)(d_2-1) = (\ell-1)^2 . \qquad (10)$$

This implies that all projective lines have the same size g and that all non-projective lines have the same size ℓ.

The proof of Proposition 2.4 will follow in a series of steps.

Step 1. There are at least two projective lines in S.
Indeed, assume that there is only one projective line G in S. Then any line through a point q outside G is a line of size ℓ and any line different from G through a point p on G is a line of size ℓ as well. Therefore

$$r_q(\ell-1) = v-1 = g-1+(r_p-1)(\ell-1).$$

Hence $\ell-1$ is a divisor of $g-1$, and so $\ell-1$ and $g-2$ are relatively prime. Now (10) implies that $(\ell-1)^2$ divides d_2-1, a contradiction.

Step 2. If all projective lines pass through a common point o, then S is an affine plane with one point o at infinity.
Indeed, since there is more than one projective line, any point $p \neq o$ has degree g. So, through any such point p there is the same number of projective lines; in particular, the set of projective lines is precisely the set of lines through o. Hence

$$v-1 = r_o(g-1) .$$

On the other hand, if p denotes a point different from o, we have

$$v-1 = g-1 + (r_p-1)(\ell-1) = g-1 + (g-1)(\ell-1) = (g-1)\ell$$

Together it follows that o has degree ℓ.

If L and L' are two disjoint lines, none of them passes through o. On the other hand, any of the ℓ lines through o is projective. Therefore $d_2 = \ell$. By (10), this implies $g = \ell+1$, hence $v = \ell^2+1$. Consider the incidence structure $S-\{o\}$, which consists of all points of S, except o. We have just seen that $S-\{o\}$ is a linear space with ℓ^2 points, in which any line has exactly ℓ points. Therefore, $S-\{o\}$ is an affine plane. Then, obviously, S itself is an affine plane with one point o at infinity.

Step 3. Suppose that for any point p of S there is a projective line not through p. Then S is a punctured projective plane.
Indeed, in the present situation, any point of S has degree g. Let us denote by π the total number of projective lines and by ρ the number of projective lines through a point. Clearly, the following equations hold :

$$\pi g = v \rho \qquad (11)$$
$$v-1 = \rho(g-1) + (g-\rho)(\ell-1) = g(\ell-1) + \rho(g-\ell) \qquad (12)$$
$$\pi-1 = g(\rho-1) \qquad (13)$$

Equations (11) and (13) imply

$$v \rho = (g(\rho-1)+1)g \qquad (14)$$

Using (12), we get

$$(\rho(g-\ell) - (g-1))(\rho-g) = 0$$

If $\rho=g$; S would be a projective plane, hence

$$\rho = (g-1)/(g-\ell) \qquad (15)$$

Next, we claim that $g = \ell+1$. In order to prove this, denote by q and n the unique non-negative integers with

$$g = q\ell+n \quad \text{and} \quad 0 \leqslant n < \ell .$$

From (15) we deduce that $g-\ell$ divides $\ell-1$. Therefore

$$q\ell + n - \ell \mid \ell-1 ,$$

in particular

$$q\ell + n-\ell \leqslant \ell-1 ,$$

which implies q=1. So, n divides $\ell-1$. Denote by t the positive integer such that $nt = \ell-1$.

From (10) we infer that $g-2 = \ell+n-2 = n(t+1)-1$ divides $(\ell-1)^2 = n^2t^2$. But

$$n(t+1)-1 \mid (n(t+1)-1)n(t-1) = n^2t^2 - n^2 - n(t-1) ,$$

therefore

$$n(t+1)-1 \mid n(n+t-1) ,$$

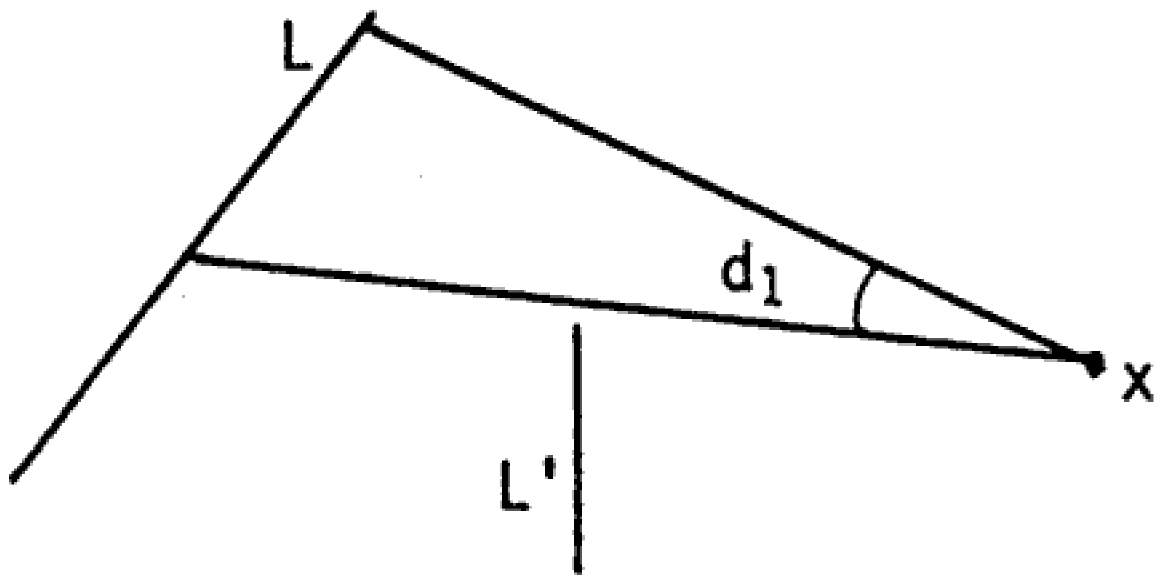
hence

$$n(t+1)-1 \mid n+t-1 ,$$

and so $n \leq 1$. Since $nt = \ell-1 \neq 0$, it follows that $n = 1$, i.e. $g = \ell+1$.

Now (10) implies $d_2 = \ell$, (15) yields $\rho=\ell$, and by (14) we have $v = \ell^2+\ell$. In particular, it follows that the lines of size ℓ form a "complete parallel class" of S. Introducing one new point which is incident precisely with the lines of size ℓ of S, it is easy to see that this new linear space is a projective plane of order ℓ. Thus S is a punctured projective plane.

5. LINEAR SPACES SATISFYING CONDITION (D1).



(D1) *there is a non-negative integer* d_1 *such that for any ordered pair of disjoint lines* L, L' *of* S *and any point* x *outside* L U L', *there are exactly* d_1 *lines through* x *intersecting* L *but not* L'.

The finite linear spaces satisfying (D1) are classified in [27]:

Theorem 3. If S *is a finite non-trivial linear space satisfying condition* (D1), *then one of the following occurs :*

(i) S *is an affine plane, an affine plane with one point at infinity, a punctured projective plane or a (possibly degenerate) projective plane,*

(ii) S *is a 3-dimensional projective space* PG(3,d_1) ,

(iii) S *is a 3-dimensional generalized projective space* P(3, k+ℓ),

(iv) S *is a degenerate projective space* PG(d,1), $d \geq 2$.

Conversely, each of these finite spaces satisfies (D1).

Comparing Theorems 2 and 3, we observe that condition (D1) is stronger than condition (D2).

Let S denote a finite linear space satisfying (D1). The proof of Theorem 3 uses the following lemmas :

Lemma 3.1. If $d_1 = 0$, *then* S *is a semi-affine plane.*

Proof. Let L be a line and x a point outside L. If x is on two lines L' and L" both disjoint from L, then any point $y \neq x$ on L" is on at least one line (namely L") intersecting L' but not L, contradicting $d_1 = 0$. Therefore, for any line L of S, any point outside L is on at most one line disjoint from L. In other words, S is a semi-affine plane, and so, since S is finite, we know by (16) that S is an affine plane, an affine plane with one point at infinity, a punctured projective plane or a (possibly degenerate) projective plane.

Lemma 3.2. If S *is the union of two of its lines and if* $d_1 \geq 1$, *then* S *is either a degenerate projective plane or a generalized projective space* P(3, k+ℓ).

The proof is very easy and will be omitted.

Thanks to these lemmas, we may assume from now on that
(A) $d_1 \geq 1$ *and for any two lines of* S *, there is at least one point outside their union.*

Lemma 3.3. Any two disjoint lines have the same size.

Proof. Let L and L' be two disjoint lines. The degree of every point $x \notin L \cup L'$ is

$$\begin{aligned} r_x &= |L| + d_1 + d_0(x,L',L) \\ &= |L'| + d_1 + d_0(x,L,L'), \end{aligned}$$

and so $\qquad |L| = |L'|$.

Lemma 3.4. If S *contains non-projective lines of distinct sizes, then for every point* x *of* S *and for every size* ℓ *of a non-projective line, there are two disjoint lines of size* ℓ *not containing* x. *Moreover,* $\ell \geq 3\, d_1$.

Since ℓ is the size of a non-projective line, we conclude from Lemma 3.3 that S contains at least two disjoint lines of size ℓ, and so there is a non-projective line L of size ℓ not containing x. Suppose that all lines disjoint from L pass through x. Since L is non-projective, there is at least one line L' disjoint from L. Thanks to the assumption (A), we know that there is at least one point $y \notin L \cup L'$ and one line L" disjoint from L passing through y.

L' and L" have size ℓ by Lemma 3.3 and have the point x in common since we have assumed that all lines disjoint from L intersect in x.
Let $h \neq \ell$ be the size of a non-projective line. We conclude again from Lemma 3.3 that S contains at least two disjoint lines H_1 and H_2 of size h. If $x \notin H_1 \cup H_2$, then it follows from the assumption $d_1 \geq 1$ that x is on at least one line H_3 disjoint from H_1, and by Lemma 3.3, H_3 has size h. Therefore there is a line H of size h passing through x. If h = 2, let H' be a bisecant of L and L' disjoint from H. By lemma 3.3, H' has size h = 2, and so H' is disjoint from L". Hence, by Lemma 3.3 again, H' has size ℓ, contradicting $\ell \neq h$. Therefore h > 2 and H contains a point $y \notin L \cup L'$. Since $d_1 \geq 1$, we conclude that y is on at least one line disjoint from L and not containing x, contradicting the assumption that all lines disjoint from L contain x.

Therefore there exist two disjoint lines L_1, L_2 of size ℓ not containing x and two disjoint lines H_1, H_2 of size h not containing x. The point x is on at least 2 d_1 lines disjoint from H_1 or H_2 and so, by Lemma 3.3, x is on at least 2 d_1 lines of size h. Moreover, x is on exactly $\ell - d_1$ bisecants of L_1 and L_2, so that, by Lemma 3.3 again, x is on at most $\ell - d_1$ lines of size h. Therefore $3 d_1 \leq \ell$.

Lemma 3.5. All non-projective lines have the same size.

Proof. Suppose that S contains non-projective lines of distinct sizes a and b, with a > b.

