INTRODUCTION

Inspired by the linearity properties which are a common feature of most classical spaces, Libois has introduced the notion of linear space around 1960. A linear space is a set of elements called points, provided with a family of subsets, called lines, such that any two points are in exactly one line, each line containing at least two points. Buekenhout introduced then the notion of planar space, that is a linear space provided with a family of distinguished linear subspaces, called planes, such that any three non-collinear points are in exactly one plane, each plane containing at least three non-collinear points.

One of the reasons for the success of these notions is that projective and affine spaces may be entirely characterized by their linear (or planar) structure. Indeed, by the classical work of Veblen and Young at the beginning of this century, the projective spaces are exactly the linear spaces (endowed with all their linear subspaces) in which any two intersecting lines are contained in a linear subspace which is a projective plane. In other words, a planar space in which all planes are projective planes consists of the points, lines and planes of some projective space. On the other hand, by a theorem of Buekenhout (1969), the affine spaces of order ≥ 4 are exactly the linear spaces (endowed with all their linear subspaces) in which any two intersecting lines are contained in a linear subspace which is an affine plane of order ≥ 4. In other words, a planar space in which all planes are affine planes of order ≥ 4 consists of the points, lines and planes of some affine space (in order to characterize the affine spaces of order ≤ 3, the only additional requirement is the transitivity of the parallelism relation on lines).

Linear and planar spaces correspond respectively to the diagrams

$$\circ\overset{L}{\text{———}}\circ \quad \text{and} \quad \circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ$$

and are particular cases of the more general notion of n-dimensional linear space (or geometric lattice or simple matroid), corresponding to the diagram

$$\circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ \quad \cdots \quad \circ\overset{L}{\text{———}}\circ$$

The use of diagrams goes back to Dynkin and Coxeter. Diagrams for geometries, appearing around 1955 in Tits'geometric interpretation of the simple groups of Lie-Chevalley type, were later generalized by Buekenhout in an attempt to associate a geometry to the sporadic simple groups. The above mentioned point-line characterizations of the Desarguesian affine and projective spaces form only a small part of a recently achieved characterization of all Lie incidence geometries in terms of points and lines. This remarkable work, associated with the names Veldkamp, Tits, Buekenhout, Shult, Cohen and Cooperstein, is fascinating in the sense that very complex geometric structures are characterized by a few simple axioms involving only points and lines.

Let us go back to the elementary notion of linear space. If X is a non-empty subset of a linear space S, the linear space induced on X is defined as follows its points are those of X and its lines are the intersections of X with the lines of S having at least two points in X.

Clearly the class of finite linear spaces is much too wide to allow a complete classification : for example, the structure induced on any non-empty subset of a finite projective space is a linear space. This raises three types of problems

(i) find sufficient conditions for a finite linear space to be embeddable in some finite projective space,

(ii) classify the finite linear spaces satisfying some given combinatorial regularity condition,

(iii) classify the finite linear spaces whose automorphism group is transitive on some given configurations of points and lines.

Of course, similar types of problems may also be formulated for planar spaces.

These lecture notes will essentially focus on problems of type (ii), and the results will have some consequences for problems of type (i). Although this will not be our topic here, let us mention that much progress was made recently on problems of type (iii), mainly because of the classification of all finite simple groups.

In Chapter I, the reader will find the definitions of most of the notions appearing later, as well as some basic theorems (numbered in the left margin) which will be used in subsequent chapters. The only original result in Chapter I concerns L^o-spaces, i.e non-trivial planar spaces all of whose planes are isomorphic to some given linear space L^o, the corresponding diagram being

$$\circ \overset{L^o}{\text{———}} \circ \overset{L}{\text{———}} \circ$$

Finite L^o-spaces have been investigated by Buekenhout, Deherder, Brouwer, Leonard and myself, but examples other than projective or affine spaces or Steiner systems $S(3,k,v)$ seem to be extremely rare. On the other hand, very little is known about planar spaces with diagram

$$\circ \overset{L}{\text{———}} \circ \overset{L^o}{\text{———}} \circ$$

even in the case where L^o is a projective plane. We prove that a finite planar space admitting diagram

