CHAPTER I. DEFINITIONS AND BASIC RESULTS

1. LINEAR SPACES AND PLANAR SPACES

A *linear space* S is a non-empty set of elements, called *points*, provided with a family of distinguished subsets, called *lines*, such that any two points x and y are in exactly one line, denoted by $<x,y>$, and each line contains at least two points. S is called *non-trivial* if it has at least two lines, and it is called *finite* if it has only a finite number of points. The *size* of a line L is the number of points of L and the *degree* of a point x is the number of lines passing through x. We shall say that two lines *intersect* if they have exactly one point in common.

A *linear subspace* S' of S is a set of points of S such that any line of S having at least two points in S' is contained in S'. A linear subspace S' of S is called *proper* if $S' \neq S$.

A *planar space* is a linear space provided with a family of distinguished linear subspaces called *planes*, such that any three non-collinear points x,y,z are contained in exactly one plane, denoted by $<x,y,z>$, each plane containing at least three non-collinear points. A planar space is called *non-trivial* if it has at least two planes. If L is a line of S and if x is a point of S outside L, the unique plane containing L and x will be denoted by $<L,x>$. Similarly, if L and L' are two intersecting lines of S, the unique plane containing L and L' will be denoted by $<L,L'>$. Note that if x,y,z are non-collinear points, the plane x,y,z is not necessarily the smallest linear subspace of S containing x,y,z. Actually, any non-trivial linear space can be a plane of some non-trivial planar space.

We shall say that a line and a plane *intersect* if they have exactly one point in common. A *planar subspace* S' of S is a linear subspace S' of S such that any plane of S having at least three non-collinear points in S is contained in S'.

If X is a non-empty subset of a linear space S, the *linear space induced on* X *by* S is the linear space whose points are the points of X and whose lines are the intersections with X of the lines of S having at least two points in X. Similarly, if S is a planar space and X is a non-empty subset of S, the *planar space induced on* X *by* S is the planar space whose points are the points of X, whose lines are the intersections with X of the lines of S having at least two points in X, and whose planes are the intersections with X of the planes of S having at least three non-collinear points in X.

A *circular space* G is a non-empty set of elements called *points*, provided with a family of distinguished subsets called *circles*, such that any three points are in exactly one circle and any circle contains at least three points. A planar space S whose lines are the unordered pairs of points may be viewed as a circular space in which the circles are the planes of S, and conversely.

2. STEINER SYSTEMS AND t-DESIGNS.

A *Steiner system* S(t,k,v) (where t,k,v are integers with $2 \leqslant t \leqslant k \leqslant v$) is a finite set of v elements, called *points*, provided with a family of distinguished k-subsets, called *blocks*, such that any t points are contained in exactly one block. The Steiner systems S(2,k,v) are the finite linear spaces of v points in which all lines (or blocks) have the same size k.

A *projective plane* is a linear space in which any two lines intersect and which has at least two lines of size $\geqslant 3$. It is well-known that in a finite projective plane S, all the lines have the same size n+1, where $n \geqslant 2$ is called the *order* of the projective plane, so that S is an $S(2, n+1, n^2 + n + 1)$. Conversely, any Steiner system with these parameters is a projective plane of order n. A *generalized projective plane* is a non-trivial linear space in which any two lines intersect. The generalized projective planes which are not projective planes are called *degenerate projective planes* and consist of a line L and a point x outside L, all lines joining x to a point of L having size 2. An *affine plane* is a non-trivial linear space in which for any line L, any point outside L is on exactly one line disjoint from L. The affine planes of *order* n are exactly the Steiner systems $S(2, n, n^2)$.

The Steiner systems S(3,k,v) are the finite circular spaces of v points in which all circles have k points. We shall also use this notation for the planar spaces in which all lines have two points, the planes being the blocks of the Steiner system. An *inversive plane of order* n is an $S(3, n+1, n^2+1)$.

More generally, a *t-design* $S = S_\lambda(t,k,v)$ (where $1 \leqslant t \leqslant k \leqslant v$ and $1 \leqslant \lambda$) is a finite set of v elements, called *points*, provided with a family of k-subsets, called *blocks*, such that any t points are in exactly λ blocks (a t-design with $\lambda=1$ is a Steiner system). For any set I of i points ($0 \leqslant i \leqslant t-1$), the i-th *derived design of* S *with respect to* I is the (t-i)-design $S_\lambda(t-i, k-i, v-i)$ whose points are the points outside I and whose blocks are the restrictions to S-I of the blocks containing I.

I1) A well-known necessary condition for the existence of an $S_\lambda(t,k,v)$ is that, for any integer i with $0 \leq i \leq t$,

$$\lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}$$

is an integer, since this expression counts the number of blocks containing any given set of i points. In particular, any point is on exactly

$$r = \lambda \binom{v-1}{t-1} / \binom{k-1}{t-1}$$

blocks, and the total number of blocks is

$$b = \lambda \binom{v}{t} / \binom{k}{t}$$

If S is a 2s-design $S_\lambda(2s, k,v)$ where $s \in \mathbb{N}_0$ and $v \geq k+s$, Wilson and Ray-Chaudhuri [49] have proved that

$$b \geq \binom{v}{s}$$

S is said to be *tight* if equality holds. Examples are given by the Steiner system S(4, 7, 23) and its complement the S_{52} (4, 16, 23).
If S is a (2s+1)-design $S_\lambda(2s+1, k,v)$ where $s \in \mathbb{N}_0$ and $v-1 \geq k+s$, then the preceding result can be applied to the first derived designs of S, so that

$$b \geq \binom{v-1}{s} v / k$$

S is said to be *tight* if equality holds. The Steiner system S(5, 8, 24) is tight.

