INTRODUCTION

Inspired by the linearity properties which are a common feature of most classical spaces, Libois has introduced the notion of linear space around 1960. A linear space is a set of elements called points, provided with a family of subsets, called lines, such that any two points are in exactly one line, each line containing at least two points. Buekenhout introduced then the notion of planar space, that is a linear space provided with a family of distinguished linear subspaces, called planes, such that any three non-collinear points are in exactly one plane, each plane containing at least three non-collinear points.

One of the reasons for the success of these notions is that projective and affine spaces may be entirely characterized by their linear (or planar) structure. Indeed, by the classical work of Veblen and Young at the beginning of this century, the projective spaces are exactly the linear spaces (endowed with all their linear subspaces) in which any two intersecting lines are contained in a linear subspace which is a projective plane. In other words, a planar space in which all planes are projective planes consists of the points, lines and planes of some projective space. On the other hand, by a theorem of Buekenhout (1969), the affine spaces of order ≥ 4 are exactly the linear spaces (endowed with all their linear subspaces) in which any two intersecting lines are contained in a linear subspace which is an affine plane of order ≥ 4. In other words, a planar space in which all planes are affine planes of order ≥ 4 consists of the points, lines and planes of some affine space (in order to characterize the affine spaces of order ≤ 3, the only additional requirement is the transitivity of the parallelism relation on lines).

Linear and planar spaces correspond respectively to the diagrams

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and are particular cases of the more general notion of n-dimensional linear space (or geometric lattice or simple matroid), corresponding to the diagram

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The use of diagrams goes back to Dynkin and Coxeter. Diagrams for geometries, appearing around 1955 in Tits'geometric interpretation of the simple groups of Lie-Chevalley type, were later generalized by Buekenhout in an attempt to associate a geometry to the sporadic simple groups. The above mentioned point-line characterizations of the Desarguesian affine and projective spaces form only a small part of a recently achieved characterization of all Lie incidence geometries in terms of points and lines. This remarkable work, associated with the names Veldkamp, Tits, Buekenhout, Shult, Cohen and Cooperstein, is fascinating in the sense that very complex geometric structures are characterized by a few simple axioms involving only points and lines.

Let us go back to the elementary notion of linear space. If X is a non-empty subset of a linear space S, the linear space induced on X is defined as follows its points are those of X and its lines are the intersections of X with the lines of S having at least two points in X.

Clearly the class of finite linear spaces is much too wide to allow a complete classification : for example, the structure induced on any non-empty subset of a finite projective space is a linear space. This raises three types of problems

(i) find sufficient conditions for a finite linear space to be embeddable in some finite projective space,

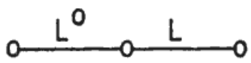
(ii) classify the finite linear spaces satisfying some given combinatorial regularity condition,

(iii) classify the finite linear spaces whose automorphism group is transitive on some given configurations of points and lines.

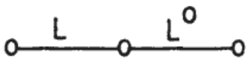
Of course, similar types of problems may also be formulated for planar spaces.

These lecture notes will essentially focus on problems of type (ii), and the results will have some consequences for problems of type (i). Although this will not be our topic here, let us mention that much progress was made recently on problems of type (iii), mainly because of the classification of all finite simple groups.

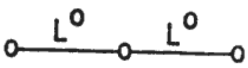
In Chapter I, the reader will find the definitions of most of the notions appearing later, as well as some basic theorems (numbered in the left margin) which will be used in subsequent chapters. The only original result in Chapter I concerns L^0-spaces, i.e non-trivial planar spaces all of whose planes are isomorphic to some given linear space L^0, the corresponding diagram being



Finite L^0-spaces have been investigated by Buekenhout, Deherder, Brouwer, Leonard and myself, but examples other than projective or affine spaces or Steiner systems $S(3,k,v)$ seem to be extremely rare. On the other hand, very little is known about planar spaces with diagram



even in the case where L^0 is a projective plane. We prove that a finite planar space admitting diagram



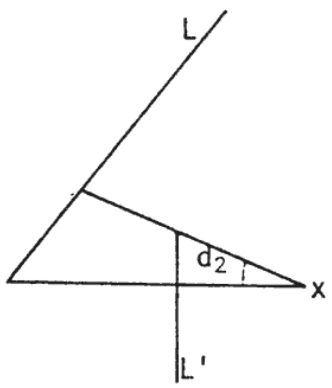
is necessarily a 3-dimensional generalized projective space.

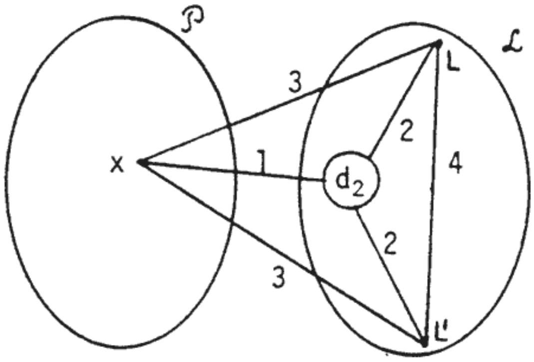
In Chapter II, starting from the observation that the generalized projective spaces of dimension ≤ 3 are characterized by the fact that for any two disjoint lines L and L', any point outside L $\cup$ L' is on exactly one line intersecting both L and L', we completely classify (in a joint work with A. Beutelspacher) the finite linear spaces S satisfying the following condition

(D2) there is an integer d_2 such that for any pair of disjoint lines L and L', any point outside L $\cup$ L' is on exactly d_2 lines intersecting both L and L'. Such a space S is a generalized projective space of dimension ≤ 3, or PG(d,1) with $d \geq 4$, or a semi-affine plane or a small exceptional space with 7 points.

Condition (D2) can be interpreted in terms of metrical regularity in the incidence graph of S, as suggested by the figures below, which are self-explanatory :

(D2)

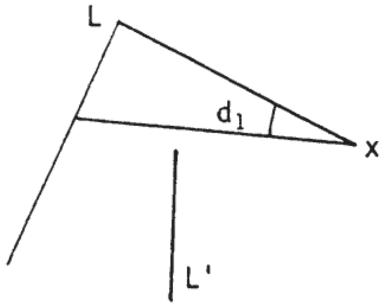


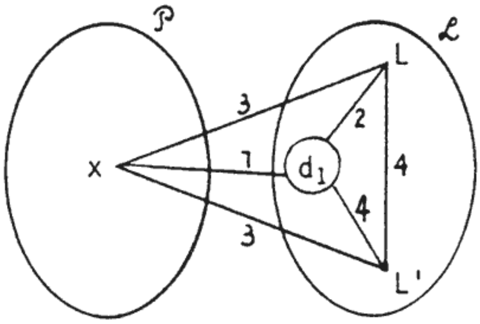


More generally, we will say that a linear space S satisfies condition $(\mathcal{S},i,j,k;\ell,m,n)$ where $\mathcal{S}$ is either the point set $\mathcal{P}$ or the line set $\mathcal{L}$ of S and i,j,k,ℓ,m,n are positive integers, if there is an integer c such that for any triple (u,v,w) of vertices of the incidence graph $\mathcal{G}$ of S where $u \in \mathcal{S}$ and $d(u,v) = i$, $d(v,w) = j$, $d(w,u) = k$, there are exactly c vertices of $\mathcal{G}$ which are simultaneously at distance ℓ from u, m from v and n from w.