$$\circ \overset{L^o}{\text{———}} \circ \overset{L^o}{\text{———}} \circ$$

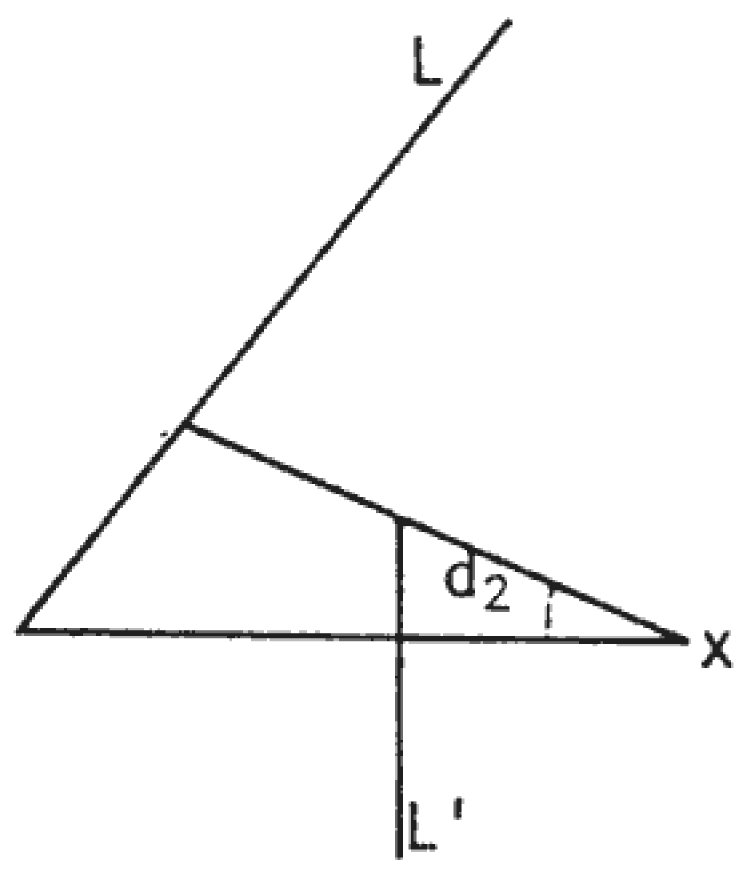
is necessarily a 3-dimensional generalized projective space.

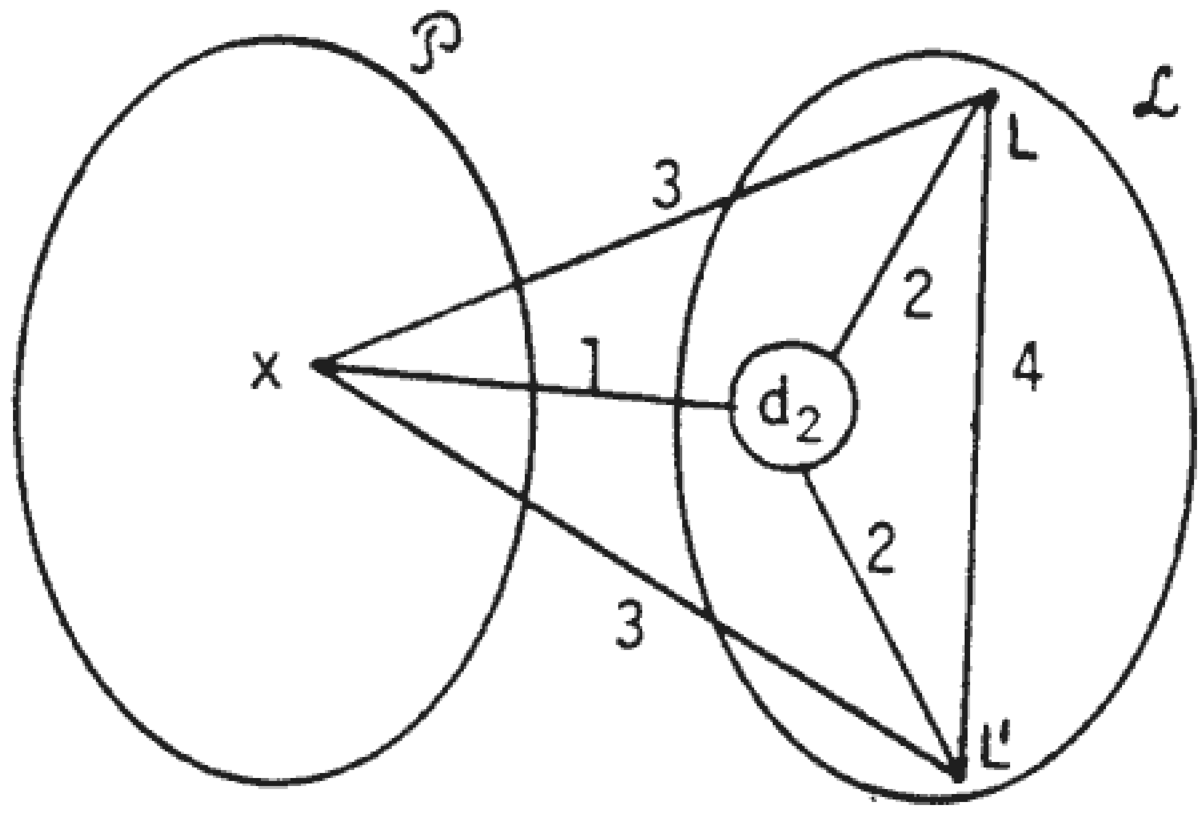
In Chapter II, starting from the observation that the generalized projective spaces of dimension ≤ 3 are characterized by the fact that for any two disjoint lines L and L', any point outside $L \cup L'$ is on exactly one line intersecting both L and L', we completely classify (in a joint work with A. Beutelspacher) the finite linear spaces S satisfying the following condition