I2) In particular, any 2-design with $v \geq k+1$ satisfies *Fisher's inequality* $b \geq v$, which is equivalent to $k \leq r$ since $bk = vr$. A *symmetric 2-design* is a tight 2-design, that is a 2-design for which v=b (or equivalently k=r). The symmetric 2-designs with $\lambda = 1$ are exactly the projective planes $S(2, n+1, n^2+n+1)$, including the degenerate projective plane of 3 points.

I3) The Bruck-Chowla-Ryser theorem gives an important necessary condition for the existence of symmetric 2-designs, namely :

(i) if v is even, $r-\lambda$ should be a square,

(ii) if v is odd, the Diophantine equation

$$(r-\lambda)x^2 + (-1)^{(v-1)/2} y^2 = z^2$$

should have a solution in integers x,y,z not all zero [37]

More generally, a t-design S is called *symmetric* if its (t-2)th derived designs are symmetric 2-designs.

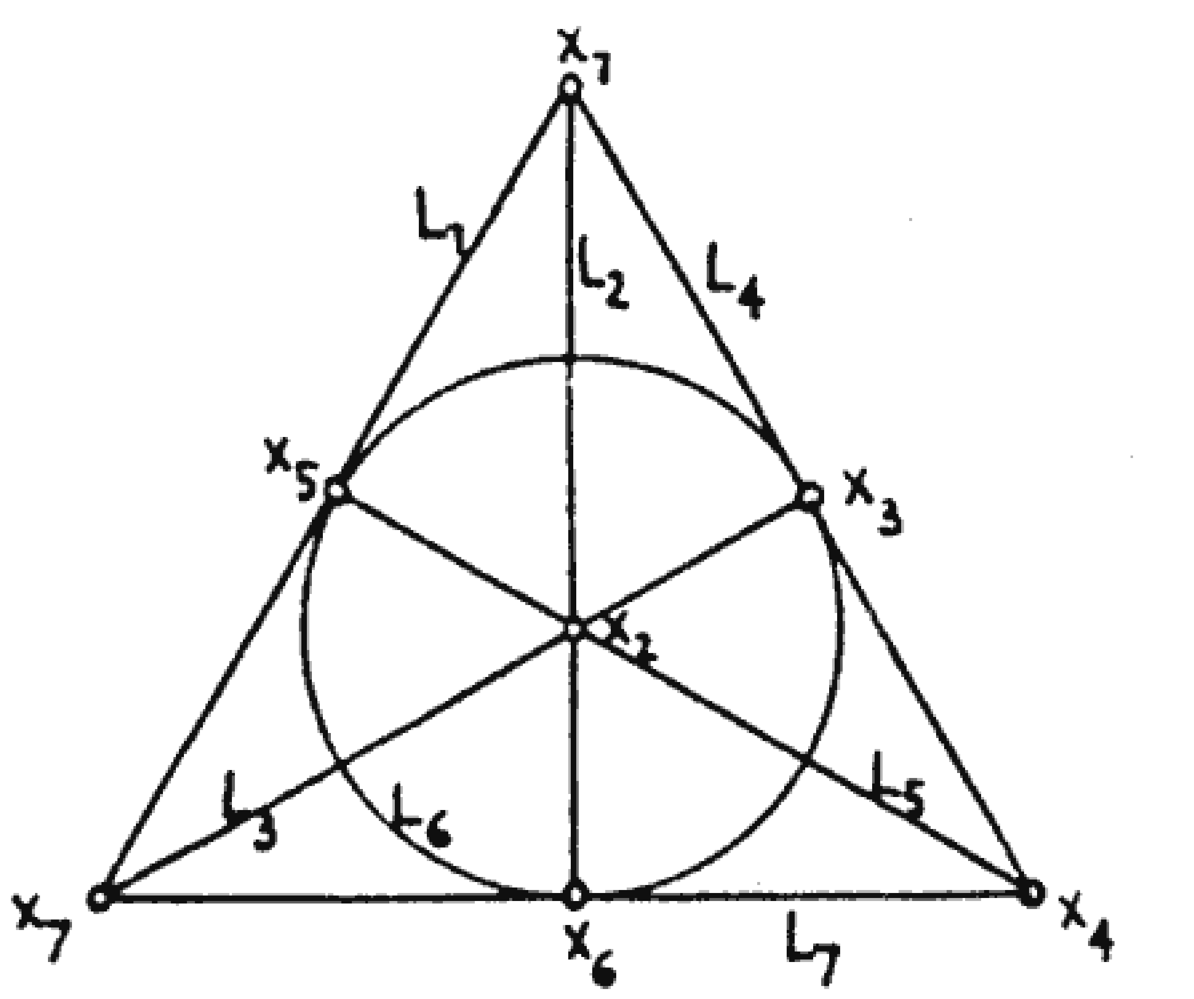
(I4) Cameron [13] has proved that the parameters of a symmetric 3-design are necessarily in the following list