We prove that condition (D1), pictured below, is stronger than (D2).

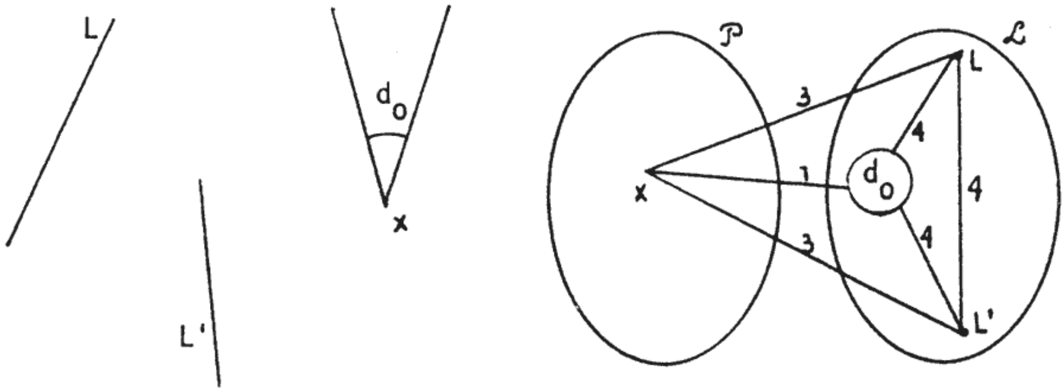
(D1)





We also prove that the finite non-trivial linear spaces satisfying

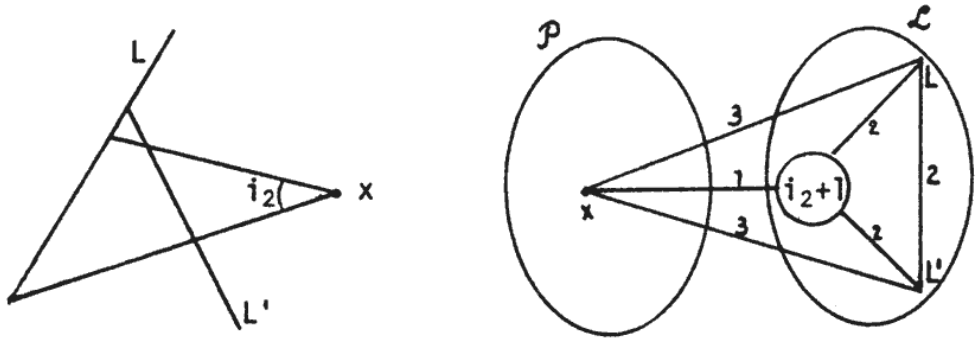
(D0)



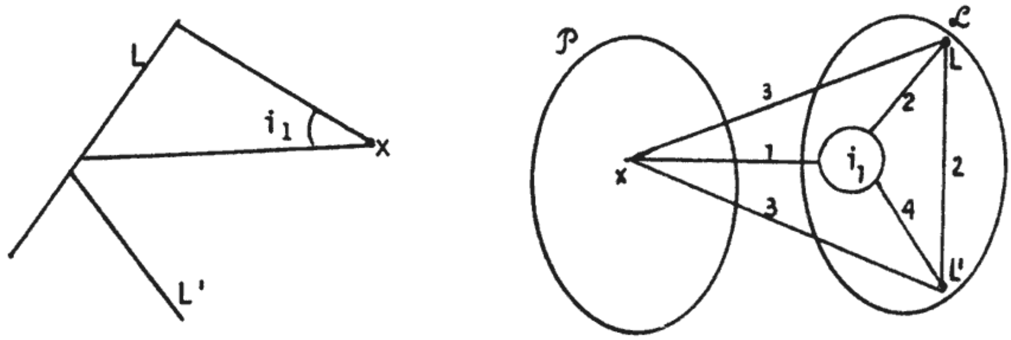
(with $d_0 > 0$) either satisfy (D2) or are obtained from an affine plane by deleting a line or a point, or is an affino-projective plane.

Generalizing a problem proposed by Cameron in 1980, we investigate the conditions obtained from the preceding ones by starting with an ordered pair of intersecting lines L and L' instead of an ordered pair of disjoint lines, that is

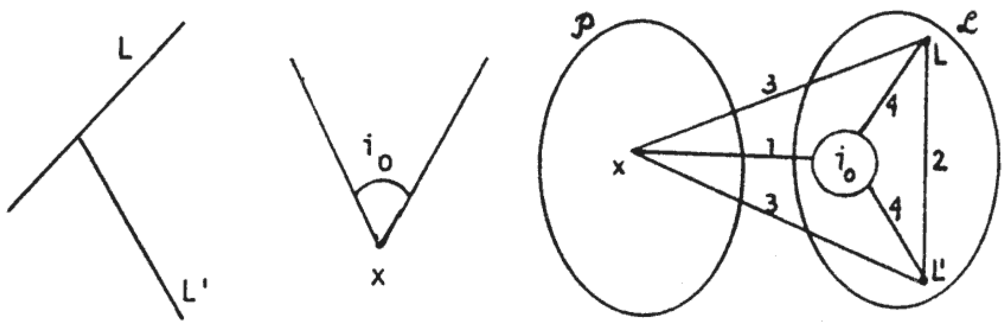
(I2)



(I1)



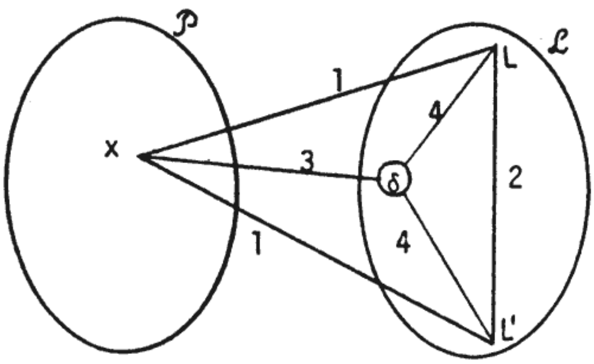
(I0)



These three conditions are equivalent in the case of Steiner systems $S(2,k,v)$. The only known examples of finite non-trivial linear spaces satisfying one of them are certain semi-affine planes and the $S(2,2,v)$'s. We prove that any other example would necessarily be a Steiner system $S(2,k,v)$ whose parameters k and i_2 satisfy very strong arithmetical conditions : for example, if $k \leqslant 100$, there remain only six admissible pairs (i_2,k).