(D2) there is an integer d_2 such that for any pair of disjoint lines L and L', any point outside $L \cup L'$ is on exactly d_2 lines intersecting both L and L'.
Such a space S is a generalized projective space of dimension ≤ 3, or PG(d,1) with $d \geq 4$, or a semi-affine plane or a small exceptional space with 7 points.

Condition (D2) can be interpreted in terms of metrical regularity in the incidence graph of S, as suggested by the figures below, which are self-explanatory :

(D2)

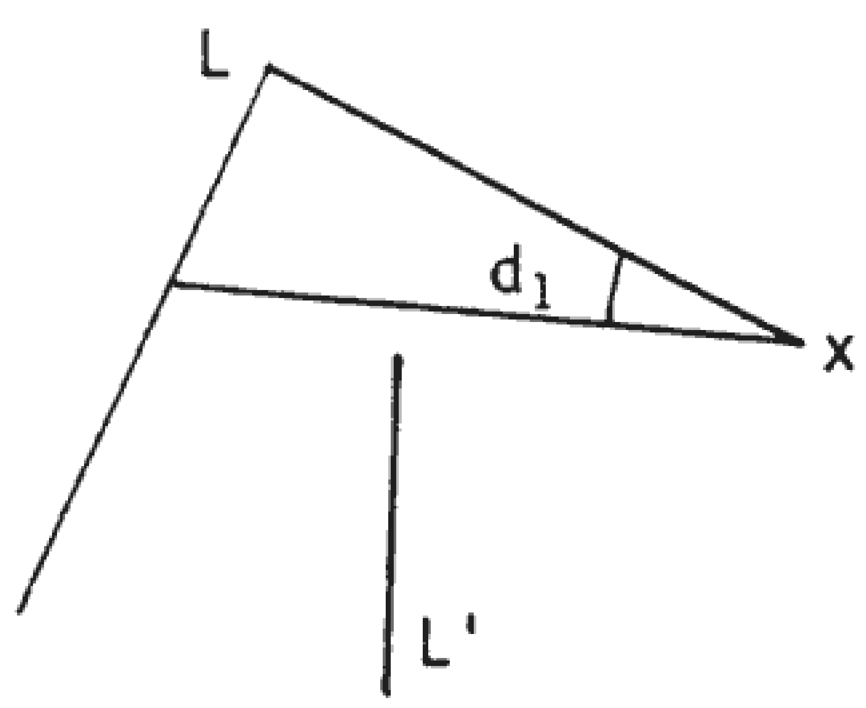


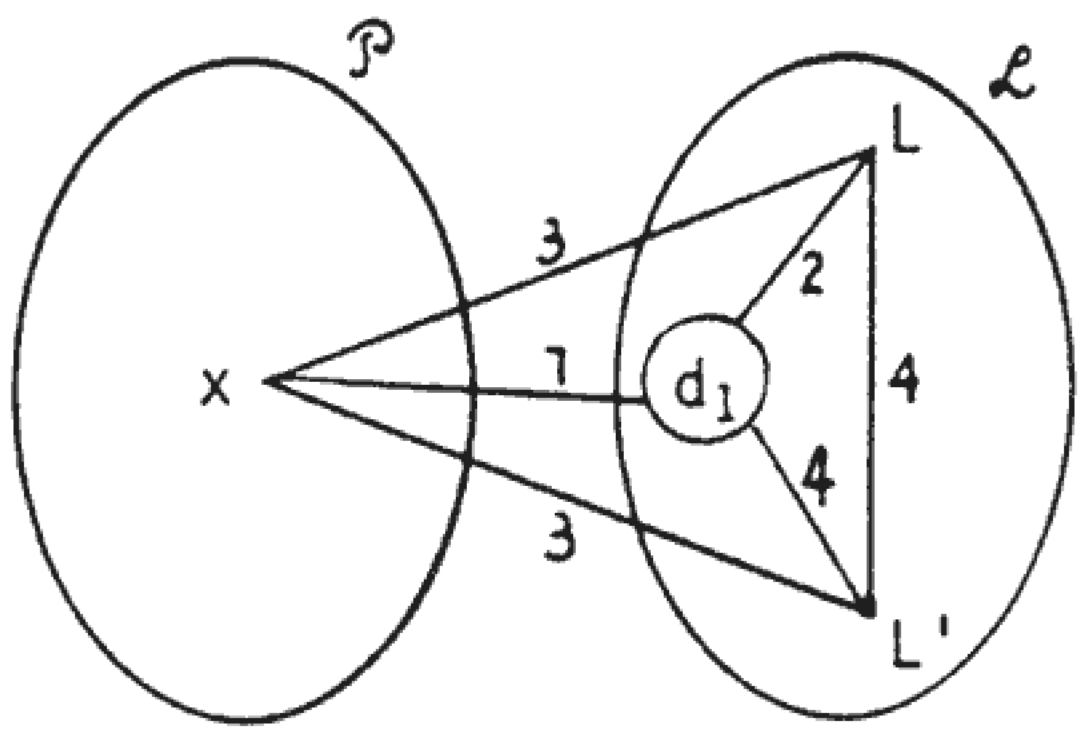


More generally, we will say that a linear space S satisfies condition $(\mathcal{S},i,j,k;\ell,m,n)$ where $\mathcal{S}$ is either the point set $\mathcal{P}$ or the line set $\mathcal{L}$ of S and i,j,k,ℓ,m,n are positive integers, if there is an integer c such that for any triple (u,v,w) of vertices of the incidence graph $\mathcal{G}$ of S where $u \in \mathcal{S}$ and $d(u,v) = i$, $d(v,w) = j$, $d(w,u) = k$, there are exactly c vertices of $\mathcal{G}$ which are simultaneously at distance ℓ from u, m from v and n from w.