If S contains three pairwise disjoint lines A, A', A" of size a, and if B is a line of size b, then, by Lemma 3.3, every line of size a (in particular every line disjoint from A) intersects B. Therefore, counting in two ways the number of lines intersecting A' but disjoint from A and from $A' \cap B$, we get

$$(b-2)d_1 = \sum_{x \in A'-(A' \cap B)} (r_x - a - 1) ,$$

where

$$r_x = a + d_1 + d_0(x,A,A'') \geq a + d_1 + 1$$

and so

$$(b-2)d_1 \geq (a-1)d_1$$

and, since $d_1 > 0$,

$b \geq a+1$, contradicting the assumption $a > b$.

Therefore S does not contain three pairwise disjoint lines of size a, and so for any triple (x,A,A') where A,A' are two disjoint lines of size a and $x \notin A \cup A'$, $d_0(x,A,A') = 0$. By Lemma 3.4, we conclude that every point x of S has degree

$$r = r_x = a + d_1 \ . \qquad (1)$$

Let B and B' be two disjoint lines of size b and let A be a line of size a. We know by Lemma 3.3 that every line of size b (in particular every line disjoint from B) intersects A. Therefore, counting in two ways the number of lines intersecting B' but disjoint from B and from $B' \cap A$, we get

$$(a-2)d_1 = \sum_{x \in B'-(B' \cap A)} (r_x - b-1) \ ,$$

and so, by (1),

$$(a-2)d_1 = (b-1)(a + d_1 - b-1)$$

or equivalently

$$(a-b-1)(d_1 - b+1) = 0 \ .$$

Since by Lemma 3.4, $b \geq 3\, d_1$ and since $d_1 \geq 1$, we have $d_1 - b+1 \neq 0$. Therefore $a = b+1$. Let A' be a line disjoint from A. Counting in two ways the number of lines intersecting A' but disjoint from A and $A' \cap B$, we get

$$(b-2)d_1 = (a-1)(r-a-1) \ ,$$

and so, using (1) and $a = b+1$, we conclude that $b = 2\, d_1$, contradicting Lemma 3.4.

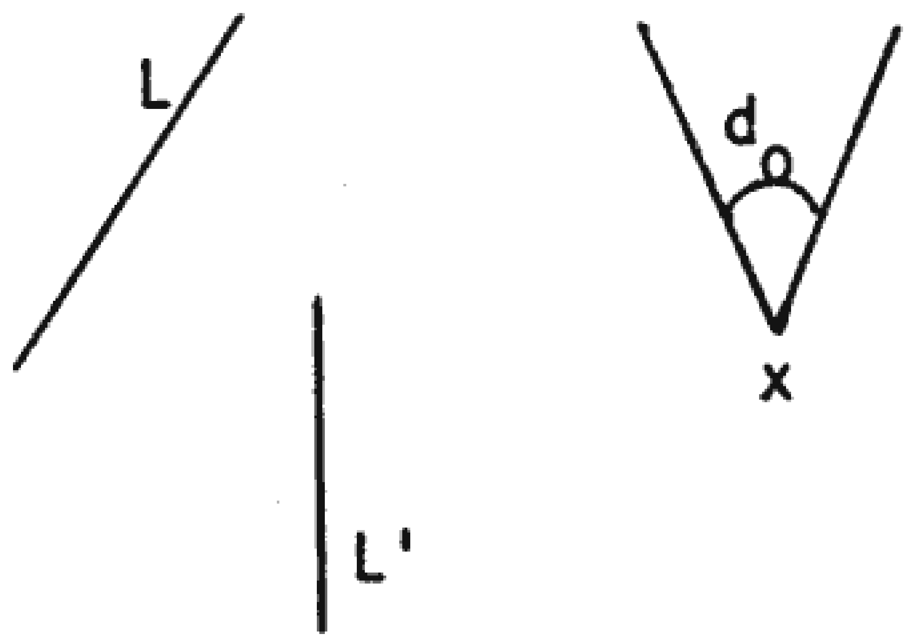
Proof of Theorem 3.

Let L, L' be two disjoint lines and let x be a point outside $L \cup L'$. Counting the lines containing x and intersecting L, we get

$$|L| = d_2(x,L,L') + d_1 \ ,$$

from which it follows, by Lemma 3.5, that $d_2(x,L,L')$ is independent of the triple (x,L,L'). In other words, condition (D2) is satisfied. Theorem 3 follows now easily from Lemmas 3.1, 3.2 and Theorem 2.

6. LINEAR SPACES SATISFYING CONDITION (D0).



(D0) *there is a non-negative integer* d_0 *such that for any two disjoint lines* L,L' *of* S *and any point* x *outside* L ∪ L', *there are exactly* d_0 *lines through* x *disjoint from* L ∪ L'.

The finite linear spaces satisfying (D0) with $d_0 > 0$ are classified in

> *Theorem 4* [24]. *If* S *is a finite non-trivial linear space satisfying condition* (D0) *with* $d_0 > 0$, *then one of the following occurs :*
>
> (i) S *is an affino-projective plane (but not an affine plane with one point at infinity),*
>
> (ii) S *is an affine plane of order* ≥ 3 *from which either one point or one line has been removed,*
>
> (iii) S *is a 3-dimensional projective space* PG(3,q),
>
> (iv) S *is a generalized projective space* P(2, k+1), P(3, k+k) *or* PG(d,1) *with* d ≥ 2
>
> *Conversely, each of these finite spaces satisfies* (D0) *with* $d_0 > 0$.

We assume here that the parameter d_0 is non-zero while in (D2) and (D1) we have also considered the case where the parameter was zero. We have no classification of the finite linear spaces which do not contain three pair-wise disjoint lines.

Proof of Theorem 4.

Let S be a finite non-trivial linear space satisfying condition (D0) with $d_0 > 0$. During the proof, we always assume that S is not the union of two of its lines, because otherwise, as it is easily seen, S would be a degenerate projective plane P(2, k+1) or a generalized projective space P(3, k+k) consisting of two lines of the same size k.

We divide the proof into three cases, according as $d_0 = 1$, or $d_0 > 1$ and S contains a projective line, or $d_0 > 1$ and S contains no projective line.

6.1. The case $d_0 = 1$.

Proposition 4.1. If $d_0 = 1$, *then one of the following occurs*

- S *is a generalized projective space of 6 points, either* P(3, 3+3) *or* PG(5,1)
- S *is an affino-projective plane (but not an affine plane with one point at infinity)*
- S *is an affine plane from which either one point or one line has been removed.*

Proof. If for any line L of S and any point $p \notin L$, p is on at most one line disjoint from L, then S is a finite semi-affine plane, and so, by (I6), S is a projective plane, a punctured projective plane, an affine plane or an affine plane with one point a infinity (the last case is easily ruled out).

Assume now that there are two intersecting lines L_1 and L_2, both disjoint from a line L of S. Since $d_0 = 1$, the line L_1 (resp. L_2) determines a partition Δ_1 (resp. Δ_2) of the points of S-L into lines. On the other hand, any two lines $L_1' \in \Delta_1$ and $L_2' \in \Delta_2$ must intersect, otherwise a point of L_2' would be on at least two lines disjoint from $L \cup L_1'$. Therefore all lines of Δ_1 (resp. Δ_2) have the same size $|L_1|$ (resp. $|L_2|$). Moreover any line $L' \notin \Delta_1 \cup \Delta_2$ distinct from L is disjoint from at most one line of Δ_i $(i = 1,2)$: indeed if on the contrary L' is disjoint from two lines L_i' and L_i'' of Δ_i, then any point of L' inside S-L is on at least two lines disjoint from $L_i' \cup L_i''$, contradicting $d_0 = 1$. Therefore $|L_1| = |L_2|$ or $|L_1| = |L_2| \pm 1$.

1°) Consider first the case $|L_1| = n$, $|L_2| = n+1$. In this case, S-L is a set of n(n+1) points partitioned by Δ_1 into n+1 lines of n points and by Δ_2 into n lines of n+1 points. Any line $L' \notin \Delta_1 \cup \Delta_2$ $(L' \neq L)$ intersects at least n lines of Δ_1 and at most n lines of Δ_2, therefore L' contains exactly n points of S-L and is disjoint from exactly one line $L_1' \in \Delta_1$. If L' is disjoint from L, then any point of L' is on at least two lines disjoint from $L \cup L_1'$, a contradiction. Therefore any line $L' \notin \Delta_1 \cup \Delta_2$ $(L' \neq L)$ intersects L and has size n+1.

Let p be a point of S-L. Counting in two different ways the number of pairs (q, L') where L' is the line $\langle p,q \rangle$, we get

$$n(n+1) - 1 + |L| = (n-1) + n + |L|n$$

and so $|L| = n$. Thus the lines of size n partition S and all other lines have size n+1. Completing the lines of size n with one new point x, we define a linear space $S \cup \{x\}$ of $(n+1)^2$ points in which all lines have size n+1, that is an affine plane of order n+1. Therefore S is an affine plane from which one point has been removed.

2°) Consider now the case $|L_1| = |L_2| = n$. Then S-L has n^2 points and the lines intersecting L have size n or n+1. Let p be a point of L. Counting in two ways the number of pairs (q,L') where $<p,q> = L' \neq L$, we get

$$n^2 = r_{p,n}\,(n-1) + r_{p,n+1}\,n \qquad (1)$$

where $r_{p,n}$ (resp. $r_{p,n+1}$) denotes the number of lines of size n (resp. n+1) intersecting L in p. It follows that n divides $r_{p,n}$ and $r_{p,n} \leq 2n$.

If there is a point $p \in L$ such that $r_{p,n} = 2n$, then (1) implies that n=2 and $r_{p,n+1} = 0$. Thus S-L consists of 4 points x,y,z and t. If there is a line L' distinct from L, of size greater than 2 (hence of size 3), say $L' = \{x,y,q\}$, then $<z,t> = \{z,t,q\}$ (otherwise there would be no line through p disjoint from the two disjoint lines $<x,y>$ and $<z,t>$), and so S is the union of the three lines $<p,q>$, $\{x,y,q\}$ and $\{z,t,q\}$, but there is no line through q disjoint from the disjoint lines $<x,z>$ and $<p,t>$, a contradiction. We conclude that all lines of S distinct from L have size 2. Therefore, since $d_o = 1$, S is the generalized projective space PG(5,1) in which all lines have size 2.

If $r_{p,n} = 0$ for every point $p \in L$, then the linear space of n^2 points induced on S-L has only lines of size n, and so it is an affine plane of order n. Therefore S is an affine plane A completed with at most n-1 points at infinity, since A has n+1 directions of lines and since the lines of at least two directions Δ_1 and Δ_2 are disjoint from L. In other words, S is an affino-projective plane of order n which is not a semi-affine plane.

We may now assume that $r_{x,n} = 0$ or n for every point x of L and that $r_{p,n} = n$ for some point $p \in L$. Hence, thanks to (1), $r_{p,n+1} = 1$.

We first examine the case n = 2. Then S consists of the points of L, together with four additional points x,y,z,t. We know that the point p is on exactly one line of size 3 distinct from L, say {p,x,y}. If L contains a point $q \neq p$ not belonging to the line $<z,t>$, then there is no line through q disjoint from $\{z,t\} \cup \{p,x,y\}$, a contradiction. Therefore S is the generalized projective space P(3, 3+3) consisting of two disjoint lines of size 3.