We end Chapter II by proving that the finite non-trivial linear spaces satisfying condition

(ID)



(which means that there is an integer δ such that for any two intersecting lines L and L', there are exactly δ lines disjoint from $L \cup L'$) are exactly the Steiner systems $S(2,k,v)$ with $k < v$.

In Chapter III, we start from the observation that the 3-dimensional generalized projective spaces are exactly the planar spaces in which

(I) for every pair (Π, Π') of planes intersecting in a line, any line intersecting Π intersects Π',

and (I') there are at least two planes intersecting in a line.

We first prove that the finite planar spaces satisfying

(II) for every pair (Π, Π') of planes intersecting in exactly one point, any line intersecting Π intersects Π',

and (II') there are at least two planes intersecting in exactly one point are essentially obtained by deleting certain points from a 3-dimensional projective space; more precisely, such a space is either obtained from $PG(3,k)$ by deleting an affino-projective (but not projective) plane of order k or by deleting k collinear points, or is obtained by adding a new point (joined to all other points by lines of size 2) to a punctured projective plane or to an affine plane with one point at infinity or to an affine plane, or is $PG(4,1)$.

We also investigate the finite planar spaces satisfying

(III) for every pair (Π, Π') of disjoint planes, any line intersecting Π intersects Π'

and (III') there are at least two disjoint planes.

We prove that up to six small uninteresting spaces and up to a possibly empty class of spaces (in which all planes admitting a disjoint plane are pairwise disjoint and which contain at least four non-coplanar points outside the union of these planes), any finite planar space satisfying (III) and (III') is either obtained from PG(3,k) by deleting a line or an affino-projective (but not affine) plane of order k or is rather unexpectedly related to the Fischer spaces F_{18} or F_{36} constructed from a Hermitian quadric in PG(3,4).

CHAPTER I. DEFINITIONS AND BASIC RESULTS

1. LINEAR SPACES AND PLANAR SPACES

A *linear space* S is a non-empty set of elements, called *points*, provided with a family of distinguished subsets, called *lines*, such that any two points x and y are in exactly one line, denoted by $\langle x,y \rangle$, and each line contains at least two points. S is called *non-trivial* if it has at least two lines, and it is called *finite* if it has only a finite number of points. The *size* of a line L is the number of points of L and the *degree* of a point x is the number of lines passing through x. We shall say that two lines *intersect* if they have exactly one point in common.

A *linear subspace* S' of S is a set of points of S such that any line of S having at least two points in S' is contained in S'. A linear subspace S' of S is called *proper* if S' $\neq$ S.

A *planar space* is a linear space provided with a family of distinguished linear subspaces called *planes*, such that any three non-collinear points x,y,z are contained in exactly one plane, denoted by $\langle x,y,z \rangle$, each plane containing at least three non-collinear points. A planar space is called *non-trivial* if it has at least two planes. If L is a line of S and if x is a point of S outside L, the unique plane containing L and x will be denoted by $\langle L,x \rangle$. Similarly, if L and L' are two intersecting lines of S, the unique plane containing L and L' will be denoted by $\langle L,L' \rangle$. Note that if x,y,z are non-collinear points, the plane x,y,z is not necessarily the smallest linear subspace of S containing x,y,z. Actually, any non-trivial linear space can be a plane of some non-trivial planar space.

We shall say that a line and a plane *intersect* if they have exactly one point in common. A *planar subspace* S' of S is a linear subspace S' of S such that any plane of S having at least three non-collinear points in S is contained in S'.

If X is a non-empty subset of a linear space S, the *linear space induced on* X *by* S is the linear space whose points are the points of X and whose lines are the intersections with X of the lines of S having at least two points in X. Similarly, if S is a planar space and X is a non-empty subset of S, the *planar space induced on* X *by* S is the planar space whose points are the points of X, whose lines are the intersections with X of the lines of S having at least two points in X, and whose planes are the intersections with X of the planes of S having at least three non-collinear points in X.

A *circular space* G is a non-empty set of elements called *points*, provided with a family of distinguished subsets called *circles*, such that any three points are in exactly one circle and any circle contains at least three points. A planar space S whose lines are the unordered pairs of points may be viewed as a circular space in which the circles are the planes of S, and conversely.

2. STEINER SYSTEMS AND t-DESIGNS.

A *Steiner system* $S(t,k,v)$ (where t,k,v are integers with $2 \leqslant t \leqslant k \leqslant v$) is a finite set of v elements, called *points*, provided with a family of distinguished k-subsets, called *blocks*, such that any t points are contained in exactly one block. The Steiner systems $S(2,k,v)$ are the finite linear spaces of v points in which all lines (or blocks) have the same size k.

A *projective plane* is a linear space in which any two lines intersect and which has at least two lines of size $\geqslant 3$. It is well-known that in a finite projective plane S, all the lines have the same size $n+1$, where $n \geqslant 2$ is called the *order* of the projective plane, so that S is an $S(2, n+1, n^2 + n + 1)$. Conversely, any Steiner system with these parameters is a projective plane of order n. A *generalized projective plane* is a non-trivial linear space in which any two lines intersect. The generalized projective planes which are not projective planes are called *degenerate projective planes* and consist of a line L and a point x outside L, all lines joining x to a point of L having size 2. An *affine plane* is a non-trivial linear space in which for any line L, any point outside L is on exactly one line disjoint from L. The affine planes of *order* n are exactly the Steiner systems $S(2, n, n^2)$.

The Steiner systems $S(3,k,v)$ are the finite circular spaces of v points in which all circles have k points. We shall also use this notation for the planar spaces in which all lines have two points, the planes being the blocks of the Steiner system. An *inversive plane of order* n is an $S(3, n+1, n^2+1)$.

More generally, a *t-design* $S = S_\lambda(t,k,v)$ (where $1 \leqslant t \leqslant k \leqslant v$ and $1 \leqslant \lambda$) is a finite set of v elements, called *points*, provided with a family of k-subsets, called *blocks*, such that any t points are in exactly λ blocks (a t-design with $\lambda=1$ is a Steiner system). For any set I of i points ($0 \leqslant i \leqslant t-1$), the i-th *derived design of* S *with respect to* I is the $(t-i)$-design $S_\lambda(t-i, k-i, v-i)$ whose points are the points outside I and whose blocks are the restrictions to S-I of the blocks containing I.