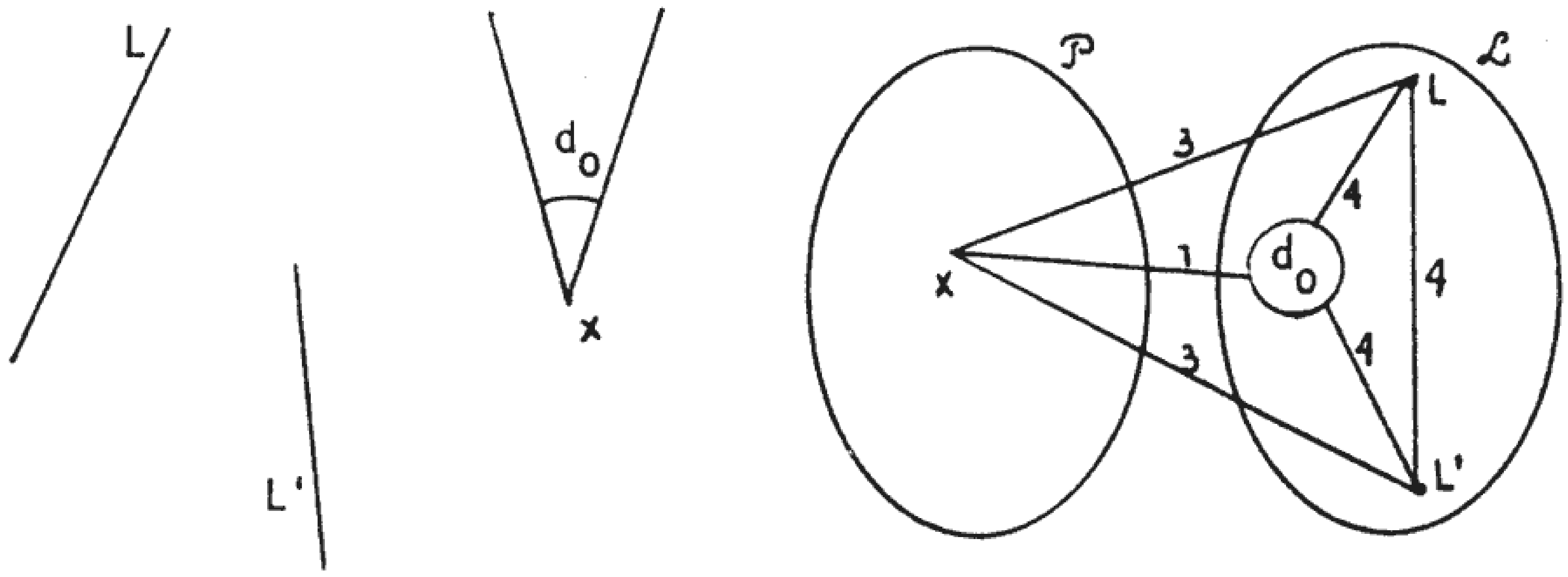
We prove that condition (D1), pictured below, is stronger than (D2).

(D1)