Finally, suppose that $n \geq 3$. Consider the linear space induced on S-L. The lines of S disjoint from L, together with the restrictions to S-L of the lines of S intersecting L in a point y for which $r_{y,n} = 0$, determine k partitions of S-L into n lines of size n ($k \geq 2$). On the other hand, the restrictions to S-L of the lines intersecting L in a point x for which $r_{x,n} = n$ determine h partitions of S-L into n lines of size n-1 and one line of size n

($h \geq 1$ is the number of points $x \in L$ for which $r_{x,n} = n$). Counting in two ways the number of ordered pairs of distinct points in the linear space S-L, we get

$$n^2(n^2-1) = k\ n^2(n-1) + h\ n(n-1)(n-2) + h\ n(n-1)$$

that is $n(n+1) = k\ n + h(n-1)$.
Therefore n divides h and, since $h \geq 1$, we have $n \leq h$, which, together with $k \geq 2$, implies

$$n(n+1) \geq 2\ n + n(n-1).$$

This inequality being in fact an equality, we conclude that k=2 and h=n. In other words, $r_{x,n} \neq 0$ for every point x of L and $|L| = h = n$.
Now we construct from S a bigger linear space in the following way : we add a new point to L, as well as to each line of Δ_1, these n+1 new points forming a new line N. We also add the new point of L to all lines of Δ_2. Finally, to each line L' of size n intersecting L, we add the new point of the unique line of Δ_1 disjoint from L'. The space S U N constructed in this way is a linear space. Indeed, if two lines of S of size n intersecting L in a given point x are both disjoint from the same line $L_1' \in \Delta_1$, then one of the lines through x must contain at least two points of L_1', contradicting the fact that S is a linear space. On the other hand, if two lines L', L" intersecting L in distinct points x', x" are both disjoint from a line $L_1' \in \Delta_1$ and intersect in a point z of S, then there is no line through x" disjoint from $L' \cup L_1'$, contradicting the hypothesis.

Since S U N is a linear space of $(n+1)^2$ points in which all lines have size n+1, it is an affine plane of order n+1. Therefore S is an affine plane from which one line has been removed.

6.2. The case $d_o > 1$ with a projective line.

From now on, we always assume $d_o > 1$.

Lemma 4.1.

(i) *If* A *and* B *are two disjoint lines and if* C *is a line disjoint from* A U B, *then*

$$(a-b)(c+d_o) = r_A - r_B$$

(ii) *If* A *and* B *are two intersecting lines and if* C *is a line disjoint from* A U B, *then*

$$(a-b)(c+d_o+1) = r_A - r_B$$

Proof. If L and L' are two disjoint lines and if p is a point outside L ∪ L', we have

$$r_p = |L| + |L'| - d_2(p,L,L') + d_o \qquad (2)$$

Counting in two ways the number of trisecants of A, B and C, we get

(i) if A and B are disjoint

$$\sum_{x\in A} d_2(x,B,C) = \sum_{y\in B} d_2(y,A,C)$$

that is, using (2),

$$a(b+c+d_o) - r_A = b(a+c+d_o) - r_B$$

or $(a-b)(c+d_o) = r_A - r_B$

(ii) if A and B intersect in z

$$\sum_{\substack{x\in A\\ x\neq z}} d_2(x,B,C) = \sum_{\substack{y\in B\\ y\neq z}} d_2(y,A,C)$$

that is, using (2),

$$(a-1)(b+c+d_o) - r_A = (b-1)(a+c+d_o) - r_B \qquad \text{or}$$

$$(a-b)(c+d_o+1) = r_A - r_B .$$

Corollary 4.1. If A *and* B *are two lines of different sizes, then all lines disjoint from* A ∪ B *have the same size.*

Corollary 4.2. If the lines A *and* B *have the same size and if there is a line disjoint from* A ∪ B, *then* $r_A = r_B$.

Lemma 4.2. If two disjoint lines A *and* B *have different sizes, then all lines disjoint from* A ∪̇ B *have the same size, equal either to* a *or to* b.

Proof. Suppose that there is a line C disjoint from A ∪ B, of size c ≠ a,b. By Corollary 4.1, all lines disjoint from A ∪ B have size c and they intersect each other (otherwise A and B would have the same size). So there are c(s-1)+1 such lines. Counting in two ways the number of pairs (p,L) where L is a line disjoint from A ∪ B and p ∈ L yields

$$(v-a-b)d_o = (c(d_o-1)+1)c$$

that is

$$(v-a-b-c^2)d_o = -c^2+c \qquad (3)$$

By the same argument, all lines disjoint from A ∪ C have b points and intersect each other, so that

$$(v-a-b^2-c)d_o = -b^2 + b \quad (4)$$

Subtracting (4) from (3), we get

$$(b^2 - c^2 + c - b)d_o = b^2 - c^2 + c - b \ .$$

Since $d_o > 1$, we must have

$$(b-c)(b+c) = b-c \ .$$

But $b \neq c$, and so $b+c = 1$, a contradiction. Therefore any line disjoint from A ∪ B has size a, or b, and Corollary 4.1 ends the proof.

Lemma 4.3. If S *contains a projective line* G *of size* g *and if* S *is not a degenerate projective plane, then* $g \geq 4$ *is the size of any projective line of* S, *as well as the degree of any point outside a projective line. Moreover, the size of a non-projective line is less than* $g-1$.

Proof. Consider two disjoint lines A and B. Since $d_o \geq 1$, there is a point p outside A ∪ B ∪ G. The degree of p is equal to g because p is outside the projective line G and it is greater than a+1 since there are at least $d_o > 1$ lines disjoint from A through the point p outside A ∪ B. So $g > a+1$ and in particular $g \geq 4$. Moreover, any projective line G' different from G has size g : indeed, the degree of any point outside G ∪ G' is $|G| = |G'| = g$.

Proposition 4.2. If S *contains a projective line, then* S *is a projective plane.*

Proof. It suffices to prove that any two lines intersect. Assume on the contrary that S contains two disjoint lines A and B. Let x and y be the points of intersection of A and B with the projective line G. Let C be a line disjoint from A ∪ B, intersecting G in a point z. Counting in two ways the number of pairs (p,L) where $p \in L \cap G$ and $L \cap (A \cup C) = \emptyset$, we get

$$(g-2)d_o = |\mathcal{L}| - (r_x+r_z-1) - (a+c-2)(g-2) + (a-1)(c-1) \quad (5)$$

where $|\mathcal{L}|$ denotes the total number of lines in S.

Considering the disjoint lines B and C, we have similarly

$$(g-2)d_o = |\mathcal{L}| - (r_y+r_z-1) - (b+c-2)(g-2) + (b-1)(c-1) \quad (6)$$

Subtracting (6) from (5), we have

$$r_x - r_y = (b-a)(g-1-c) \quad (7)$$

On the other hand, Lemma 4.1 gives

$$(a-b)(c+d_o) = (a-1)g + r_x - (b-1)g - r_y$$

or $\quad r_x - r_y = (a-b)(c+d_o-g) \qquad (8)$

Subtracting (8) from (7) yields

$$(b-a)(d_o-1) = 0$$

which implies b=a since $d_o > 1$, moreover $r_x = r_y$ thanks to (8). Thus any two disjoint lines have necessarily the same size and all points of G have the same degree r.

If n denotes the common size of two disjoint lines,

$$(g-2)s = |\mathcal{L}| - (2r-1) - (2n-2)(g-2) + (n-1)^2 \qquad (9)$$

Solving for n, we get

$$n = g-1 \pm \sqrt{\delta}$$

where δ is the discriminant of equation (9).
Since $n < g-1$ by Lemma 4.3, n is uniquely determined, and so all non-projective lines have the same size n.

α) Suppose first that G is the only projective line in S. The total number of points in S is easily seen to be

$$g + (r-1)(n-1) = 1 + g(n-1)$$

(count the points on the lines passing through a point of G, or through a point outside G). It follows that n-1 divides g-1.

Given two disjoint lines A and B and a point p of G outside $A \cup B$, the number of bisecants of A and B through p does not depend on $p \in G$ and is equal to $t = d_2(p,A,B) = 2n + d_o - r \leq n$. Counting in two ways the number of pairs (p,L) where $p \in G \cap L$, $p \notin A \cup B$ and $L \cap A \neq \phi \neq L \cap B$, we get

$$(n-1)^2 = (g-2)(t-1)$$

Since n-1 and g-2 are relatively prime, $(n-1)^2$ divides t-1, in contradiction with $t \leq n$.

β) Suppose now that there are at least two projective lines in S. If all projective lines have a point p in common, then, since $d_o > 0$, there is at least one non-projective line L through p. Let G be one of the projective lines. Consider two points x and y distinct from p and lying respectively on L and G. Counting in two ways the number of pairs (q,L') where $q \neq x$ and $\langle q,x \rangle = L'$ (resp. $q \neq y$ and $\langle q,y \rangle = L'$), we get

$$v-1 = (n-1) + (g-1)(n-1) = g-1 + (g-1)(n-1)$$

and so $n = g$, contradicting $n < g-1$. Therefore the projective lines of S have no point in common. This implies that all points of S have the same degree g.

Thus for any two disjoint lines A and B and for any point p outside $A \cup B$, the number of bisecants of A and B passing through p is $d_2(p,A,B) = 2n + d_0 - g$, which is independent *of* the triple (p,A,B), and so S satisfies condition (D2). Proposition 4.2 follows now easily from Theorem 2.

6.3. The case $d_0 > 1$ with no projective line.

Lemma 4.4. If there is no projective line in S, *then two disjoint lines have always the same size.*

Proof. Suppose on the contrary that A and B are two disjoint lines with different sizes a and b. By Lemma 4.2, we know that all lines disjoint from $A \cup B$ have the same size, equal either to a or to b. It is no loss of generality to assume that this size is b. Then Lemma 4.1 yields

$$(a-b)(b+d_0) = r_A - r_B \tag{14}$$

Now we shall give a proof in three steps :

Step 1. All lines disjoint from A have size b.

Suppose on the contrary that there is a line C disjoint from A and of size $c \neq b$. Lemma 4.2 implies that C intersects B, as well as any line disjoint from $A \cup B$. Thus, counting in two ways the number of pairs (p,L) where $p \in L$ and L is disjoint from $A \cup B$, we get

$$(c-1)d_0\, b = (v-a-b)d_0$$

that is

$$cb = v-a \tag{15}$$

Therefore c is uniquely determined and the only possible sizes for the lines disjoint from A are b and c. This implies that the size of any line disjoint from $A \cup C$ is either b or c. We show that b is impossible. Indeed, if $a \neq c$, this is obvious by Lemma 4.2; if $a = c$, suppose on the contrary that there is a line B' of size b disjoint from $A \cup C$. Then Lemma 4.1 gives

$$(a-b)(a+d_0) = r_A - r_{B'}$$

but Corollary 4.2 implies that

$$r_B = r_{B'}$$

and so

$$(a-b)(a+d_o) = r_A - r_B$$

contradicting (14).