I1) A well-known necessary condition for the existence of an $S_\lambda(t,k,v)$ is that, for any integer i with $0 \leq i \leq t$,

$$\lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}$$

is an integer, since this expression counts the number of blocks containing any given set of i points. In particular, any point is on exactly

$$r = \lambda \binom{v-1}{t-1} / \binom{k-1}{t-1}$$

blocks, and the total number of blocks is

$$b = \lambda \binom{v}{t} / \binom{k}{t}$$

If S is a 2s-design $S_\lambda(2s, k,v)$ where $s \in \mathbb{N}_0$ and $v \geq k+s$, Wilson and Ray-Chaudhuri [49] have proved that

$$b \geq \binom{v}{s}$$

S is said to be *tight* if equality holds. Examples are given by the Steiner system S(4, 7, 23) and its complement the S_{52} (4, 16, 23).
If S is a (2s+1)-design $S_\lambda(2s+1, k,v)$ where $s \in \mathbb{N}_0$ and $v-1 \geq k+s$, then the preceding result can be applied to the first derived designs of S, so that

$$b \geq \binom{v-1}{s} v / k$$

S is said to be *tight* if equality holds. The Steiner system S(5, 8, 24) is tight.

I2) In particular, any 2-design with $v \geq k+1$ satisfies *Fisher's inequality* $b \geq v$, which is equivalent to $k \leq r$ since $bk = vr$. A *symmetric 2-design* is a tight 2-design, that is a 2-design for which v=b (or equivalently k=r). The symmetric 2-designs with $\lambda = 1$ are exactly the projective planes $S(2, n+1, n^2+n+1)$, including the degenerate projective plane of 3 points.

I3) The Bruck-Chowla-Ryser theorem gives an important necessary condition for the existence of symmetric 2-designs, namely :

(i) if v is even, $r-\lambda$ should be a square,

(ii) if v is odd, the Diophantine equation

$$(r-\lambda)x^2 + (-1)^{(v-1)/2} y^2 = z^2$$

should have a solution in integers x,y,z not all zero [37]

More generally, a t-design S is called *symmetric* if its (t-2)th derived designs are symmetric 2-designs.

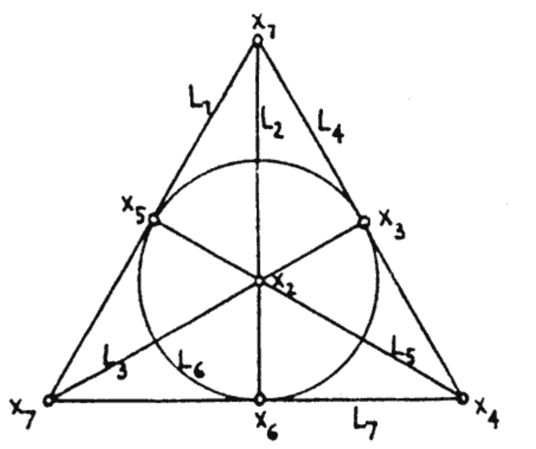
(I4) Cameron [13] has proved that the parameters of a symmetric 3-design are necessarily in the following list

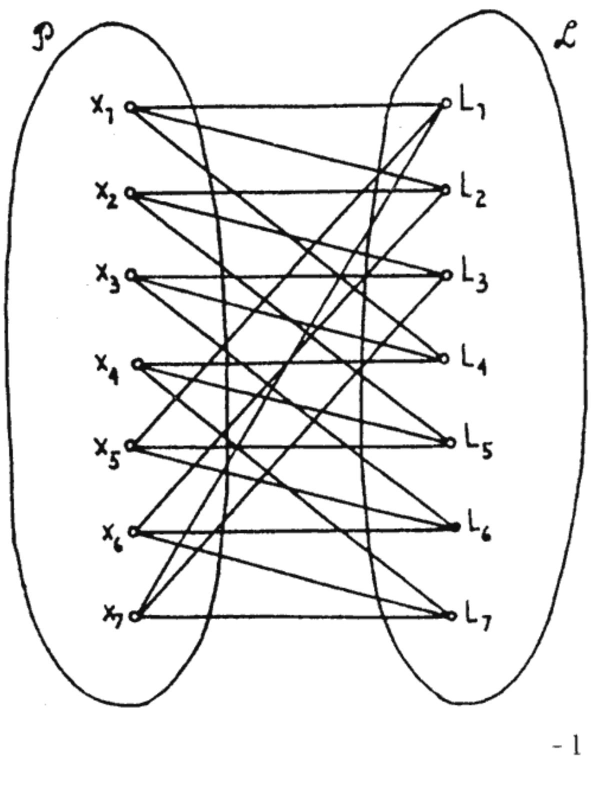
(a) $v = \lambda+3$, $k = \lambda+2$
(b) $v = 4(\lambda+1)$, $k = 2(\lambda+1)$
(c) $v = (\lambda+1)(\lambda^2 + 5\lambda+5)$, $k = (\lambda+1)(\lambda+2)$
(d) $v = 112$, $k = 12$, $\lambda = 1$
(e) $v = 496$, $k = 40$, $\lambda = 3$.

3. INCIDENCE STRUCTURES AND DIAGRAMS.

A linear space may be viewed as an incidence structure $S = (\mathcal{P}, \mathcal{L}, I)$ where $\mathcal{P}$ denotes the set of points of S, $\mathcal{L}$ denotes the set of lines of S and I is a symmetric relation between $\mathcal{P}$ and $\mathcal{L}$: a point x and a line L are incident, which is written x I L, if and only if $x \in L$. A *flag* (resp. *antiflag*) of a linear space S is a point-line incident (resp. non-incident) pair. In the definition of a linear space given in the first section, we have used the same notation for the linear space and for its set of points and we have identified every line L with the set of points incident with L. We shall use both points of view and terminologies.

The *incidence graph* $\mathcal{G}(S)$ of a linear space S is the bipartite graph whose vertices are the elements of $\mathcal{P} \cup \mathcal{L}$, two vertices being adjacent if and only if they are incident. The figure below shows the incidence graph of the projective plane PG(2,2).





The definition of a non-trivial linear space can of course be translated in terms of its incidence graph $\mathcal{G}$. In particular, any two points are at distance 2 in $\mathcal{G}$ and any vertex of $\mathcal{G}$ has degree ≥ 2. Note that the Steiner systems $S(2,k,v)$ may be defined as the linear spaces such that in their incidence graph, all lines (and therefore all points) have the same degree k (resp. r).