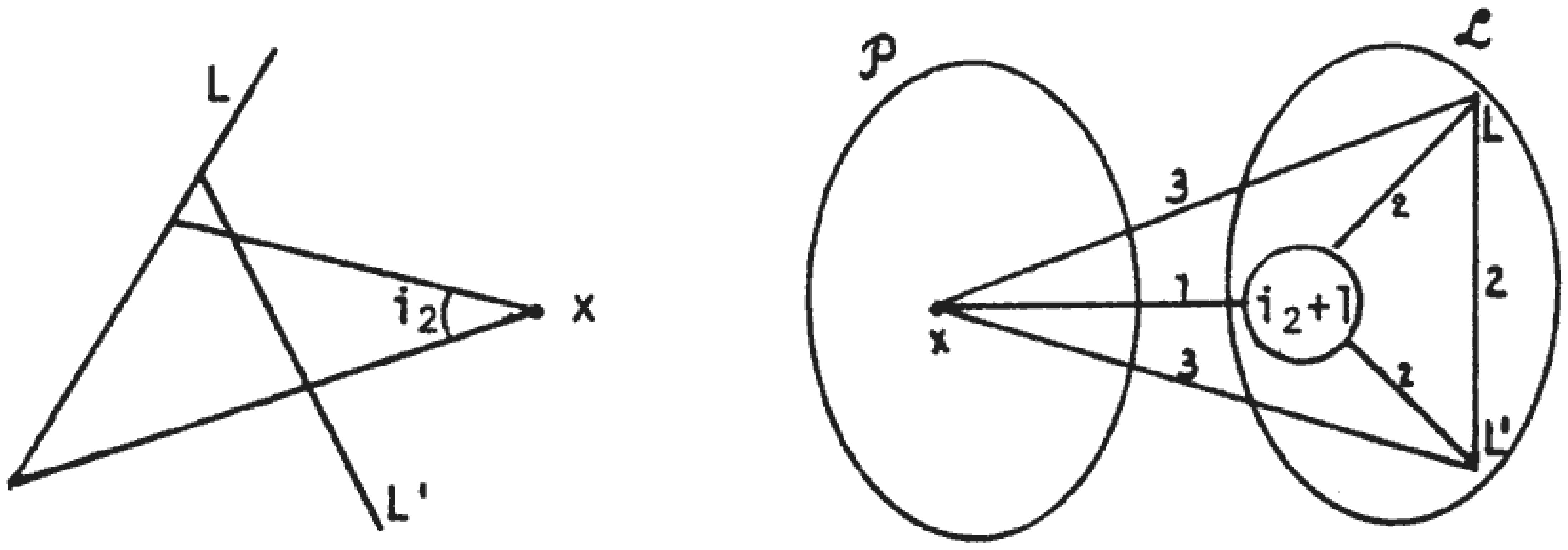


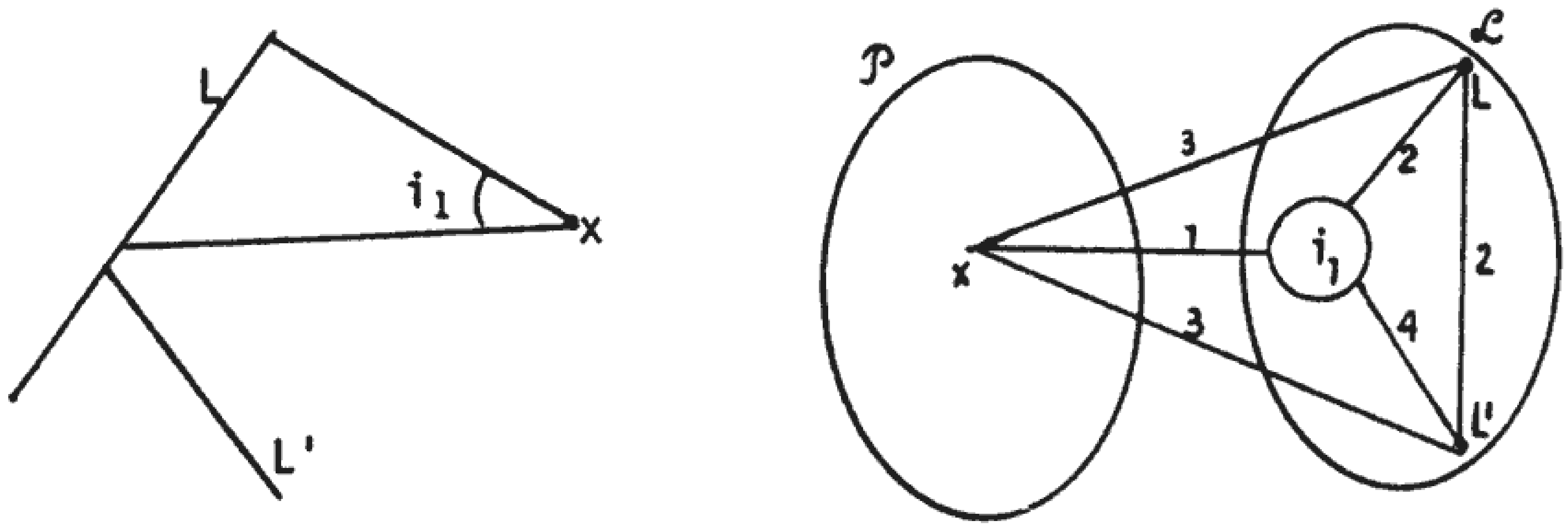
We also prove that the finite non-trivial linear spaces satisfying