Since any line disjoint from A U C has size c, by Lemma 4.1,

$$(a-c)(c+d_o) = r_A - r_C \tag{16}$$

On the other hand, since A is disjoint from the two intersecting lines B and C, Lemma 4.1 yields

$$(b-c)(a+d_o+1) = r_B - r_C \tag{17}$$

Subtracting (17) from (16) and using (14), we get

$$(a-c)(c+d_o) - (b-c)(a+d_o+1) = (a-b)(b+d_o)$$

that is

$$(b-c)(b+c-1-2a) = 0$$

and, since $b \neq c$

$$2a = b+c-1 \tag{18}$$

Now (15) becomes

$$v = 2\,ab - b^2 + a + b \tag{19}$$

or $$v = 2\,ac - c^2 + a + c \tag{20}$$

Let D be a line of size d intersecting A. We shall prove that d = a. If on the contrary $d \neq a$ and if there is a line B' of size b disjoint from A U D, then, by Corollary 4.1, D intersects all lines of size c disjoint from A. Counting in two ways the number of pairs (p,L) where L is disjoint from A U C and $p \in L$, we get

$$(d-2)d_o\, c = (v-a-c)d_o$$

Therefore

$$d = (v-a+c)/c = b + 1 \quad \text{thanks to (15).}$$

According to Lemma 4.2, the size of all lines disjoint from B' U D is either b or d = b+1. In particular, there are lines disjoint from B' U D and intersecting A, which have necessarily size a or b+1 because they intersect A and we have just seen that if $d \neq a$ is the size of a line intersecting A, then d = b+1.

Therefore, since $a \neq b$, all lines disjoint from $B' \cup D$ have size $b+1$. On the other hand, A being disjoint from the lines B and B' of the same size b, Corollary 4.2. yields

$$r_B = r_{B'}$$

which, together with (14) and Lemma 4.1, implies that all lines disjoint from $A \cup B'$ have size b. Therefore A intersects any line disjoint from $B' \cup D$. Counting in two ways the number of flags (p,L) where L is a line disjoint from $B' \cup D$, we get

$$(a-1)d_o\,(b+1) = (v - 2b - 1)d_o$$

and, using (19),

$$b(a-b) = 0, \text{ a contradiction.}$$

This implies that if $d \neq a$, D intersects all lines of size b disjoint from A. In particular, D intersects B and all lines disjoint from $A \cup B$. Then, counting in two ways the number of flags (p,L) where L is a line disjoint from $A \cup B$, we get

$$(d-2)d_o\, b = (v-a-b)d_o$$

Therefore

$$d = (v-a+b)/b = c+1 \quad \text{thanks to (15).}$$

We have seen that if a line of size $d \neq a$ intersects A, C and all lines disjoint from $A \cup C$, then $d = b+1$. Since $b \neq c$, we conclude that there is a line C' of size c disjoint from A and D. By Lemma 4.2, the size of all lines disjoint from $D \cup C'$ is either c or $d = c+1$. But some of these lines meet A and therefore have size a or $c+1$. If $a \neq c$, then all lines disjoint from $D \cup C'$ have size $c+1$, and so cannot be disjoint from $A \cup C'$. Therefore all these lines intersect A. Counting in two ways the number of flags (p,L) where L is disjoint from $D \cup C'$, we get

$$(a-1)d_o\,(c+1) = (v-2c-1)d_o$$

and, using (20),

$$c(a-c) = 0, \text{ a contradiction.}$$

Therefore $a = c$ and, by (18),

$$b = a+1 = c+1 \qquad (21)$$

It follows that a and b are the only line sizes in S. For any point p of S, we denote by α_p the number of lines of size a through p. Counting in two ways the number of pairs (q,L) where $q \neq p$ and $\langle q,p \rangle = L$, we get

$$v-1 = \alpha_p (a-1) + (r_p - \alpha_p)(b-1)$$

which, using (19) and (21), can be written

$$a(a+2) - 1 = \alpha_p (a-1) + (r_p - \alpha_p)a$$

or $$a(a+2-r_p) = 1 - \alpha_p \tag{22}$$

Therefore a divides α_p-1. Let m be the positive integer such that

$$m\, a = \alpha_p - 1 \tag{23}$$

(22) yields

$$r_p = a + 2 + m \tag{24}$$

Consider a line B' of size b = a+1 not passing through p and note that p is on at least d_o lines disjoint from B'. We have

$$r_p = a+2+m \geq b+d_o = a+1+d_o \geq a+3 \tag{25}$$

and so $m \geq 1$. (23) and (24) imply that the number of lines of size b through p is

$$r_p - \alpha_p = 1+m+a(1-m) \geq 0$$

Therefore if $a \geq 4$, then m = 1 and, using (25) and (24), we get

$$r_p = a+3 = b+d_o$$

for any point p of S. We conclude that any line intersecting A meets B. Therefore, if $p \in A$, then A is the only line disjoint from B through p, contradicting $d_o \geq 2$.

If a = 2, then b = 3 and (19) gives v = 8. Similarly, if a = 3, then b = 4 and v = 15. But these values of v are incompatible with the fact that through any point $p \notin A \cup B$, there are at least two lines of size b disjoint from $A \cup B$.

Thus we have proved that all lines meeting A have a points. Let A' be a line meeting A. If A' is disjoint from a line B' of size b (necessarily disjoint from A), Corollary 4.2 gives

$$r_A = r_{A'}$$

since B' is disjoint from $A \cup A'$, and

$$r_B = r_{B'}$$

since A is disjoint from $B \cup B'$.

Then Lemma 4.1 and (14) imply that all lines disjoint from $A' \cup B'$ have size b, contradicting the fact that some of these lines meet A.

Therefore A' meets B and all lines disjoint from $A \cup B$. Counting the number of flags (p,L) where L is disjoint from $A \cup B$, we get

$$(a-2)d_0\, b = (v-a-b)d_0$$

and so, using (19), $a = b-2$
and, using (18), $c = b-3$

In particular, $a \neq c$. Now we claim that A' meets C and all lines disjoint from $A \cup C$ (indeed, suppose, on the contrary, that one of these lines, say C', is disjoint from A'. Corollary 4.2 and relation (16) imply that all lines disjoint from $A' \cup C'$ have size c, contradicting the fact that some of these lines meet A and that $a \neq c$). Counting in two ways the number of pairs (p,L) where L is disjoint from $A \cup C$ and $p \in L$, we get

$$(a-2)d_0\, c = (v-a-c)d_0$$

and so, using (20), $a = c-2$
which, together with (21) implies $b = c$, a contradiction.

This ends the proof of step 1. From now on, we may assume that
(*) in any pair of disjoint lines of different sizes, one of the lines has the property that all lines disjoint from it have the same size.

Step 2. a and b are the only two line sizes in S.

It suffices to prove that any line D intersecting A has size a or b. Suppose on the contrary that the size d of D is different from a and b. We shall prove that D intersects all lines disjoint from A. Indeed, if there is a line B' disjoint from $A \cup D$, then the size of all lines disjoint from $D \cup B'$ is either b or d. Assume first that this size is b. Then Lemma 4.1 yields

$$(d-b)(b+d_0) = r_D - r_B$$

Subtracting this from (14), we get

$$(a-d)(b+d_0) = r_A - r_D$$

On the other hand, Lemma 4.1 gives

$$(a-d)(b+d_0+1) = r_A - r_D$$

since B' is disjoint from the two intersecting lines A and D. These last two relations are contradictory. Therefore all lines disjoint from $D \cup B'$ have size d, and so they intersect A. Counting in two ways the number of flags (p,L) where L is disjoint from D U B', we get

$$(a-1)d_o\ d = (v-b-d)d_o\ .$$

Note that all lines disjoint from A U B' intersect D; then a similar counting argument yields

$$(d-1)d_o\ b = (v-a-b)d_o$$

Subtracting these two relations and simplifying by d_o, we get

$$d(a-b) = a-b$$

Hence

d=1, a contradiction.

Thus D intersects any line disjoint from A. In particular, D intersects B and all lines disjoint form A U B. Counting in two ways the number of flags (p,L) where the line L is disjoint from A U B, we get

$$(d-2)\ b = v-a-b$$

which shows that d is uniquely determined. Therefore there are exactly three line sizes in S, namely a, b and d.

Since D meets all lines disjoint from A, any line C disjoint from D intersects A. We shall prove that C intersects also B. This has already been proved if C is of size d. Suppose now that C has size a and is disjoint from B. Then Corollary 4.2 yields

$$r_A = r_C$$

Together with (14), this implies

$$(a-b)(b+s) = r_C - r_B$$

which means, by Lemma 4.1, that all lines disjoint from C U B have size b. Thanks to assumption (*), we conclude that all lines disjoint from C have size b, contradicting the fact that D and C are disjoint.

Thus if C is disjoint from B, C has size b. Since there is a line disjoint from B U C, Corollary 4.2 yields

$$r_B = r_C\ .$$

Applying Lemma 4.1 to the line B disjoint from the two intersecting lines A and C, we get

$$(a-b)(b+d_o+1) = r_A - r_C$$

These two relations contradict (14), and so any line disjoint from D intersects both A and B. Let C be such a line. Counting in two ways the number of lines disjoint from C U D, we get

$$(a-2)d_o = (b-2)d_o$$

Hence a = b, a contradiction. Therefore any line intersecting A has size a or b.

Step 3. Let C be a line intersecting both A and B. Any line D disjoint from C and intersecting A U B meets A and B.
In order to prove this, we shall consider several cases, according to the sizes of C and D.

(i) If D has size a, then we already know that D intersects A.

(ii) If C has size a and if D, intersecting A, has size b, then D meets B. Indeed, suppose on the contrary that D and B are disjoint. Lemma 4.1, applied to the line B disjoint from the two intersecting lines A and D, yields

$$(a-b)(b+d_o+1) = r_A - r_D$$

and Corollary 4.2 implies that

$$r_D = r_B$$

but these two relations contradict (14).

(iii) If C has size a and if D, intersecting B, has size b, then D meets A. Indeed, suppose on the contrary that D and A are disjoint. Corollary 4.2 yields $r_A = r_C$ and $r_B = r_D$.
Let E be a line disjoint from C and D, through a point p ∈ A. These last two relations, together with (14), imply that E has size b. Moreover, by Corollary 4.2,

$$r_E = r_D = r_B \ .$$

Lemma 4.1 applied to the line D disjoint from C U E (resp. A U E) yields

$$(a-b)(b+d_o) = r_C - r_E = r_A - r_B$$

and $$(a-b)(b+d_o+1) = r_A - r_E = r_A - r_B$$

These two relations contradict each other.

(iv) If C has size a and if D, intersecting A, has size a, then D meets B. Indeed, if on the contrary B is disjoint from D, Corollary 4.2 yields

$$r_A = r_D$$

which, toegether with (14) and Lemma 4.1 applied to the disjoint lines B and D, implies that all lines disjoint from B ∪ D have size b. Therefore, thanks to assumption (*), we conclude that any line disjoint from D has size b, contradicting the fact that C is disjoint from D.