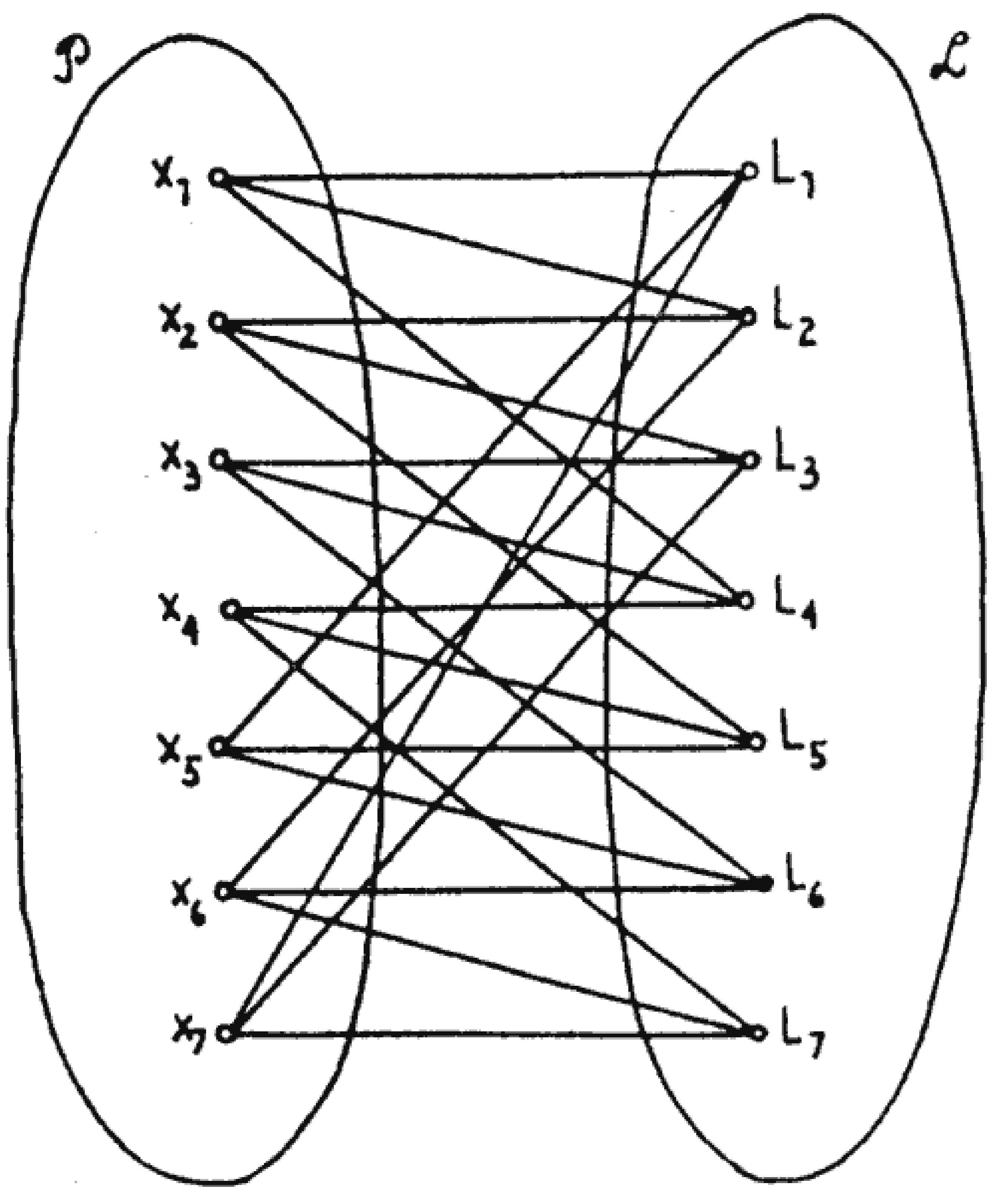
(a) $v = \lambda+3$, $k = \lambda+2$
(b) $v = 4(\lambda+1)$, $k = 2(\lambda+1)$
(c) $v = (\lambda+1)(\lambda^2 + 5\lambda+5)$, $k = (\lambda+1)(\lambda+2)$
(d) $v = 112$, $k = 12$, $\lambda = 1$
(e) $v = 496$, $k = 40$, $\lambda = 3$.

3. INCIDENCE STRUCTURES AND DIAGRAMS.

A linear space may be viewed as an incidence structure $S = (\mathcal{P}, \mathcal{L}, I)$ where $\mathcal{P}$ denotes the set of points of S, $\mathcal{L}$ denotes the set of lines of S and I is a symmetric relation between $\mathcal{P}$ and $\mathcal{L}$: a point x and a line L are incident, which is written x I L, if and only if $x \in L$. A *flag* (resp. *antiflag*) of a linear space S is a point-line incident (resp. non-incident) pair. In the definition of a linear space given in the first section, we have used the same notation for the linear space and for its set of points and we have identified every line L with the set of points incident with L. We shall use both points of view and terminologies.

The *incidence graph* $\mathcal{G}(S)$ of a linear space S is the bipartite graph whose vertices are the elements of $\mathcal{P} \cup \mathcal{L}$, two vertices being adjacent if and only if they are incident. The figure below shows the incidence graph of the projective plane PG(2,2).





The definition of a non-trivial linear space can of course be translated in terms of its incidence graph $\mathcal{G}$. In particular, any two points are at distance 2 in $\mathcal{G}$ and any vertex of $\mathcal{G}$ has degree ≥ 2. Note that the Steiner systems $S(2,k,v)$ may be defined as the linear spaces such that in their incidence graph, all lines (and therefore all points) have the same degree k (resp. r).