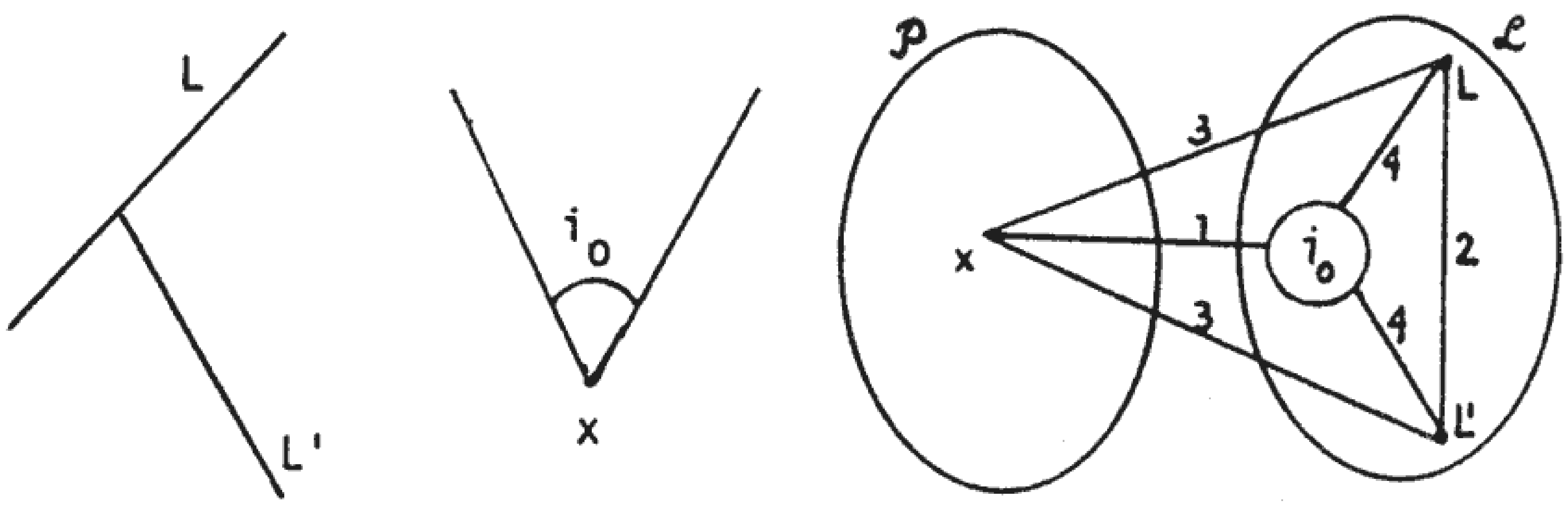
(D0) 

(with $d_0 > 0$) either satisfy (D2) or are obtained from an affine plane by deleting a line or a point, or is an affino-projective plane.

Generalizing a problem proposed by Cameron in 1980, we investigate the conditions obtained from the preceding ones by starting with an ordered pair of intersecting lines L and L' instead of an ordered pair of disjoint lines, that is

(I2) 