(v) If C and D have size b and if D intersects B, then D meets A. Indeed, if on the contrary A and D are disjoint, Corollary 4.2 yields

$$r_B = r_D$$

and

$$r_C = r_D$$

Lemma 4.1 applied to the line D disjoint from the two intersecting lines C and A yields

$$(a-b)(b+d_o+1) = r_A - r_C = r_A - r_B$$

in contradiction with (14).

(vi) If C and D have size b and if D meets A, then D meets B. Indeed, if on the contrary D and B are disjoint, then

$$r_B = r_D$$

and, using Lemma 4.1, we get

$$(a-b)(b+d_o+1) = r_A - r_D \ .$$

These two relations contradict (14).

(vii) Finally, if C has size b and if D has size a and intersects A, then D intersects B. Indeed, suppose on the contrary that B and D are disjoint, and let B' be a line intersecting D and disjoint from A ∪ B. Corollary 4.2 yields

$$r_A = r_D$$

By Lemma 4.1, we get

$$(a-b)(b+d_o) = r_A - r_{B'}$$

since B is disjoint from the two disjoint lines A and B', and

$$(a-b)(b+d_o+1) = r_D - r_{B'} = r_A - r_{B'}$$

since B is disjoint from the two intersecting lines D and B'.
These two relations contradict each other.

Now, take a line D disjoint from C and intersecting A ∪ B.
Thanks to step 3, counting in two ways the number of lines disjoint from C ∪ D

and intersecting A ∪ B, we have

$$(a-2)d_o = (b-2)d_o$$

which implies a = b, a contradiction. This proves Lemma 4.4.

Lemma 4.5. If there is no projective line in S *and if any two disjoint lines of* S *have the same size, then all lines of* S *have the same size.*

Proof. Suppose on the contrary that there are at least two distinct line sizes n and ℓ. Consider two disjoint lines A and B of size n and let L be a line of size ℓ. L and all lines disjoint from L intersect A and B. Counting in two ways the number of bisecants of A and B passing neither through $p \in A \cap L$ nor through $q \in B \cap L$, we get

$$(n-1)^2 = \sum_{\substack{x \in L \\ x \neq p,q}} [d_2(x,A,B)-1] + \delta$$

where δ denotes the number of lines disjoint from L.
Since

$$d_2(x,A,B) = 2n + d_o - r_x$$

we get

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_p + r_q \tag{26}$$

Through any point $y \neq p,q$ on L, consider a line C disjoint from A ∪ B. The same counting argument applied to the pairs of lines {A,C} and {B,C} instead of {A,B} yields

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_p + r_y \tag{27}$$

and

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_q + r_y \tag{28}$$

(26), (27) and (28) imply that $r_p = r_q = r_y$, and so all points of L have the same degree. One proves in the same way that all points of any line of size n have the same degree. Finally, since any line of size n intersects L, all points of S have the same degree r.

Counting in two ways the number of pairs (y,C) where C is disjoint from A ∪ B and $y \in C \cap L$, we get

$$(\ell-2)d_o = |\mathcal{L}| - (2r-1) - 2(n-1)(r-2) + (n-1)^2$$

Similarly,

$$(n-2)d_0 = |\mathcal{L}| - (2r-1) - 2(\ell-1)(r-2) + (\ell-1)^2$$

Subtracting these two relations, we have

$$(\ell-n)d_0 = 2(\ell-n)(r-2) + n^2 - \ell^2 + 2(\ell-n)$$

Therefore, since $\ell \neq n$,

$$2r = \ell + n + d_0 + 2 \ . \tag{29}$$

Suppose $\ell > n$ and let L' be a line disjoint from L. The number of lines intersecting L but not L', and passing through a point $p \notin L \cup L'$, is

$$r - \ell - d_0 \geq 0$$

Multiplying by 2 and using (29), we get

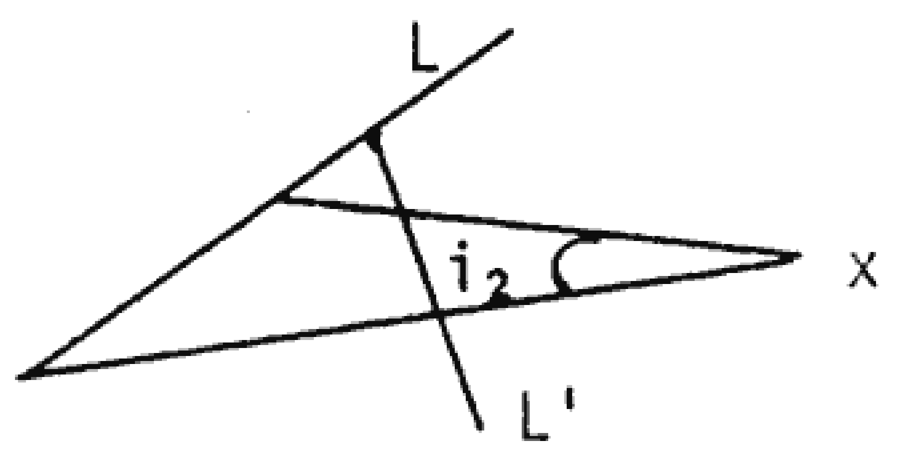
$$2 - d_0 \geq \ell - n$$

in contradiction with $d_0 \geq 2$ and $\ell \geq n+1$.

Proposition 4.3. If S *contains no projective line, then* S *is an affine plane, a projective space of dimension 3 or a generalized projective space in which all lines have size 2.*

Proof. Thanks to Lemmas 4.4 and 4.5, we know that all lines of S have the same size n, and so that all points have the same degree r. Thus, for any two disjoint lines L, L' and any point p outside $L \cup L'$, $d_2(p,L,L') = 2n + d_0 - r$ is independent of the choice of p, L and L'. Therefore by Theorem 2 (more precisely by Propositions 2.1 and 2.2), we have the desired conclusion.

7. LINEAR SPACES SATISFYING CONDITION (I2).



(I2) *there is a non-negative integer* i_2 *such that for any two intersecting lines* L, L' *and any point* x *outside* $L \cup L'$, *there are exactly* i_2 *bisecants of* L *and* L' *passing through* x.

The only finite linear spaces which are known to satisfy condition (I2) are the (possibly degenerate) projective planes, the affine planes and the Steiner systems S(2,2,v). We will prove that other examples (if any) should be

Steiner systems $S(2,k,v)$ satisfying some rather restrictive arithmetical conditions on i_2 and k.

Theorem 5 [20]. *If* S *is a finite non-trivial linear space satisfying condition* (I2), *then*

(i) S *is a degenerate projective plane* (and $i_2 = 1$),

or (ii) S *is a Steiner system* $S(2,k,v)$ *with* $i_2 < k < v$, *such that*

(1) $i_2 | (k-1)(k-2)$

(2) $k | [(k-1)(k-2)/i_2+2][(k-1)(k-2)/i_2+1]$

(3) $i_2(2k-2-i_2) | k(k-1)^2(k-2)$

(4) *if* $(k-1)(k-2)/i_2$ *is odd, then* $k-1-i_2$ *is a square*
if $(k-1)(k-2)/i_2$ *is even, then the Diophantine equation*

$$(k-1-i_2)x^2 + (-1)^{(k-1)(k-2)/2i_2}\, i_2\, y^2 = z^2$$

has a solution in integers x,y,z *not all zero.*
Moreover $i_2 = k-1$ *iff* S *is a projective plane of order* $k-1$
$i_2 = k-2$ *iff* S *is an affine plane of order* k.

Note that this theorem gives a partial answer to an open problem mentioned by Cameron in [15, p.54].

Proof. The proof of (3) (resp. (4)) is based on the construction of certain partial geometries (resp. symmetric 2-designs) associated with S, whence all other statements follow directly from the linear structure of S. Thus we shall divide the proof into three parts :

7.1. The linear space S.

First of all, note that if $i_2 = 0$, any bisecant of two intersecting lines has size 2, and so all lines of S have size 2. Therefore, from now on, we shall suppose $i_2 \geq 1$.

It is easy to check that if S is the union of two intersecting lines, then condition (I2) is satisfied iff S is a degenerate projective plane. Hence we may assume, in what follows, that any point of S is on at least three lines.

Let L and L' be two lines intersecting in x. Consider a third line L" passing through x. Counting in two different ways the number of trisecants of L, L' and L", we get

$$(|L| - 1)i_2 = (|L'| - 1).i_2$$

that is $|L| = |L'|$.

Therefore any two intersecting lines have the same size, and so all lines of S have the same size, which we denote by k. Counting in two different ways the number of pairs (p, L_1) where $p \notin L \cup L'$ and L_1 is a bisecant of L and L' passing through p, we get

$$(v - 2k + 1)i_2 = (k-1)^2(k-2) \tag{5}$$

If r denotes the degree of a point of S, we have

$$v-k = (r-1)(k-1) \tag{6}$$

(5) and (6) yield

$$(r-2)i_2 = (k-1)(k-2) \tag{7}$$

which implies $i_2|(k-1)(k-2)$ (1).

On the other hand, the number of lines of S is vr/k, which must be an integer. Thanks to (6), we deduce that

$$k|r(r-1) \tag{8}$$

which, using (7), gives

$$k|[(k-1)(k-2)/i_2 + 2][(k-1)(k-2)/i_2+1] \qquad (2)$$

For $i_2 = k-2$ (resp. k-1), (5) yields $v = k^2$ (resp. k^2-k+1) and so S is an affine (resp. projective) plane.

This ends the first part of the proof.

Note that (2) implies

$$k|2(i_2+1)(i_2+2) \tag{9}$$

Thanks to (1), (2), (3) it is easily shown that $k \neq 2(i_2+1)(i_2+2)$ and that $k = (i_2+1)(i_2+2)$ is admissible only for $i_2 = 2$. So (9) implies that $k \leq 2(i_2+1)(i_2+2)/3$ for $i_2 > 2$ and we conclude that $\sqrt{k} < i_2$ for any $i_2 > 3$.

7.2. Partial geometries and association schemes.

Given a line L of S, the point set S-L, provided with the restrictions to S-L of the lines of S intersecting L, forms a partial geometry with para-

meters $(R,K,T) = (k, k-1, i_2)$ having $V = v-k = (k-1)^2(k-2)/i_2+k-1$ points and $B = k(k-1)(k-2)/i_2+k$ lines.
The point graph (resp. line graph) of a partial geometry is defined by calling two points (resp. two lines) adjacent iff they are collinear (resp. concurrent). The line graph is also the point graph of the dual partial geometry. Thus four strongly regular graphs are associated with every partial geometry, namely the point graph G_p, the line graph G_L and their complements $\bar{G}_p$ and $\bar{G}_L$.
In our problem, the parameters of G_p are

$$v_p = V = (k-1)^2(k-2)/i_2 + k - 1 \qquad k_p = k(k-2)$$

$$\lambda_p = (k-1)(i_2-1) + k - 3 \qquad \mu_p = ki_2$$

Besides the trivial eigenvalue k_p, the other eigenvalues of G_p are

$$r_p = k - i_2 - 2 \quad \text{with multiplicity } f_p = (k-2)(k-1)^2k/i_2(2k - 2 - i_2)$$

and

$$s_p = -k \qquad \text{with multiplicity } g_p = v_p - 1 - f_p$$

For the line graph G_L,

$$v_L = k(k-1)(k-2)/i_2+k \qquad k_L = (k-1)^2$$

$$\lambda_L = (k-2)i_2 \qquad \mu_L = (k-1)i_2$$

$$r_L = k - i_2 - 1 \qquad f_L = f_p$$

$$s_L = -(k-1) \qquad g_L = v_L-1-f_L = k-1 + \frac{k(k-1)(k-2)(k-1-i_2)}{i_2(2k-2-i_2)}$$

Multiplicities of eigenvalues being integers, we get

$$i_2(2k-2-i_2) \mid (k-2)(k-1)^2 \qquad (3)$$

The other known necessary conditions for the existence of a strongly regular graph, namely the Krein condition (I10), the absolute bound (I11), the μ-bound (I12) and the claw-bound (I13), are tedious but easy to check : they give nothing more than the previous conditions.