Similarly, a planar space S may be viewed as an incidence structure $S = (\mathcal{P}, \mathcal{L}, \Pi, I)$ where $\mathcal{P}$, $\mathcal{L}$ and Π denote respectively the set of points, lines and planes of S, and where I is the incidence relation, two elements of $\mathcal{P} \cup \mathcal{L} \cup \Pi$ being incident if and only if, considered as point sets, one of them is strictly contained in the other. The incidence graph $\mathcal{G}(S)$ of a planar space is then a tripartite graph. A *flag* of S is a set of pairwise incident elements of $\mathcal{P} \cup \mathcal{L} \cup \Pi$. A *maximal flag* is a set $\{x, L, \pi\}$ where $x \in \mathcal{P}$, $L \in \mathcal{L}$, $\pi \in \Pi$ and $x \; I \; L \; I \; \pi$.

The *residue* S_v of a vertex v of $\mathcal{G}(S)$ is the incidence structure whose incidence graph is the restriction of $\mathcal{G}(S)$ to the neighbourhood of v in $\mathcal{G}(S)$. For example, the residue S_π of a plane π of S is nothing else than the linear space π itself. The residue S_L of a line L of S consists of the points of L and the planes containing L, each of these planes being automatically incident with each of these points, so that the incidence graph of S_L is a complete bipartite graph. Such incidence structures are called *generalized digons*. The residue S_x of a point x is a linear space whose points are the lines containing x, whose lines are the planes containing x and in which a point L_x corresponding to a line L and a line π_x corresponding to a plane π are incident if and only if $L \subset \pi$. The fact that S_x is a linear space follows essentially from the fact that any two intersecting lines of S are contained in exactly one plane.

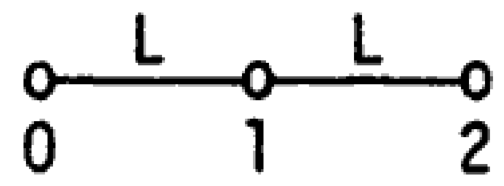
Linear spaces and planar spaces may be represented by diagrams in the sense of Buekenhout [11]. For definitions and conventions concerning incidence structures admitting diagrams, we refer the reader to [11]. In particular, the class of non-trivial linear spaces is represented by the diagram

$$\circ\overset{L}{\rule{3cm}{0.4pt}}\circ$$

where the left dot represents the set $\mathcal{P}$ of points and the right dot represents the set $\mathcal{L}$ of lines. The class of generalized digons is represented by the diagram

$$\circ \qquad\qquad \circ$$

Then, according to the conventions of Buekenhout, the class of non-trivial planar spaces is represented by the diagram



expressing the fact that non-trivial planar spaces are incidence structures with three sets of objets $\mathcal{P}$, $\mathcal{L}$ and Π, represented respectively by the three dots 0, 1 and 2, such that the residue of any element of $\mathcal{P} \cup \Pi$ is a non-trivial linear space and that the residue of any element of $\mathcal{L}$ is a generalized digon.

This symbolic representation suggests several problems by considering various subclasses of the class of linear spaces. For example, if the class of projective planes of order n is denoted by

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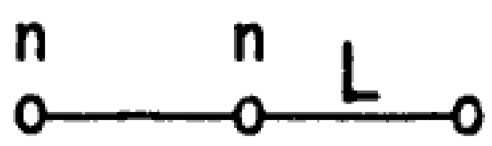
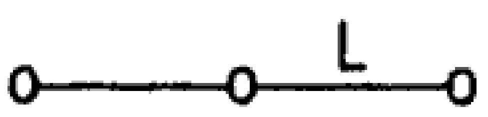
then the incidence structures admitting the diagram



are the non-trivial planar spaces all of whose planes are projective planes of order n. Thanks to the classical work of Veblen and Young, we know that these are exactly the Desarguesian projective spaces PG(d,n) of dimension $d \geq 3$, endowed with all their planes. More generally, the class of generalized projective planes is denoted by

so that the diagram

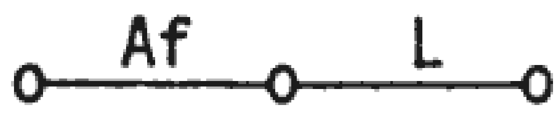


represents exactly the generalized projective spaces of dimension ≥ 3, endowed with all their planes. Remember that a *generalized projective space* is a linear space such that for every pair of lines L and L' intersecting in a point x, any two lines not passing through x and intersecting each of the lines L and L' intersect (*Pasch's axiom*).These spaces are unions of projective spaces of any dimension ≥ 0 joined together by lines of size 2. The planes of a generalized projective space are the smallest linear subspaces containing three non-collinear points. A generalized projective space has *dimension* n if there are n+1 points which are contained in no proper linear subspace of S and if any n points are contained in a proper linear subspace of S. We shall denote by P(2, k+1) the degenerate projective plane of k+1 points and by P(3, k+ℓ) the 3-dimensional generalized projective space consisting of two disjoint lines of size k and ℓ respectively, all the other lines having size 2.

From now on, PG(d,q) will not only denote the Desarguesian projective space of dimension d over GF(q) for $q \geq 2$, but also, for q=1, the generalized projective space PG(d,1) with d+1 points, in which all lines have size 2, which may be viewed as the d-dimensional projective space over "GF(1)". However AG(d,q) will always denote the d-dimensional Desarguesian affine space over GF(q) $(q \geq 2)$.