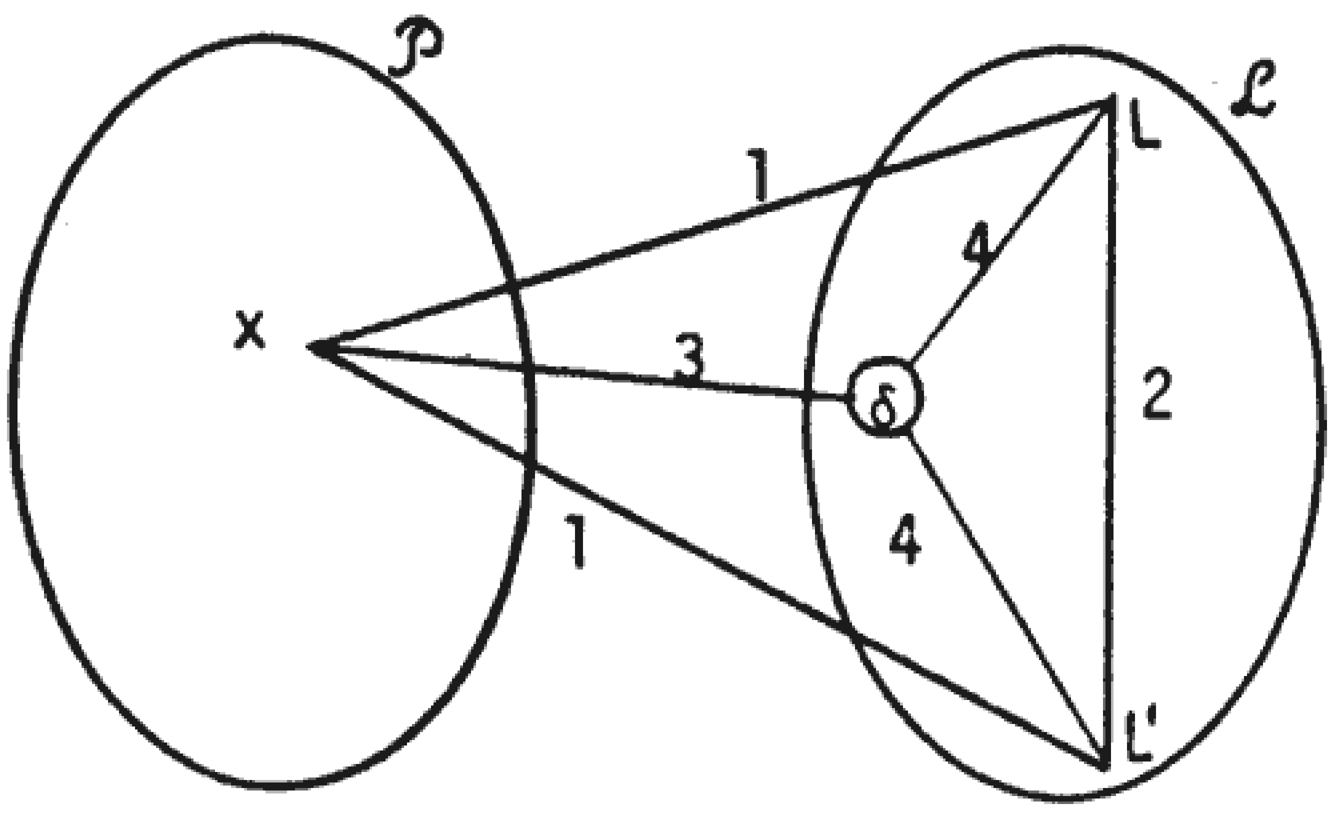
(I1) 

(I0) 

These three conditions are equivalent in the case of Steiner systems $S(2,k,v)$. The only known examples of finite non-trivial linear spaces satisfying one of them are certain semi-affine planes and the $S(2,2,v)$'s. We prove that any other example would necessarily be a Steiner system $S(2,k,v)$ whose parameters k and i_2 satisfy very strong arithmetical conditions : for example, if $k \leq 100$, there remain only six admissible pairs (i_2,k).

We end Chapter II by proving that the finite non-trivial linear spaces satisfying condition

(ID)



(which means that there is an integer δ such that for any two intersecting lines L and L', there are exactly δ lines disjoint from $L \cup L'$) are exactly the Steiner systems $S(2,k,v)$ with $k < v$.

In Chapter III, we start from the observation that the 3-dimensional generalized projective spaces are exactly the planar spaces in which

(I) for every pair (Π, Π') of planes intersecting in a line, any line intersecting Π intersects Π',

and (I') there are at least two planes intersecting in a line.

We first prove that the finite planar spaces satisfying

(II) for every pair (Π, Π') of planes intersecting in exactly one point, any line intersecting Π intersects Π',

and (II') there are at least two planes intersecting in exactly one point

are essentially obtained by deleting certain points from a 3-dimensional projective space; more precisely, such a space is either obtained from $PG(3,k)$ by deleting an affino-projective (but not projective) plane of order k or by deleting k collinear points, or is obtained by adding a new point (joined to all other points by lines of size 2) to a punctured projective plane or to an affine plane with one point at infinity or to an affine plane, or is $PG(4,1)$.

We also investigate the finite planar spaces satisfying

(III) for every pair (Π, Π') of disjoint planes, any line intersecting Π intersects Π'

and (III') there are at least two disjoint planes.

We prove that up to six small uninteresting spaces and up to a possibly empty class of spaces (in which all planes admitting a disjoint plane are pairwise disjoint and which contain at least four non-coplanar points outside the union of these planes), any finite planar space satisfying (III) and (III') is either obtained from PG(3,k) by deleting a line or an affino-projective (but not affine) plane of order k or is rather unexpectedly related to the Fischer spaces F_{18} or F_{36} constructed from a Hermitian quadric in PG(3,4).