Similarly, a planar space S may be viewed as an incidence structure $S = (\mathcal{P}, \mathcal{L}, \Pi, I)$ where $\mathcal{P}$, $\mathcal{L}$ and Π denote respectively the set of points, lines and planes of S, and where I is the incidence relation, two elements of $\mathcal{P} \cup \mathcal{L} \cup \Pi$ being incident if and only if, considered as point sets, one of them is strictly contained in the other. The incidence graph $\mathcal{G}(S)$ of a planar space is then a tripartite graph. A *flag* of S is a set of pairwise incident elements of $\mathcal{P} \cup \mathcal{L} \cup \Pi$. A *maximal flag* is a set $\{x, L, \pi\}$ where $x \in \mathcal{P}$, $L \in \mathcal{L}$, $\pi \in \Pi$ and $x \, I \, L \, I \, \pi$.

The *residue* S_v of a vertex v of $\mathcal{G}(S)$ is the incidence structure whose incidence graph is the restriction of $\mathcal{G}(S)$ to the neighbourhood of v in $\mathcal{G}(S)$. For example, the residue S_π of a plane π of S is nothing else than the linear space π itself. The residue S_L of a line L of S consists of the points of L and the planes containing L, each of these planes being automatically incident with each of these points, so that the incidence graph of S_L is a complete bipartite graph. Such incidence structures are called *generalized digons*. The residue S_x of a point x is a linear space whose points are the lines containing x, whose lines are the planes containing x and in which a point L_x corresponding to a line L and a line π_x corresponding to a plane π are incident if and only if $L \subset \pi$. The fact that S_x is a linear space follows essentially from the fact that any two intersecting lines of S are contained in exactly one plane.

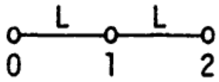
Linear spaces and planar spaces may be represented by diagrams in the sense of Buekenhout [11]. For definitions and conventions concerning incidence structures admitting diagrams, we refer the reader to [11]. In particular, the class of non-trivial linear spaces is represented by the diagram

$$\circ \overset{L}{\text{———————}} \circ$$

where the left dot represents the set $\mathcal{P}$ of points and the right dot represents the set $\mathcal{L}$ of lines. The class of generalized digons is represented by the diagram

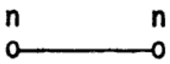
$$\circ \qquad\qquad \circ$$

Then, according to the conventions of Buekenhout, the class of non-trivial planar spaces is represented by the diagram



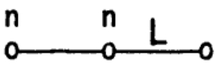
expressing the fact that non-trivial planar spaces are incidence structures with three sets of objets $\mathcal{P}$, $\mathcal{L}$ and Π, represented respectively by the three dots 0, 1 and 2, such that the residue of any element of $\mathcal{P} \cup \Pi$ is a non-trivial linear space and that the residue of any element of $\mathcal{L}$ is a generalized digon.

This symbolic representation suggests several problems by considering various subclasses of the class of linear spaces. For example, if the class of projective planes of order n is denoted by



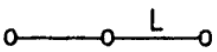
,

then the incidence structures admitting the diagram



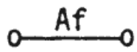
are the non-trivial planar spaces all of whose planes are projective planes of order n. Thanks to the classical work of Veblen and Young, we know that these are exactly the Desarguesian projective spaces PG(d,n) of dimension $d \geq 3$, endowed with all their planes. More generally, the class of generalized projective planes is denoted by

so that the diagram

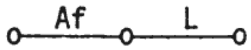


represents exactly the generalized projective spaces of dimension ≥ 3, endowed with all their planes. Remember that a *generalized projective space* is a linear space such that for every pair of lines L and L' intersecting in a point x, any two lines not passing through x and intersecting each of the lines L and L' intersect (*Pasch's axiom*). These spaces are unions of projective spaces of any dimension ≥ 0 joined together by lines of size 2. The planes of a generalized projective space are the smallest linear subspaces containing three non-collinear points. A generalized projective space has *dimension* n if there are n+1 points which are contained in no proper linear subspace of S and if any n points are contained in a proper linear subspace of S. We shall denote by P(2, k+1) the degenerate projective plane of k+1 points and by P(3, k+ℓ) the 3-dimensional generalized projective space consisting of two disjoint lines of size k and ℓ respectively, all the other lines having size 2.

From now on, PG(d,q) will not only denote the Desarguesian projective space of dimension d over GF(q) for $q \geq 2$, but also, for q=1, the generalized projective space PG(d,1) with d+1 points, in which all lines have size 2, which may be viewed as the d-dimensional projective space over "GF(1)". However AG(d,q) will always denote the d-dimensional Desarguesian affine space over GF(q) ($q \geq 2$).

If 

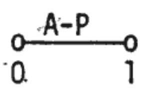
denotes the class of affine planes, then the diagram



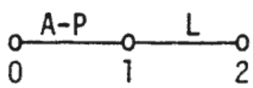
represents the class of non-trivial planar spaces all of whose planes are
(I5) affine planes. Buekenhout [9] has proved that these are exactly the affine spaces of order ≥ 4 if one of the planes has order ≥ 4. However, if the planes have order 3, these spaces, called *Hall triple systems*, are not necessarily affine spaces, as shown by Hall [36]. Finally, if the planes have order 2, then all Steiner systems S(3,4,v) yield planar spaces admitting the above diagram.

Let us mention a generalization of this situation. An *affino-projective plane* is a linear space Π obtained from a projective plane by deleting a set of points of a line L. The *order* of a finite affino-projective plane is the order of the initial projective plane. In particular, Π is a projective plane if we delete no point, Π is a *punctured projective plane* if we delete just one point, Π is an *affine plane with one point at infinity* if we delete all points of L except one, and Π is an affine plane if we delete all points of L. The planes of these four types are examples of *semi-affine planes*, that is non-trivial linear spaces in which for any line L, any point outside L is on
(I6) at most one line disjoint from L. Dembowski [30] and Kuiper [31, p.310] have proved that conversely any finite semi-affine plane is of one of these four types or is a degenerate projective plane. Actually, Dembowski assumes that all lines have size ≥ 3 and that all points have degree ≥ 3, but the classification of all finite semi-affine planes containing a line of size 2 or a point of degree 2 is very easy.

Let



represent the class of affino-projective planes. Teirlinck [53] has proved that if S corresponds to the diagram



,

if S contains at least one plane of order $\geqslant 4$ and contains a finite subset of points which is not contained in any proper linear subspace of S, then S is an *affino-projective space*, that is a projective space from which a subset of a hyperplane has been deleted. Note that a plane of an affino-projective space is not necessarily an affino-projective plane.

Dual problems arise in a natural way. For example, what can be said about the planar spaces with diagram

$$\circ\overset{L}{\text{———}}\overset{n}{\circ}\text{————}\overset{n}{\circ} \quad ,$$

that is the non-trivial planar spaces in which the residue of every point is a projective plane of order n (these spaces are sometimes said to be locally a projective plane) ? This question seems to be rather difficult to answer. Kantor [11] has conjectured that a finite non-trivial planar space which is locally a projective plane is necessarily obtained by deleting a set of points from a 3-dimensional projective space.