If 

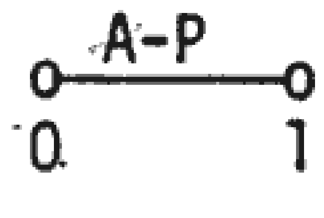
denotes the class of affine planes, then the diagram



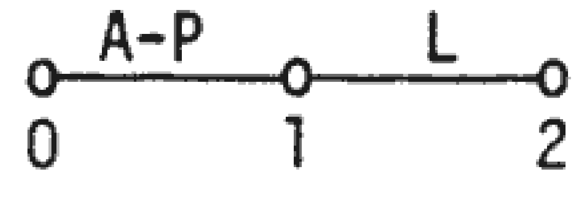
represents the class of non-trivial planar spaces all of whose planes are
(I5) affine planes. Buekenhout [9] has proved that these are exactly the affine spaces of order ≥ 4 if one of the planes has order ≥ 4. However, if the planes have order 3, these spaces, called *Hall triple systems*, are not necessarily affine spaces, as shown by Hall [36]. Finally, if the planes have order 2, then all Steiner systems S(3,4,v) yield planar spaces admitting the above diagram.

Let us mention a generalization of this situation. An *affino-projective plane* is a linear space Π obtained from a projective plane by deleting a set of points of a line L. The *order* of a finite affino-projective plane is the order of the initial projective plane. In particular, Π is a projective plane if we delete no point, Π is a *punctured projective plane* if we delete just one point, Π is an *affine plane with one point at infinity* if we delete all points of L except one, and Π is an affine plane if we delete all points of L. The planes of these four types are examples of *semi-affine planes*, that is non-trivial linear spaces in which for any line L, any point outside L is on
(I6) at most one line disjoint from L. Dembowski [30] and Kuiper [31, p.310] have proved that conversely any finite semi-affine plane is of one of these four types or is a degenerate projective plane. Actually, Dembowski assumes that all lines have size ≥ 3 and that all points have degree ≥ 3, but the classification of all finite semi-affine planes containing a line of size 2 or a point of degree 2 is very easy.

Let



represent the class of affino-projective planes. Teirlinck [53] has proved that if S corresponds to the diagram



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if S contains at least one plane of order ≥ 4 and contains a finite subset of points which is not contained in any proper linear subspace of S, then S is an *affino-projective space*, that is a projective space from which a subset of a hyperplane has been deleted. Note that a plane of an affino-projective space is not necessarily an affino-projective plane.

Dual problems arise in a natural way. For example, what can be said about the planar spaces with diagram

$$\circ\overset{L}{\text{———}}\overset{n}{\circ}\text{———}\overset{n}{\circ}\quad ,$$

that is the non-trivial planar spaces in which the residue of every point is a projective plane of order n (these spaces are sometimes said to be locally a projective plane) ? This question seems to be rather difficult to answer. Kantor [11] has conjectured that a finite non-trivial planar space which is locally a projective plane is necessarily obtained by deleting a set of points from a 3-dimensional projective space.

(I7) A partial answer to this question has been given by Doyen and Hubaut [32] who proved that if S is a finite planar space with diagram

$$\circ\overset{L}{\text{———}}\circ\text{———}\circ$$

and if all lines of S have the same size k, then S is either PG(3, k-1) or AG(3,k) or a *Lobachevsky space* of type k^2-k+1 (resp. k^3+1), that is a planar space such that for any line L and any point x outside L, the number of lines of <x,L> which are disjoint from L and contain x is equal to k^2-k+1 (resp. k^3+1). The only known example of this latter class of spaces is the planar space S(3, 6, 22).

Similarly, the problem of classifying the planar spaces S with diagram

$$\circ\overset{L}{\text{———}}\circ\overset{Af}{\text{———}}\circ$$

has been solved by Cameron [14], Brouwer and Wilbrink [8] under the additional assumptions that S is finite and that all lines of S have the same size k : then k=2 and S is a planar space S(3, n+1, n^2+1), in other words an inversive plane of order n.

Observe that, though the class of incidence structures admitting the diagram

$$\circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ$$

is very wide and contains rather wild spaces, the incidence structures admitting the diagram

$$\underset{\mathcal{P}}{\circ}\overset{J}{\text{———}}\underset{\mathcal{L}}{\circ}\overset{L}{\text{———}}\underset{\Pi}{\circ}$$

have been classified by Sprague [52] under the additional assumption that some point is on finitely many lines or some line has finitely many points. Such an incidence structure, in which the residue of each point is a linear space, the residue of each line is a generalized digon and the residue of each plane is the dual of a linear space is, for some integer i, the incidence structure having as varieties all (i-1)-, i- and (i+1)- dimensional subspaces of some generalized projective space, and inclusion as incidence.

Finally, note that, on the contrary, the problem of classifying the incidence structures admitting the diagram

$$\circ\overset{L}{\text{———}}\circ\overset{J}{\text{———}}\circ$$

is still more general than that of classifying the incidence structures admitting diagram

$$\circ\overset{L}{\text{———}}\circ\text{———}\circ$$

and so, seems to be hopeless.

4. PLANAR SPACES WITH ISOMORPHIC PLANES.

Let L^o be a given linear space, which we shall represent by

$$\circ\overset{L^o}{\text{———}}\circ$$

Then

$$\circ\overset{L^o}{\text{———}}\circ\overset{L}{\text{———}}\circ$$

denotes the non-trivial planar spaces in which every plane is isomorphic to L^o. These spaces, called L^o-*spaces*, were introduced by Buekenhout and Deherder [12] and have been investigated by Brouwer [7], Leonard [42] and myself [19],[22].