The line graph G_L may also be viewed as a 2-class association scheme Σ_2 if we say that two lines are first associates when they are distinct and adjacent (i.e. when the corresponding lines of S form a triangle with L) and second associates otherwise. Actually we can define a 3-class association scheme Σ_3 by subdividing the second class of Σ_2. The vertices of Σ_3 are the lines of S intersecting L, two vertices are first associates if the corresponding lines intersect in a point outside L,

, second associates if the corresponding lines intersect in a point of L and third associates if the corresponding lines are disjoint. Indeed, the number n_i of i-th associates of a vertex x is independent of x , and for any two i-th associates x and y, the number p^i_{jk} of vertices which are j-th associates of x and k-th associates of y does not depend on the pair x,y.

Consider the $v_L \times v_L$ association matrices $A_j = (a^j_{xy})$ with entries $a^j_{xy} = 1$ if the vertices x and y are j-th associates and $a^j_{xy} = 0$ otherwise. We denote by λ_{jk} the (not necessarily distinct) eigenvalues of A_j and by μ_k the multiplicity of λ_{jk} (it can be shown that μ_k does not depend on j). The parameters of Σ_3 are

$n_1 = (k-1)^2$, $\quad n_2 = (k-1)(k-2)/i_2$, $\quad n_3 = (k-1)(k-2)(k-1-i_2)/i_2$

$p^1_{11} = (k-2)i_2$, $\quad p^1_{12} = k-2$, $\quad p^1_{13} = (k-2)(k-1-i_2)$

$p^1_{22} = 0$, $\quad p^1_{23} = (k-2)(k-1-i_2)/i_2$, $\quad p^1_{33} = (k-2)(k-1-i_2)(k-2-i_2)/i_2$

$p^2_{11} = (k-1)i_2$, $\quad p^2_{12} = 0$, $\quad p^2_{13} = (k-1)(k-1-i_2)$

$p^2_{22} = (k-1)(k-2)/i_2-1$, $\quad p^2_{23} = 0$, $\quad p^2_{33} = (k-1)(k-2-i_2)(k-1-i_2)/i_2$

$p^3_{11} = (k-1)i_2$, $\quad p^3_{12} = k-1$, $\quad p^3_{13} = (k-1)(k-2-i_2)$

$p^3_{22} = 0$, $\quad p^3_{23} = (k-1)(k-2-i_2)/i_2$, $\quad p^3_{33} = (k-1)[(k-1-i_2)(k-3-i_2)+i_2+1]/i_2-1$

$\lambda_{11} = k-1-i_2$, $\quad \lambda_{12} = -(k-1)$, $\quad \lambda_{13} = -(k-1)$

$\lambda_{21} = -1$, $\quad \lambda_{22} = -1$, $\quad \lambda_{23} = (k-1)(k-2)/i_2$

$\lambda_{31} = -(k-1-i_2)$, $\quad \lambda_{32} = k-1$, $\quad \lambda_{33} = -(k-2)(k-1-i_2)/i_2$

$\mu_1 = (k-2)(k-1)^2k/i_2(2k-2-i_2)$, $\quad \mu_2 = k(k-1)(k-2)(k-1-i_2)/i_2(2k-2-i_2)$, $\quad \mu_3 = k-1$

Condition (3) and $1 + \mu_1 + \mu_2 + \mu_3 = v_L$ imply that all multiplicities are integers. Moreover it is not difficult to check that the Krein condition as well as the condition given by Mathon (I14) are satisfied for all pairs (i_2,k) satisfying (1), (2), (3) with $i_2 \leq k-3$.

7.3. Symmetric 2-designs.

If $i_2 \leq k-2$, a non-trivial symmetric 2-design D(p,q) can be associated with any pair (p,q) of distinct points of S as follows : the points of D(p,q) are the lines through p distinct from the line <p,q>, the blocks of D(p,q) are the lines through q distinct from <p,q>, a point and a block being incident iff the corresponding lines intersect. Each point of $D(p,q)$ is on $k-1$ blocks and each block contains $k-1$ points, any two points are on i_2 blocks and any two blocks have i_2 points in common. Thus $D(p,q)$ is an $S_{i_2}(2,k-1,(k-1)(k-2)/i_2+1)$. The Bruck-Chowla-Ryser theorem (I3) gives a necessary condition for the existence of such a design, namely (4). This ends the proof of Theorem 5.

Unfortunately, there remain infinitely many pairs of parameters (i_2,k) with $i_2 \leq k-3$ satisfying the conditions (1), (2), (3) and (4) : for instance, all pairs (i_2,k) with $i_2 = 9^n$ $(n \geq 1)$ and $k = (i_2+1)(i_2+2)/2$. Indeed, it is easy to check that these pairs satisfy the first three conditions. We shall prove that the fourth condition is also satisfied by using the following

Theorem [37], [40]. The equation

$$b\,x^2 + c\,y^2 = z^2 \qquad (10)$$

has solutions in integers x,y,z not all zero if and only if for every prime p as well as for $p = \infty$, the Hilbert norm-residue symbol $(b,c)_p$ is equal to $+1$.

The symbol $(b,c)_p$ is defined to be $+1$ or -1 according as the congruence

$$b\,x^2 + c\,y^2 \equiv z^2 \pmod{p^m}$$

does have solutions in integers x,y,z, not all multiples of p, for every power p^m of the prime p, or not, and $(b,c)_\infty = +1$ or -1 according as (10) does or does not have solutions in real numbers $x,y,z \neq 0$. Thus $(b,c)_\infty = +1$ unless both b and c are negative. We shall use the following properties of the Hilbert norm-residue symbol :

(P1) $\prod_p (b,c)_p = 1$ (the product being over all primes p, including $p = \infty$)

(P2) $(b,c^2)_p = 1$,

(P3) if p is an odd prime and $b,c \not\equiv 0 \pmod p$, then $(b,c)_p = 1$,

(P4) if p is an odd prime and if $c_1 \equiv c_2 \not\equiv 0 \pmod p$, then $(b,c_1)_p = (b,c_2)_p$.

If $i_2 = 9^n$ $(n \geq 1)$ and $k = (i_2+1)(i_2+2)/2$, then $k \equiv (9^n+2)(9^n+1)/2 \equiv 3 \pmod 4$, and so $(k-1)(k-2) \equiv 2 \pmod 4$. Since i_2 is odd and divides $k-1$, we conclude that $(k-1)(k-2)/2\,i_2$ is an odd integer and (4) becomes

$$\frac{9^n+1}{2}\,9^n\,x^2 - 9^n\,y^2 = z^2$$

has a solution in integers x,y,z not all zero, or equivalently

$$\frac{9^n+1}{2} x^2 - y^2 = m^2 \qquad (11)$$

has a solution in integers x,y,m not all zero. Since (11) has solutions in real numbers $x,y,m \neq 0$, thanks to the above theorem, we have only to prove that $(\frac{9^n+1}{2}, -1)_p = +1$ for every prime p. Moreover thanks to (P1) and to $(\frac{9^n+1}{2}, -1)_\infty = +1$, it is sufficient to check this for the odd primes. Thus, let p be any odd prime.

If $p \nmid \frac{9^n+1}{2}$, then $(\frac{9^n+1}{2}, -1)_p = 1$ by (P3). If on the contrary $p \mid \frac{9^n+1}{2}$, then $9^n \equiv -1 \pmod p$ and we deduce from (P4) and (P2) that

$$(\frac{9^n+1}{2}, -1)_p = (\frac{9^n+1}{2}, 9^n)_p = 1 .$$

Therefore condition (4) is satisfied for the above values of i_2 and k.

For $k \leqslant 100$ there are only six pairs (i_2,k) with $i_2 \leqslant k-3$ satisfying conditions (1) to (4), namely (2,12), (24,65), (20,66), (3,10), (7,36) and (9,55) (for the last three, $k = (i_2+1)(i_2+2)/2$). The existence of a symmetric 2-design $S_{i_2}(2, k-1, (k-1)(k-2)/i_2+1)$ is known [57] for only two of these pairs, namely (2,12) and (3,10).

Note that if S satisfies one further hypothesis, the symmetric 2-designs D(p,q) are extendable and (I4) rules out all pairs (i_2,k) with $i_2 \leqslant k-3$, except one. Given a point p and a line L not through p, let $\mathcal{L}(p,L)$ denote the set of lines passing through p and intersecting L and let S(p,L) denote the set of points distinct from p and belonging to a line of $\mathcal{L}(p,L)$. Note that if $i_2 < k-1$, two lines L, L' such that $\mathcal{L}(p,L) = \mathcal{L}(p,L')$ are necessarily disjoint.

Proposition 5.1. Let S *be a finite linear space satisfying condition* (I2) *with* $1 \leqslant i_2 \leqslant k-3$. *If there is a point* $p \in S$ *such that for any line* L *not through* p, *any point of* S(p,L) *is on a line* L' *such that* $\mathcal{L}(p,L) = \mathcal{L}(p,L')$, *then* $i_2 = 2$ *and* k = 12.

Proof. Consider the design D(p) whose points are the lines through p, whose blocks are the distinct sets $\mathcal{L}(p,L)$, incidence of points and blocks being given by set inclusion. Thus any block has exactly k points and the total number of points is $r = (k-1)(k-2)/i_2+2$. By hypothesis, given a line L not through p, any point of S(p,L) is on one and only one line L' such that $\mathcal{L}(p,L) = \mathcal{L}(p,L')$ (only one because we have seen before that such lines are necessarily disjoint).

It follows that any three points of D(p) are in exactly i_2 blocks and that the total number of blocks is $r(r-1)/k$. Therefore, D(p) is a symmetric 3-design $S_{i_2}(3,k,(k-1)(k-2)/i_2+2)$. Note that for any point q distinct from p, the 2-design D(p,q) is isomorphic to the derived design of D(p) relative to <p,q>. Theorem 5 and the theorem of Cameron (I4) listing the admissible parameters of symmetric 3-designs imply immediately that $i_2 = 2$ and $k = 12$. The pair $(i_2,k) = (2,12)$ corresponds to a hypothetical linear space S(2, 12, 628), the existence of which is unsettled.