(I7) A partial answer to this question has been given by Doyen and Hubaut [32] who proved that if S is a finite planar space with diagram

$$\circ\overset{L}{\text{———}}\circ\text{————}\circ$$

and if all lines of S have the same size k, then S is either PG(3, k-1) or AG(3,k) or a *Lobachevsky space* of type k^2-k+1 (resp. k^3+1), that is a planar space such that for any line L and any point x outside L, the number of lines of <x,L> which are disjoint from L and contain x is equal to k^2-k+1 (resp. k^3+1). The only known example of this latter class of spaces is the planar space S(3, 6, 22).

Similarly, the problem of classifying the planar spaces S with diagram

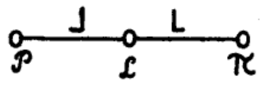
$$\circ\overset{L}{\text{———}}\circ\overset{Af}{\text{———}}\circ$$

has been solved by Cameron [14], Brouwer and Wilbrink [8] under the additional assumptions that S is finite and that all lines of S have the same size k : then k=2 and S is a planar space S(3, n+1, n^2+1), in other words an inversive plane of order n.

Observe that, though the class of incidence structures admitting the diagram

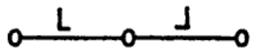
$$\circ\overset{L}{\text{———}}\circ\overset{L}{\text{———}}\circ$$

is very wide and contains rather wild spaces, the incidence structures admitting the diagram

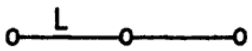


have been classified by Sprague [52] under the additional assumption that some point is on finitely many lines or some line has finitely many points. Such an incidence structure, in which the residue of each point is a linear space, the residue of each line is a generalized digon and the residue of each plane is the dual of a linear space is, for some integer i, the incidence structure having as varieties all (i-1)-, i- and (i+1)- dimensional subspaces of some generalized projective space, and inclusion as incidence.

Finally, note that, on the contrary, the problem of classifying the incidence structures admitting the diagram



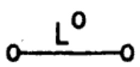
is still more general than that of classifying the incidence structures admitting diagram



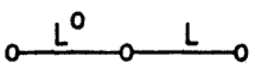
and so, seems to be hopeless.

4. PLANAR SPACES WITH ISOMORPHIC PLANES.

Let L^o be a given linear space, which we shall represent by



Then



denotes the non-trivial planar spaces in which every plane is isomorphic to L^o. These spaces, called L^o-*spaces*, were introduced by Buekenhout and Deherder [12] and have been investigated by Brouwer [7], Leonard [42] and myself [19],[22].

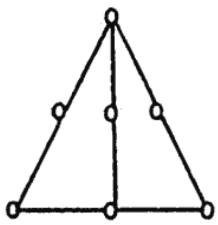
The only known finite L^o-spaces are the Desarguesian projective or affine spaces of any dimension ≥ 3, the Hall triple systems, the planar spaces $S(3, |L^o|, v)$ where all lines have size 2, and finally the 3-dimensional generalized projective spaces $P(3, k+k)$ consisting of two disjoint lines of size k (here all planes are degenerate projective planes $P(2, k+1)$). The spaces $P(3, k+k)$ are also the only known finite L^o-spaces with lines of different sizes. In [19] and [42], some rather restrictive relations on the parameters

of a finite L^0-space with different line sizes have been given, which are particularly strong if there are more than two line sizes, so that this possibility seems to be very improbable.
In particular, it was shown that the number v of points of a finite L^0-space having at least two distinct line sizes k and ℓ is uniquely determined by the following parameters of L^0 : $v' = |L^0|$, k, ℓ, the number b'_k (resp. b'_ℓ) of lines of size k (resp. ℓ) in L^0, the number n_{kk} of ordered pairs of intersecting lines of size k and the number $n_{k\ell}$ of pairs of intersecting lines (one of size k and the other of size ℓ) in L^0 :

$$v = v' + (v'-\ell)(v'-k)\,\frac{b'_k\, b'_\ell\, k\, \ell + n_{kk}\, b'_\ell\, \ell - n_{k\ell}\, b'_k\, k}{n_{k\ell}\, b'_k\, k(v'-k) - n_{kk}\, b'_\ell\, \ell(v'-\ell)}$$

This situation is very different from what we get if L^0 is a Desarguesian projective or affine plane.

The smallest linear space L^0 having at least two line sizes for which the existence of an L^0-space is still unsettled is represented below.



(the lines of size 2 are not drawn)

We have proved in [31] that the corresponding L^0-space, which would have 47 points, is rigid, i.e. has no other automorphism than the identity.

(I8) We shall use later the fact, proved in [7], [19] and in [42], that in any finite L^0-space, all points are on the same number r_k of lines of a given size k.

Very little is known about the dual problem, namely the classification of planar spaces with diagram

$$\circ \overset{L}{\text{——}} \circ \overset{L^0}{\text{——}} \circ \quad ,$$

even in the finite case. However, the finite spaces with diagram

$$\circ \overset{L^0}{\text{——}} \circ \overset{L^0}{\text{——}} \circ$$

can be classified completely :

Theorem 1. [21] *Let* L^0 *be a given linear space. If* S *is a finite planar space in which every plane is isomorphic to* L^0 *and the residue of every point is isomorphic to* L^0*, then* L^0 *is a (possibly degenerate) projective plane and* S *is either a projective space* PG(3,q) *or a generalized projective space* P(3, k+k).

Lemma 1.1. The set K *of sizes of lines of* L^0 *is equal to the set of degrees of points of* L^0.

Proof. Let x be any point of S. Since S_x is isomorphic to L^0, K coincides with the set of degrees of x in the planes of S passing through x. Therefore, since all planes through x are isomorphic to L^0 and since K does not depend on x, K coincides with the set of degrees of points of L^0.

Lemma 1.2. If B *is a line of size* k *in* S *and if* $x \in B$*, then the point* B_x *of* S_x *has degree* k *in* S_x.

Proof. Let v (resp. v') denote the number of points in S (resp. in L^0). The degree of the point B_x in S_x is equal to the number of planes of S containing the line B, that is to $(v-k)/(v'-k) = f(k)$ which depends only on the size k of B. Since L^0 is isomorphic to S_x, Lemma 1.1 implies that $f(K) = K$, and so $f(k) = k$ for every $k \in K$ because $f:K \to K:k \to f(k)$ is an increasing function and K is a finite set.

Proof of the theorem.

If $|K| = 1$, then all lines of L^0 have the same size k and, by Lemma 1.1, all points of L^0 have degree k. Therefore L^0 is a projective plane of order k-1 (or a degenerate projective plane with 3 points if k=2), and so S is PG(3, k-1) with $k-1 \geq 1$.