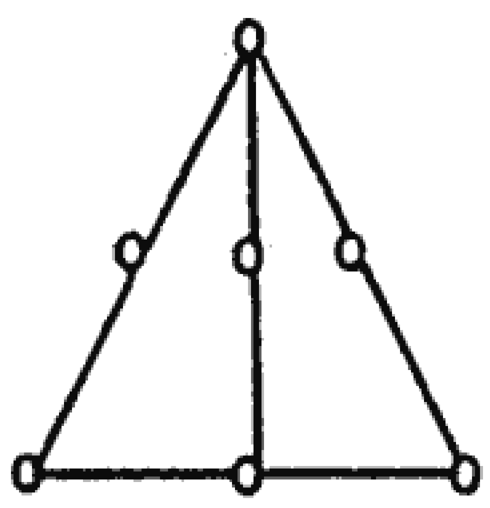
The only known finite L^o-spaces are the Desarguesian projective or affine spaces of any dimension ≥ 3, the Hall triple systems, the planar spaces $S(3, |L^o|, v)$ where all lines have size 2, and finally the 3-dimensional generalized projective spaces $P(3, k+k)$ consisting of two disjoint lines of size k (here all planes are degenerate projective planes $P(2, k+1)$). The spaces $P(3, k+k)$ are also the only known finite L^o-spaces with lines of different sizes. In [19] and [42], some rather restrictive relations on the parameters

of a finite L^o-space with different line sizes have been given, which are particularly strong if there are more than two line sizes, so that this possibility seems to be very improbable.
In particular, it was shown that the number v of points of a finite L^o-space having at least two distinct line sizes k and ℓ is uniquely determined by the following parameters of L^o : $v' = |L^o|$, k, ℓ, the number b'_k (resp. b'_ℓ) of lines of size k (resp. ℓ) in L^o, the number n_{kk} of ordered pairs of intersecting lines of size k and the number $n_{k\ell}$ of pairs of intersecting lines (one of size k and the other of size ℓ) in L^o :

$$v = v' + (v'-\ell)(v'-k)\,\frac{b'_k\, b'_\ell\, k\, \ell + n_{kk}\, b'_\ell\, \ell - n_{k\ell}\, b'_k\, k}{n_{k\ell}\, b'_k\, k(v'-k) - n_{kk}\, b'_\ell\, \ell(v'-\ell)}$$

This situation is very different from what we get if L^o is a Desarguesian projective or affine plane.

The smallest linear space L^o having at least two line sizes for which the existence of an L^o-space is still unsettled is represented below.



(the lines of size 2 are not drawn)

We have proved in [31] that the corresponding L^o-space, which would have 47 points, is rigid, i.e. has no other automorphism than the identity.

(I8) We shall use later the fact, proved in [7], [19] and in [42], that in any finite L^o-space, all points are on the same number r_k of lines of a given size k.

Very little is known about the dual problem, namely the classification of planar spaces with diagram

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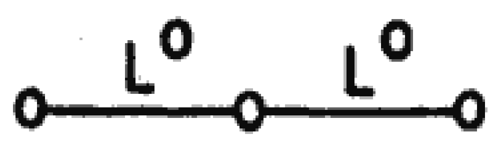
even in the finite case. However, the finite spaces with diagram



can be classified completely :

Theorem 1. [21] *Let* L^0 *be a given linear space. If* S *is a finite planar space in which every plane is isomorphic to* L^0 *and the residue of every point is isomorphic to* L^0*, then* L^0 *is a (possibly degenerate) projective plane and* S *is either a projective space* PG(3,q) *or a generalized projective space* P(3, k+k).

Lemma 1.1. The set K *of sizes of lines of* L^0 *is equal to the set of degrees of points of* L^0.

Proof. Let x be any point of S. Since S_x is isomorphic to L^0, K coincides with the set of degrees of x in the planes of S passing through x. Therefore, since all planes through x are isomorphic to L^0 and since K does not depend on x, K coincides with the set of degrees of points of L^0.

Lemma 1.2. If B *is a line of size* k *in* S *and if* $x \in B$*, then the point* B_x *of* S_x *has degree* k *in* S_x.

Proof. Let v (resp. v') denote the number of points in S (resp. in L^0). The degree of the point B_x in S_x is equal to the number of planes of S containing the line B, that is to $(v-k)/(v'-k) = f(k)$ which depends only on the size k of B. Since L^0 is isomorphic to S_x, Lemma1.1 implies that $f(K) = K$, and so $f(k) = k$ for every $k \in K$ because $f:K \to K:k \to f(k)$ is an increasing function and K is a finite set.

Proof of the theorem.

If $|K| = 1$, then all lines of L^0 have the same size k and, by Lemma1.1, all points of L^0 have degree k. Therefore L^0 is a projective plane of order k-1 (or a degenerate projective plane with 3 points if k=2), and so S is PG(3, k-1) with $k-1 \geq 1$.

If $|K| \geq 2$, let B be a line of maximal size $k \geq 3$ in L^0. All points of L^0 outside B have a degree greater than or equal to k, and so, by Lemma1.1, their degree is equal to k.