Finally, let us mention the following

Proposition 5.2. The only finite linear spaces S(2,k,v) *with* k *a prime power that satisfy condition* (I2) *are projective or affine planes.*

Proof. Let S be a finite linear space S(2,k,v) satisfying condition (I2) which is neither a projective plane nor an affine plane, so that

$$k \geq i_2 + 3 \qquad (12)$$

Let $k = p^n$ with p a prime number. (2) implies that

$$k \mid 2(i_2+1)(i_2+2) .$$

Since i_2+1 and i_2+2 are relatively prime, we conclude that

$$p^n \mid 2(i_2+1) \quad \text{or} \quad p^n \mid 2(i_2+2) .$$

Therefore, by (12),

$$p^n = 2(i_2+1) \quad \text{or} \quad p^n = 2(i_2+2) ,$$

and so $p^n = 2^n$ with $n \geq 2$.

If $p^n = 2(i_2+2)$, then (1) becomes

$$2^{n-2} -1 \mid (2^n-1)(2^{n-1} -1)$$

and so

$$2^{n-2} -1 \mid 2^{(n,n-2)} -1 = 1 \text{ or } 3.$$

Therefore $(i_2,k) = (2,8)$ or $(6, 16)$, contradicting (2).

If $k = 2(i_2+1)$, then (3) becomes

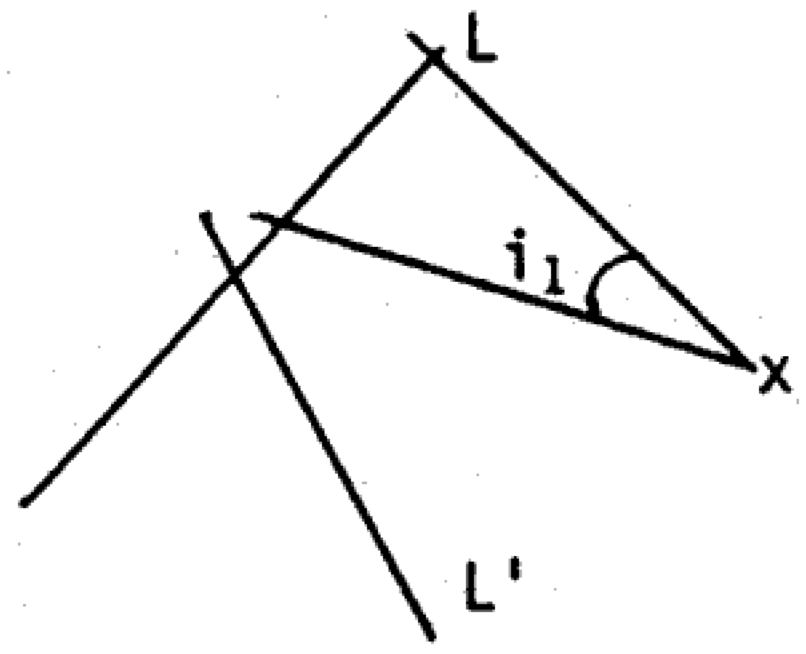
$$2(2^n -1) - (2^{n-1} -1) \mid 2^{n+1}(2^n -1)(2^{n-1} -1) .$$

Since 2^{n+1} and 2^n-1 are each relatively prime with $2(2^n - 1)-(2^{n-1} -1)$, this implies that

$$2(2^n -1)-(2^{n-1} -1) \mid 2^{n-1} - 1 > 0$$

and so $2(2^n -1) \leq 2(2^{n-1} -1)$, a contradiction.

8. LINEAR SPACES SATISFYING CONDITION (I1).



(I1) *there is a non-negative integer* i_1 *such that for any ordered pair of intersecting lines* L, L' *and for any point* x *outside* L ∪ L', *there are exactly* i_1 *lines through* x *which intersect* L *but not* L'.

The study of finite linear spaces satisfying condition (I1) reduces exactly to that of finite linear spaces satisfying condition (I2), thanks to the following result :

Theorem 6 [26]. *The finite non-trivial linear spaces satisfying* (I1) *are the finite degenerate projective planes and the Steiner systems* S(2,k,v) (k < v) *satisfying condition* (I2).

We have seen in Theorem 5 that the finite non-trivial linear spaces satisfying condition (I2) are necessarily Steiner systems S(2,k,v) or degenerate projective planes, so that conditions (I1) and (I2) are equivalent.

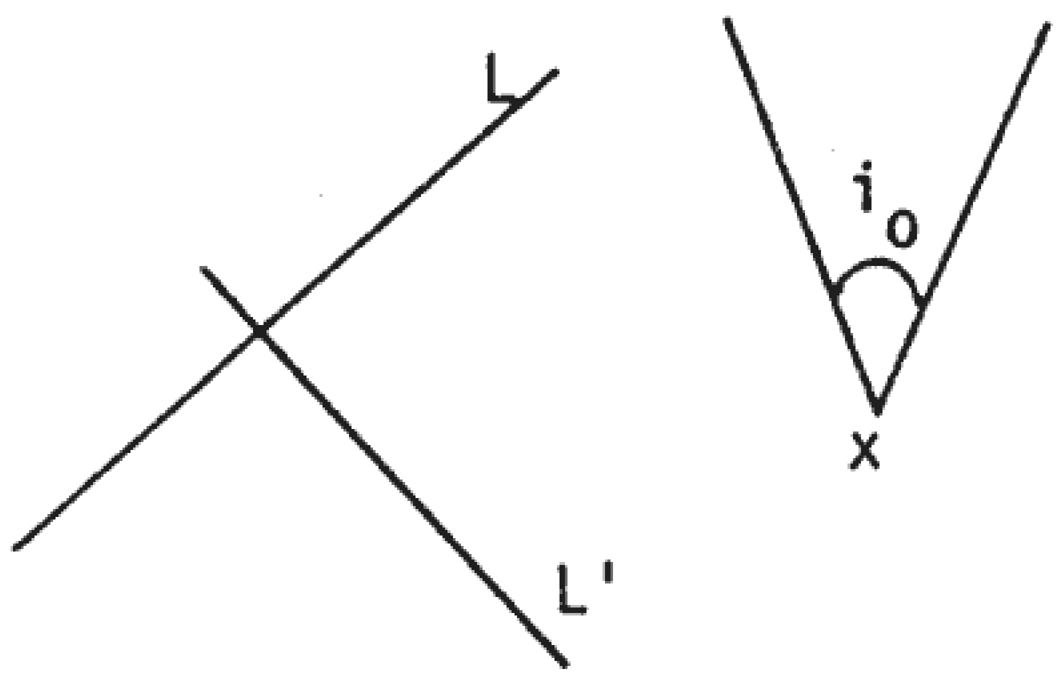
Proof of Theorem 6.

Let S be a finite non-trivial linear space satisfying condition (I1). It is easily seen that if S is the union of two intersecting lines, then S is a degenerate projective plane. Suppose now that S is not the union of two intersecting lines. Then, for any two intersecting lines L and L', the degree of any point x outside L ∪ L' is

$$r_x = |L| + i_1 + i_0(x,L,L') = |L'| + i_1 + i_0(x,L,L'),$$

and so L and L' have the same size. Therefore all lines of S have the same size. Since conditions (I1) and (I2) are equivalent for Steiner systems S(2,k,v), the theorem is proved.

9. LINEAR SPACES SATISFYING CONDITION (I0).



(I0) *there is a non-negative integer* i_0 *such that for any two intersecting lines* L, L' *of* S *and for any point* x *outside* L ∪ L', *there are exactly* i_0 *lines through* x *which are disjoint from* L ∪ L'.

The study of finite linear spaces satisfying condition (I0) reduces essentially to that of finite linear spaces satisfying condition (I2), thanks to the following

Theorem 7 [26]. *If* S *is a finite non-trivial linear space satisfying condition* (I0), *then one of the following occurs*

(i) S *is a punctured projective plane or an affine plane with one point at infinity,*

(ii) S *is a degenerate projective plane or a Steiner system* S(2,k,v) (k<v) *satisfying condition* (I2)

Conversely, each of these finite spaces satisfies condition (I0).

In particular, the only known finite linear spaces satisfying condition (I0) are, besides the trivial examples P(2, k+1) and PG(d,1), the finite semi-affine planes (in which i_0 is always zero).

Proof. Let S be a finite non-trivial linear space satisfying condition (I0). Note first that if S is the union of two intersecting lines, then S is either a degenerate projective plane or AG(2,2) with one point at infinity. On the other hand, as mentioned in section 3, conditions (I0) and (I2) are equivalent for Steiner systems $S(2,k,v)$, so that we may assume from now on that S contains two lines of distinct sizes and that every point of S has degree ≥ 3. The following lemmas will show that, under these assumptions, $i_o = 0$; this will make the proof of Theorem 7 easier.

Lemma 7.1. If S *contains a point* x *of degree 3, then* $i_o = 0$.

Proof. Let A,B,C be the three lines passing through x. If $i_o \neq 0$, there would be a line D disjoint from $B \cup C$ through a point y of A distinct from x, so that $r_x \geq 4$, a contradiction.

In the following lemmas, we assume that every point of S has degree ≥ 4.

Lemma 7.2. If A,B,C *are three pairwise intersecting lines of size* a,b,c *respectively, then*

$$r_A - r_B = (a-b)(c+i_o) + r_{A\cap C} - r_{B\cap C}$$

In particular, if A,B,C *are concurrent, then*

$$r_A - r_B = (a-b)(c+i_o)$$

Proof. Counting in two ways the number of bisecants of A and B which are disjoint from C, we get

$$\sum_{\substack{y\in A\\ y\notin B\cup C}} i_1(y,B,C) = \sum_{\substack{z\in B\\ z\notin A\cup C}} i_1(z,A,C)$$

from which we immediately deduce the desired formulas by using

$$i_1(y,B,C) = r_y - c - i_o$$

and

$$i_1(z,A,C) = r_z - c - i_o .$$

Corollary 7.2.

(i) *If* A *and* B *are two intersecting lines of the same size, then* $r_A = r_B$.

(ii) *For any point* x *of* S, *all lines passing through* x, *except possibly one, have the same size.*

(iii) *If three points* x,y,z *are such that the lines* A = <x,z> *and* B = <y,z> *are distinct and have the same size, then* $r_x = r_y$.

Proof. Since we have assumed that all points of S have degree ≥ 4, there is a line C concurrent with the two intersecting lines A and B. (i) follows immediately from Lemma 7.2 applied to A, B and C.

In order to prove (ii), suppose that A and B are two lines of distinct sizes a and b passing through x (which, by hypothesis, has degree ≥ 4). If C and C' are two lines passing through x and distinct from A and B, it follows immediately from Lemma 7.2 that C and C' have the same size c. On the other hand, if $c \neq b$, then the same argument applied to C' and A which are distinct from the two lines B and C of distinct sizes shows that $c = a$, and so (ii) is proved.

In order to prove (iii), note first that $r_A = r_B$ by (i). Then Lemma 7.2 applied to the three lines A, B and C = <x,y> yields $r_{A\cap C} - r_{B\cap C} = 0$, that is $r_x = r_y$.