If $|K| \geq 2$, let B be a line of maximal size $k \geq 3$ in L^0. All points of L^0 outside B have a degree greater than or equal to k, and so, by Lemma 1.1, their degree is equal to k.

If there are at least two points of L^0 outside B, then, by Lemma 1.2, every point of S is on at least two lines of size k. These two lines are in a plane of S isomorphic to L^0, and so L^0 contains two lines of size k intersecting in a point y. Since every point of L^0 outside a line of size k is of degree k, the point y is the only point of L^0 having a degree $\ell < k$, and $K = \{k,\ell\}$ by Lemma 1.1. Lemma 1.2 again implies that every point of S is on exactly one line of size ℓ, and so the lines of size ℓ are necessarily disjoint in L^0. Therefore, since the degree of any point z of L^0 distinct from y is $k \geq 3$, and since z is on at most one line of size ℓ, there is at least one line of size k containing z but not y, and so the degree ℓ of y is at least k, contradicting $\ell < k$.

This proves that there is only one point of L^0 outside B. In other words, L^0 is a degenerate projective plane with k+1 points. By Lemma 1.2, every point of S is on exactly one line of size k and k lines of size 2. This implies that S is the union of two lines of size k, all the other lines having size 2.

5. n-DIMENSIONAL LINEAR SPACES.

Linear and planar spaces belong to the wider class of incidence structures with diagram

$$\underset{0}{\circ}\overset{L}{\text{——}}\underset{1}{\circ}\overset{L}{\text{——}}\underset{2}{\circ}\ \cdots\ \underset{n-2}{\circ}\overset{L}{\text{——}}\underset{n-1}{\circ} \qquad (n \geq 2)$$

corresponding exactly to what Buekenhout calls n-*dimensional linear spaces* [11]; they will appear in the last chapter of this thesis. An n-*dimensional linear space* is a linear space S provided with n disjoint families $\mathcal{S}_i$ (i = 0,1, ..., n-1) of non-empty proper linear subspaces, called i-*subspaces*, or more generally *subspaces*, such that

(i) points and lines are the 0-subspaces and the 1-subspaces respectively,

(ii) if V is an i-subspace ($i \leq n-2$) and $x \in S-V$, then there is a unique (i+1)-subspace containing V and x, denoted by $\langle V,x \rangle$,

(iii) if V is an i-subspace and if W is a j-subspace containing V, then $i < j$,

(iv) any intersection of subspaces is a subspace or the empty set or S itself.

The 2-dimensional and 3-dimensional linear spaces are nothing else than the non-trivial linear spaces and planar spaces respectively. The notion of n-dimensional linear space is essentially the same as the notion of *geometric lattice of dimension* n (Birkhoff [3]), that is also the lattice of flats of a *simple matroid of rank* n+1 (Welsh [56]). The (n-1)-spaces of an n-dimensional linear space are called *hyperplanes*.

Note that the use of the word dimension is a little confusing here. For example, a projective space P of dimension $d \geq 3$ is at the same time a (2-dimensional) linear space if we consider only its points and lines, a planar space (dimension 3) if we consider only its points, lines and planes, and an n-dimensional linear space ($2 \leq n \leq d$) if we consider all its linear subspaces of dimension $< n$. On the other hand, P could be a plane of some non-trivial planar space !

6. METRICALLY REGULAR GRAPHS, ASSOCIATION SCHEMES, GENERALIZED n-GONS AND PARTIAL GEOMETRIES.

We briefly recall here a few definitions and results which will be used in Chapter II.

A graph G (always assumed to be finite, undirected, without loops and multiple edges) is called *strongly regular* with parameters (v, k, λ, μ) (Bose [4]) if v denotes the number of vertices of G, if every vertex is adjacent to exactly k vertices and if the number of vertices adjacent to any two adjacent (resp. non-adjacent) vertices is λ (resp. μ). Moreover, we always assume the non-degeneracy condition $2 \leq k \leq v-3$. The adjacency matrix of G has eigenvalues k, $n-m$, $-m$ with multiplicities 1, f, $v-1-f$, where

$$n = \sqrt{(\mu-\lambda)^2 + 4(k-\mu)} \quad ,$$

$$m = (n + \mu - \lambda)/2 \quad ,$$

$$f = \frac{1}{2}\left(v-1 - \frac{2k-(v-1)(\mu-\lambda)}{n}\right)$$

(I9) An important consequence of this is the *integrality condition* (Bose [4]) f is a non-negative integer. Moreover, m and n are integers unless $v = 4\mu+1$, $k = 2\mu$, $\lambda = \mu-1$ (conference graphs)

Four other necessary conditions for the existence of a strongly regular graph G such that both G and its complement are connected (i.e. $1 < m < n$) are the following

(I10) *Krein condition* (Higman [39], Delsarte [28])
If $1 < m < n$, then

$$\mu(n-m(m-1)) \leq (m-1)(n-m)(n+m(m-1))$$

(I11) *Absolute bound* (Delsarte, Goethals and Seidel [29], Neumaier [47])
If $1 < m < n$, then

$$v \leq \frac{1}{2} f(f+3)$$

Moreover, if G is not a Smith graph [16], then $v \leq \frac{1}{2} f(f+1)$.

(I12) *μ-bound* (Neumaier [46])
If a strongly regular graph has smallest eigenvalue $-m$, $1 < m < n$, then

$$\mu \leq m^3 (2m - 3)$$

(I13) *Claw-bound* (Neumaier [46])
If G is a strongly regular graph with smallest eigeinvalue $-m$ (where $m > 1$ is an integer) and if $\mu \neq m(m-1), m^2$, then

$$n \leq \frac{1}{2} m (m-1)(\mu+1) + m-1$$

Metrically regular graphs are a direct generalization of strongly regular graphs. A connected graph G is called *metrically regular* (or *distance-regular* (Biggs [2])) if the number of vertices at distance i from a vertex x depends only on i and not on x and if for any two vertices x and y at distance i, the number of vertices which are at distance j from x and at distance k from y depends only on the three distances i,j,k but not on the choice of x and y. It is easily shown that the metrically regular graphs of diameter 2 are exactly the connected strongly regular graphs. In turn, the notion of metrically regular graph can be generalized in two directions : that of t-metrically regular graph and that of association scheme, which we shall need later.

An *association scheme with* m *classes* on v objects (Bose and Shimamoto [6]) is a family of m binary symmetric relations satisfying the following conditions :

(i) any two objects are either 1-st, 2-nd, ..., or m-th associates

(ii) the number n_i of i-th associates of an object x is independent of x

(iii) for any two i-th associates x and y, the number p^i_{jk} of objects which are j-th associates of x and k-th associates of y is independent of the two i-th associates x and y.