If there are at least two points of L^0 outside B, then, by Lemma1.2, every point of S is on at least two lines of size k. These two lines are in a plane of S isomorphic to L^0, and so L^0 contains two lines of size k intersecting in a point y. Since every point of L^0 outside a line of size k is of degree k, the point y is the only point of L^0 having a degree $\ell < k$, and $K = \{k,\ell\}$ by Lemma1.1. Lemma1.2 again implies that every point of S is on exactly one line of size ℓ, and so the lines of size ℓ are necessarily disjoint in L^0. Therefore, since the degree of any point z of L^0 distinct from y is $k \geq 3$, and since z is on at most one line of size ℓ, there is at least one line of size k containing z but not y, and so the degree ℓ of y is at least k, contradicting $\ell < k$.

This proves that there is only one point of L^0 outside B. In other words, L^0 is a degenerate projective plane with k+1 points. By Lemma 1.2, every point of S is on exactly one line of size k and k lines of size 2. This implies that S is the union of two lines of size k, all the other lines having size 2.

5. n-DIMENSIONAL LINEAR SPACES.

Linear and planar spaces belong to the wider class of incidence structures with diagram

$$\underset{0}{\circ}\overset{L}{\text{———}}\underset{1}{\circ}\overset{L}{\text{———}}\underset{2}{\circ}\ \cdots\ \underset{n-2}{\circ}\overset{L}{\text{———}}\underset{n-1}{\circ} \qquad (n \geq 2)$$

corresponding exactly to what Buekenhout calls n-*dimensional linear spaces* [11]; they will appear in the last chapter of this thesis. An n-*dimensional linear space* is a linear space S provided with n disjoint families $\mathcal{S}_i$ ($i = 0,1, \ldots, n-1$) of non-empty proper linear subspaces, called i-*subspaces*, or more generally *subspaces*, such that

(i) points and lines are the 0-subspaces and the 1-subspaces respectively,

(ii) if V is an i-subspace ($i \leq n-2$) and $x \in S-V$, then there is a unique (i+1)-subspace containing V and x, denoted by $\langle V,x \rangle$,

(iii) if V is an i-subspace and if W is a j-subspace containing V, then $i < j$,

(iv) any intersection of subspaces is a subspace or the empty set or S itself.

The 2-dimensional and 3-dimensional linear spaces are nothing else than the non-trivial linear spaces and planar spaces respectively. The notion of n-dimensional linear space is essentially the same as the notion of *geometric lattice of dimension* n (Birkhoff [3]), that is also the lattice of flats of a *simple matroid of rank* n+1 (Welsh [56]). The (n-1)-spaces of an n-dimensional linear space are called *hyperplanes*.

Note that the use of the word dimension is a little confusing here. For example, a projective space P of dimension $d \geq 3$ is at the same time a (2-dimensional) linear space if we consider only its points and lines, a planar space (dimension 3) if we consider only its points, lines and planes, and an n-dimensional linear space ($2 \leq n \leq d$) if we consider all its linear subspaces of dimension $< n$. On the other hand, P could be a plane of some non-trivial planar space !

6. METRICALLY REGULAR GRAPHS, ASSOCIATION SCHEMES, GENERALIZED n-GONS AND PARTIAL GEOMETRIES.

We briefly recall here a few definitions and results which will be used in Chapter II.

A graph G (always assumed to be finite, undirected, without loops and multiple edges) is called *strongly regular* with parameters (v, k, λ, μ) (Bose[4]) if v denotes the number of vertices of G, if every vertex is adjacent to exactly k vertices and if the number of vertices adjacent to any two adjacent (resp. non-adjacent) vertices is λ (resp. μ). Moreover, we always assume the non-degeneracy condition $2 \leq k \leq v-3$. The adjacency matrix of G has eigenvalues k, $n-m$, $-m$ with multiplicities 1, f, v-1-f, where

$$n = \sqrt{(\mu-\lambda)^2 + 4(k-\mu)} \quad ,$$

$$m = (n + \mu - \lambda)/2 \quad ,$$

$$f = \frac{1}{2}\left(v-1 - \frac{2k-(v-1)(\mu-\lambda)}{n}\right)$$

(I9) An important consequence of this is the *integrality condition* (Bose[4]) f is a non-negative integer. Moreover, m and n are integers unless $v = 4\mu+1$, $k = 2\mu$, $\lambda = \mu-1$ (conference graphs)

Four other necessary conditions for the existence of a strongly regular graph G such that both G and its complement are connected (i.e. $1 < m < n$) are the following

(I10) *Krein condition* (Higman [39], Delsarte [28])
If $1 < m < n$, then

$$\mu(n-m(m-1)) \leq (m-1)(n-m)(n+m(m-1))$$

(I11) *Absolute bound* (Delsarte, Goethals and Seidel [29], Neumaier [47])
If $1 < m < n$, then

$$v \leq \frac{1}{2} f(f+3)$$

Moreover, if G is not a Smith graph [16], then $v \leq \frac{1}{2} f(f+1)$.

(I12) *μ-bound* (Neumaier [46])

If a strongly regular graph has smallest eigenvalue $-m$, $1 < m < n$, then

$$\mu \leq m^3 (2m - 3)$$

(I13) *Claw-bound* (Neumaier [46])

If G is a strongly regular graph with smallest eigeinvalue $-m$ (where $m > 1$ is an integer) and if $\mu \neq m(m-1)$, m^2, then

$$n \leq \frac{1}{2} m (m-1)(\mu+1) + m-1$$

Metrically regular graphs are a direct generalization of strongly regular graphs. A connected graph G is called *metrically regular* (or *distance-regular* (Biggs [2])) if the number of vertices at distance i from a vertex x depends only on i and not on x and if for any two vertices x and y at distance i, the number of vertices which are at distance j from x and at distance k from y depends only on the three distances i,j,k but not on the choice of x and y. It is easily shown that the metrically regular graphs of diameter 2 are exactly the connected strongly regular graphs. In turn, the notion of metrically regular graph can be generalized in two directions : that of t-metrically regular graph and that of association scheme, which we shall need later.