Lemma 7.3. Let p *be a point of* S. *If* B *is the only line of size* b *passing through* p, *then*

(i) *any two points outside* B *have the same degree,*

(ii) *any two lines distinct from* B *and concurrent with* B *have the same size.*

Proof. Let x and y be two points outside B such that x, y and p are not collinear. By Corollary 7.2 (ii) all lines distinct from B through p have the same size a, and so we deduce from Corollary 7.2. (iii) that x and y have the same degree. This implies that all points outside B have the same degree r.

Now let C and D be two lines distinct from B and intersecting in a point $z \in B$. If $z = p$, we already know that C and D have the same size. If $z \neq p$, let A (resp. A') be a line through p, distinct from B and intersecting C (resp. D). Then, by Lemma 7.2 and Corollary 7.2 (i), we get

$$r_A - r_B = (a-b)(c+i_0) + r - r_z$$

and

$$r_A - r_B = (a-b)(d+i_0) + r - r_z .$$

Since $a \neq b$, these equalities imply $c = d$.

Lemma 7.4. If every point of S *has degree* ≥ 4, *then* $i_0 = 0$

Proof. As we have seen at the beginning, we may assume that S contains two lines A and B having distinct sizes a and b respectively and intersecting in a point p.

Moreover, by Corollary 7.2 (ii), we may also assume that all lines distinct from B and containing the point p have size a.

Suppose first that every line intersecting B and not passing through p has size $\neq$ a. Then, if $b > 2$, any point $x \notin B$ is on at least two lines of size $\neq$ a. Therefore, by Corollary 7.2 (ii), $A(x) = \langle x,p \rangle$ is the only line through x having size a. From Lemma 7.3 (ii), we deduce that any two lines distinct from A(x) and concurrent with A(x) have the same size. Since p has degree ≥ 4 and $p \in A(x)$, this contradicts the fact that B is the only line of size b containing p. This shows at the same time that $b = 2$ (let $B = \{p,q\}$) and that any point $x \notin B$ is on at least two lines of size a. Since by hypothesis, the size c(x) of the line $C(x) = \langle x,q \rangle$ is distinct from a, we deduce from Corollary 7.2 (ii) that C(x) is the only line of size c(x) containing x. Therefore, by Lemma 7.3 (i), any point $y \notin C(x)$ has degree

$$r_y = r_p.$$

Since B is the only line of size $\neq$ a containing p, we have

$$v-1 = (r_p-1)(a-1) + (b-1) ,$$

and since C(y) is the only line of size $\neq$ a containing y, we have

$$v-1 = (r_y-1)(a-1) + (c(y)-1) .$$

Since $b = 2$, these three equations imply that

$$c(y) = b = 2 ,$$

from which we deduce that all lines containing q have size 2, all other lines of S having size a. Therefore $S - \{q\}$ is a Steiner system $S(2,a,v-1)$ with point degree $r' = (v-2)/(a-1)$.
Let $C \neq B$ be a line of S passing through q. If u is a point of A outside $B \cup C$, then $i_0(u,B,C) = r'-1$; but if v is a point outside $A \cup B \cup C$, then $i_0(v,B,C) = r'-2$, and so condition (IO) is not satisfied, a contradiction.

Suppose now that there is a line intersecting B, not passing through p and having size a. Then, by Lemma 7.3 (ii), there is a bisecant C of A and B having size a. Lemma 7.2 applied to the triple (A,B,C) yields

$$r_A - r_B = (a-b)(a+i_0) + r - r_{B \cap C} \qquad (1)$$

where $r = r_{A \cap C}$ is, by Lemma 7.3 (i), the common degree of the points outside B. On the other hand, let A' be a line distinct from A and B and passing through $A \cap B$. Lemma 7.2 applied to the triple (A,B,A') yields

$$r_A - r_B = (a-b)(a+i_o) \qquad (2)$$

(1) and (2) together give

$$r = r_{B \cap C} \qquad (3)$$

Since A' and B are concurrent with A and have different sizes, Lemma 7.3 (ii) implies that every point $x \in A$ distinct from p is on at least r-1 lines of size a, so that

$$v-1 = (r-1)(a-1) + c-1 \qquad (4)$$

where c = a or is the size of the unique line of size $\neq$ a passing through x. On the other hand, we know by Lemma 7.3 (ii) that all lines distinct from B and passing through $B \cap C$ have size a, so that

$$v-1 = (r_{B \cap C} -1)(a-1) + b-1 \qquad (5)$$

(3), (4) and (5) together imply b = c. Therefore a $\neq$ c and (4) shows that every point outside B is on exactly one line of size b, all the other lines having size a. Either the lines of size b are pairwise disjoint or they are concurrent in a point $y \in B$. We shall successively consider these two possibilities.

If the lines of size b are pairwise disjoint, then by Lemma 7.3 (i), all points of S have the same degree r.Lemma 7.2 applied to the triple (A,B,A') yields

$$r(a-b) = (a-b)(a+i_o) \text{ ,}$$

and so, since a $\neq$ b, we get $r = a + i_o$, which means that for any point x outside A, every line through x which is disjoint from A is also disjoint from every line intersecting A and not passing through x. Therefore every line through x intersects A and $i_o = 0$.

If the lines of size b are concurrent in a point $y \in B$, then, by Lemma 7.3 (i) all points distinct from y have the same degree r. Lemma 7.2, applied to the triple (A,B,B'), where B' $\neq$ B is a line of size b intersecting A, yields

$$a\ r - (b-1)r - r_y = (a-b)(b+i_o) + r - r_y \text{ ,}$$

and so, since a $\neq$ b, we get $r = b + i_o$.
Thus, every line through a point outside B intersects B and $i_o = 0$.

We will now end easily the proof of Theorem 7. Indeed we have seen in Lemmas 7.1 and 7.4 that $i_o = 0$, which means that S contains no line disjoint from two intersecting lines. Therefore, for any line L, every point outside L is on at most one line disjoint from L, in other words, S is a semi-affine plane. Since we know (I6) that the finite semi-affine planes are, besides the finite affine planes and the (possibly degenerate) finite projective planes, the finite punctured projective planes and the finite affine planes with one point at infinity, and since each of these planes satisfies condition (I0), Theorem 7 is proved.

10. LINEAR SPACES SATISFYING CONDITION (ID).

We shall end this chapter with the study of finite linear spaces satisfying condition (ID), that is ($\mathcal{P}$, 1,2,1; 3,4,4) according to our conventions in section 2. Actually (ID) can be expressed in a simpler way (in terms of lines only) :

(ID) *there is a non-negative integer* δ *such that for any two intersecting lines* L *and* L', *there are exactly* δ *lines disjoint from* $L \cup L'$.

Using the work done in section 9, we shall prove the following

> *Theorem* 8[26]. *The finite non-trivial linear spaces satisfying condition* (ID) *are exactly the finite semi-affine planes and the Steiner systems* S(2,k,v) *with* $k < v$.

Proof. Let A and B be two intersecting lines of a finite non-trivial linear space S, and let $\delta(A,B)$ denote the number of lines disjoint from both A and B. The proof is based on the following counting argument

$$|\mathcal{L}| = (r_A-a+1) + (r_B-b+1) - (r_{A\cap B} + (a-1)(b-1)) + \delta(A,B) \qquad (1)$$

where the first (resp. second) term on the right hand side counts the number of lines having a non-empty intersection with A (resp. B) and the third term counts the number of lines having a non-empty intersection with both A and B.

We immediately deduce from (1) that all Steiner systems S(2,k,v) satisfy condition (ID), since all terms different from $\delta(A,B)$ are independent of the choice of the two intersecting lines A and B. Moreover, it is obvious from their definition that the semi-affine planes are exactly the non-trivial linear spaces satisfying condition (ID) with $\delta = 0$. Therefore, it remains only to prove that if S is a finite non-trivial linear space satisfying condition (ID) and containing two lines of different sizes, then necessarily $\delta = 0$. We shall prove this in the following lemmas, which are similar to those of section 9.

Lemma 8.1. If S *contains a point* x *of degree* ≤ 3, *then* $\delta = 0$.

Proof. Let A and B be two lines intersecting in x. If $\delta \neq 0$, then there is a line C disjoint from A ∪ B, so that $r_x \geq 4$, a contradiction.
In the following lemmas, we assume that every point of S has degree ≥ 4.

Lemma 8.2. If C *intersects* A *and* B, *then*

$$r_A - r_B = (a-b)c + r_{A \cap C} - r_{B \cap C}$$

In particular, if A, B, C *are concurrent, then*

$$r_A - r_B = (a-b)c.$$

Proof. Since A and C intersect, (1) yields

$$|\mathcal{L}| = (r_A - a + 1) + (r_C - c + 1) - (r_{A \cap C} + (a-1)(c-1)) + \delta \qquad (2)$$

Since B and C intersect, we have similarly

$$|\mathcal{L}| = (r_B - b + 1) + (r_C - c + 1) - (r_{B \cap C} + (b-1)(c-1)) + \delta \qquad (3)$$

By subtracting (2) from (3), the lemma is proved.

Corollary 8.2. Identical to Corollary 7.2, both in statement and proof (it suffices to replace "Lemma 7.2" by "Lemma 8.2").

Lemma 8.3. Identical to Lemma 7.3, both in statement and proof.

Lemma 8.4. If all points of S *have degree* ≥ 4, *then* $\delta = 0$.

Proof. Since it is very similar to that of Lemma 7.4, we shall only indicate what has to be changed. At the end of the second paragraph, we conclude that $S' = S - \{q\}$ is a Steiner system $S(2,a,v-1)$ with point degree $r' = (v-2)/(a-1)$. Then $\delta(B,C)$ is the number of lines in S' which are disjoint from A ∩ B and A ∩ C, and $\delta(B,A)$ is the number of lines in S' which are disjoint from A. Therefore, if $b' = (v-1)(v-2)/a(a-1)$ denotes the total number of lines in S', we have

$$\delta(B,C) = b' - 2r' + 1$$

and

$$\delta(B,A) = b' - a\,r' + a-1$$

Since $\delta(B,C) = \delta(B,A) = \delta$ and $r' > 1$, we conclude that $a = 2$, so that all lines of S have the same size, a contradiction.

The third paragraph of the proof of Lemma 7.4 remains valid here, if we replace "Lemma 7.2" by "Lemma 8.2" and if we delete "i_0" in (1) and (2).

In the fourth paragraph, we suppose that the lines of size b are pairwise disjoint, so that, by Lemma 8.3 (i), all points of S have the same degree r. Then, Lemma 8.2 applied to the triple (A,B,A') yields

$$r(a-b) = (a-b)a\ ,$$

and so, since $a \neq b$, we get $r = a$. This means that, for any point x outside A, every line through x intersects A. Therefore $\delta = 0$.

Finally, in the fifth paragraph, we suppose that the lines of size b are concurrent in a point $y \in B$, and so, by Lemma 8.3 (i), all points distinct from y have the same degree r. Lemma 8.2, applied to the triple (A,B,B'), where $B' \neq B$ is a line of size b intersecting A, yields

$$a\,r - (b-1)r - r_y = (a-b)b + r - r_y\ .$$

Therefore, since $a \neq b$, we get $r = b$, which means that for any point x outside B, all lines through x intersect B, and so $\delta = 0$. This ends the proof.