An *association scheme with* m *classes* on v objects (Bose and Shimamoto [6]) is a family of m binary symmetric relations satisfying the following conditions :

(i) any two objects are either 1-st, 2-nd, ..., or m-th associates

(ii) the number n_i of i-th associates of an object x is independent of x

(iii) for any two i-th associates x and y, the number p^i_{jk} of objects which are j-th associates of x and k-th associates of y is independent of the two i-th associates x and y.

Obviously, any metrically regular graph of diameter d may be seen as a d-class association scheme by calling i-th associates any two vertices which are at distance i.

Here again, we recall some necessary conditions for the existence of an m-class association scheme on v points, with parameters n_i and p^i_{jk}. Consider the vxv association matrices $A_j = (a^j_{xy})$ with entries $a^j_{xy} = 1$ if the vertices x and y are j-th associates and $a^j_{xy} = 0$ otherwise.

Let us denote by λ_{jk} the (not necessarily distinct) eigenvalues of A_j and by μ_k the multiplicity of λ_{jk} (it can be shown that μ_k does not depend on j).

I14) Then the following conditions are necessary for the existence of an association scheme with the above parameters :

(1) *Integrality condition* (Connor-Clatworthy [17])

$$\mu_k = v \,/\, (1 + \sum_{\ell=1}^{m} \lambda_{\ell k}^2 / n_\ell) \text{ are positive integers for } k = 1,\dots,m$$

(2) *Krein condition* (Higman [39], Delsarte [28])

$$0 \leq 1 + \sum_{k=1}^{m} \lambda_{kr}\, \lambda_{ks}\, \lambda_{kt} \,/\, n_k^2 \leq v^2/(\mu_r\, \mu_s)$$

for $1 \leq r,s,t \leq m$

(3) (Mathon [44]). If $n_i \geq 2$ and $p_{ij}^i \geq 1$, then

$$1 + \sum_{\ell=1}^{m} \max\,\{2\, p_{i\ell}^i - n_\ell + 2\, \delta_{i\ell}\,(1 - \delta_{ij}), 0\} \leq p_{ii}^j$$

Metrically regular graphs are deeply related to some interesting incidence structures. For instance, a *generalized n-gon* S (Tits [55]) is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ whose incidence graph $\mathcal{G}(S)$ is a bipartite graph on the two sets of vertices $\mathcal{P}$ and $\mathcal{L}$, in which any vertex has degree ≥ 2 and which has diameter n and girth 2n (i.e. the circuits of minimal length have length 2n). The elements of $\mathcal{P}$ are called *points* and the elements of $\mathcal{L}$ are called *lines*. It is well-known [55],[58] that if S is finite and if any vertex of $\mathcal{G}(S)$ has degree ≥ 3, then there is an ordered pair of integers (s,t), called the *order* of S, such that any line of S has size s+1 and any point of S is on exactly t+1 lines. The *point graph* (resp. *line graph*) of $S = (\mathcal{P}, \mathcal{L}, I)$ has vertex set $\mathcal{P}$ (resp. $\mathcal{L}$) and is obtained by calling two points (resp. two lines) adjacent if and only if they are collinear (resp. intersecting). The line graph of S is also the point graph of the dual of S. The point graph and the line graph of a finite generalized
I15) n-gon of order (s,t) are metrically regular with diameter $[\frac{n}{2}]$, so that the above-mentioned necessary conditions apply to generalized n-gons. For example, the Feit-Higman non-existence theorem for generalized polygons [33], stating that a finite generalized n-gon of order (s,t) with $s,t \geq 2$ does not exist if $n \neq 2,3,4,6$ or 8, can be deduced from the integrality condition [51] and some classical inequalities between powers of t and s follow from the Krein condition [51], [39].

On the other hand, a finite *partial geometry* (R,K,T) where $R \geq 2$, $K \geq 2$, $T \geq 1$ are integers is a set of elements called *points* together with a family of distinguished subsets called *lines* such that

(i) each point belongs to exactly R lines and every pair of points is contained in at most one line,

(ii) each line contains exactly K points

(iii) for any line L, any point outside L is on exactly T lines intersecting L.

In particular, the partial geometries with T=1 are exactly the generalized
(I16) quadrangles (or 4-gons) of order (s,t) = (K-1, R-1). It is obvious that the point graph and the line graph of a partial geometry are strongly regular, so that the above-mentioned necessary conditions apply to partial geometries.

Finally, let us mention another generalization of metrically regular graphs. A graph G is called t-*metrically regular* if for any $n \leq t$ and for any n-tuple of vertices $(x_1, x_2, \ldots, x_n)$, the number of vertices which are at distance δ_1 from x_1, δ_2 from $x_2, \ldots$, δ_n from x_n, depends only on the distances $\delta_1, \ldots, \delta_n$ and on the distances between the x_i's, but not on the choice of the n-tuple. So, the 2-metrically regular graphs are exactly the metrically regular graphs.