Obviously, any metrically regular graph of diameter d may be seen as a d-class association scheme by calling i-th associates any two vertices which are at distance i.
Here again, we recall some necessary conditions for the existence of an m-class association scheme on v points, with parameters n_i and p^i_{jk}. Consider the vxv association matrices $A_j = (a^j_{xy})$ with entries $a^j_{xy} = 1$ if the vertices x and y are j-th associates and $a^j_{xy} = 0$ otherwise.
Let us denote by λ_{jk} the (not necessarily distinct) eigenvalues of A_j and by μ_k the multiplicity of λ_{jk} (it can be shown that μ_k does not depend on j).

I14) Then the following conditions are necessary for the existence of an association scheme with the above parameters :

(1) *Integrality condition* (Connor-Clatworthy [17])

$$\mu_k = v \,/\, (1 + \sum_{\ell=1}^{m} \lambda_{\ell k}^2 / n_\ell) \text{ are positive integers for } k = 1,\ldots,m$$

(2) *Krein condition* (Higman [39], Delsarte [28])

$$0 \leq 1 + \sum_{k=1}^{m} \lambda_{kr}\, \lambda_{ks}\, \lambda_{kt} \,/\, n_k^2 \leq v^2/(\mu_r\, \mu_s)$$

for $1 \leq r,s,t \leq m$

(3) (Mathon [44]). If $n_i \geq 2$ and $p_{ij}^i \geq 1$, then

$$1 + \sum_{\ell=1}^{m} \max \{2\, p_{i\ell}^i - n_\ell + 2\, \delta_{i\ell}\, (1 - \delta_{ij}),0\} \leq p_{ii}^j$$

Metrically regular graphs are deeply related to some interesting incidence structures. For instance, a *generalized n-gon* S (Tits [55]) is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ whose incidence graph $\mathcal{J}(S)$ is a bipartite graph on the two sets of vertices $\mathcal{P}$ and $\mathcal{L}$, in which any vertex has degree ≥ 2 and which has diameter n and girth 2n (i.e. the circuits of minimal length have length 2n). The elements of $\mathcal{P}$ are called *points* and the elements of $\mathcal{L}$ are called *lines*. It is well-known [55],[58] that if S is finite and if any vertex of $\mathcal{J}(S)$ has degree ≥ 3, then there is an ordered pair of integers (s,t), called the *order* of S, such that any line of S has size s+1 and any point of S is on exactly t+1 lines. The *point graph* (resp. *line graph*) of $S = (\mathcal{P}, \mathcal{L}, I)$ has vertex set $\mathcal{P}$ (resp. $\mathcal{L}$) and is obtained by calling two points (resp. two lines) adjacent if and only if they are collinear (resp. intersecting). The line graph of S is also the point
15) graph of the dual of S. The point graph and the line graph of a finite generalized n-gon of order (s,t) are metrically regular with diameter $[\frac{n}{2}]$, so that the above-mentioned necessary conditions apply to generalized n-gons. For example, the Feit-Higman non-existence theorem for generalized polygons [33], stating that a finite generalized n-gon of order (s,t) with $s,t \geq 2$ does not exist if $n \neq 2,3,4,6$ or 8, can be deduced from the integrality condition [51] and some classical inequalities between powers of t and s follow from the Krein condition [51], [39].

On the other hand, a finite *partial geometry* (R,K,T) where $R \geq 2$, $K \geq 2$, $T \geq 1$ are integers is a set of elements called *points* together with a family of distinguished subsets called *lines* such that

(i) each point belongs to exactly R lines and every pair of points is contained in at most one line,

(ii) each line contains exactly K points

(iii) for any line L, any point outside L is on exactly T lines intersecting L.

(I16) In particular, the partial geometries with T=1 are exactly the generalized quadrangles (or 4-gons) of order (s,t) = (K-1, R-1). It is obvious that the point graph and the line graph of a partial geometry are strongly regular, so that the above-mentioned necessary conditions apply to partial geometries.

Finally, let us mention another generalization of metrically regular graphs. A graph G is called *t-metrically regular* if for any $n \leq t$ and for any n-tuple of vertices $(x_1, x_2, \ldots, x_n)$, the number of vertices which are at distance δ_1 from x_1, δ_2 from $x_2, \ldots$, δ_n from x_n, depends only on the distances $\delta_1, \ldots, \delta_n$ and on the distances between the x_i's, but not on the choice of the n-tuple. So, the 2-metrically regular graphs are exactly the metrically regular graphs.

CHAPTER II. FINITE LINEAR SPACES WITH METRICAL REGULARITIES IN THEIR INCIDENCE GRAPHS.

1. INTRODUCTION.

In a linear space, the classical axiom of Pasch may be reformulated as follows

(*) *for any two disjoint lines* L *and* L', *any point outside* L ∪ L' *is on at most one line intersecting both* L *and* L'.

Indeed, suppose that condition (*) is satisfied. Let A and A' be two distinct lines intersecting in a point p and denote by L and L' two lines intersecting A and A' such that neither L nor L' passes through p. If L and L' were disjoint, we would have at least two lines through p intersecting L and L' : a contradiction. Thus L and L' have a point in common and Pasch's axiom is satisfied. The converse is obvious.

It follows that condition (*) characterizes the generalized projective spaces. If "at most one" is replaced by "exactly one" in (*), we get a characterization of the generalized projective spaces of dimension ≤ 3.

Note that the finite affine planes of order n have a similar property : for any two disjoint (hence parallel) lines L and L', any point outside L ∪ L' is on exactly n lines intersecting both L and L'.

These examples suggest the problem of classifying the linear spaces which satisfy the following condition :

(D2) *there is a non-negative integer* d_2 *such that for any two disjoint lines* L, L' *and any point* x *outside* L ∪ L', *there are exactly* d_2 *lines through* x *intersecting the two lines* L *and* L'.

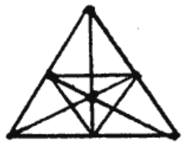
In the finite case, the answer is given by the following result :

Theorem 2. (A. Beutelspacher and A. Delandtsheer [1])
If S *is a finite linear space satisfying condition* (D2), *then one of the following occurs :*

(i) S *is a generalized projective space, and if the dimension of* S *is at least 4, then any line of* S *has exactly two points,*

(ii) S *is an affine plane, an affine plane with one point at infinity, or a punctured projective plane,*

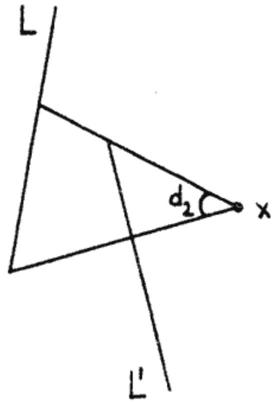
(iii) S *is the Fano quasi-plane, obtained from* PG(2,2) *by "breaking" one of its lines into three lines of size 2.*

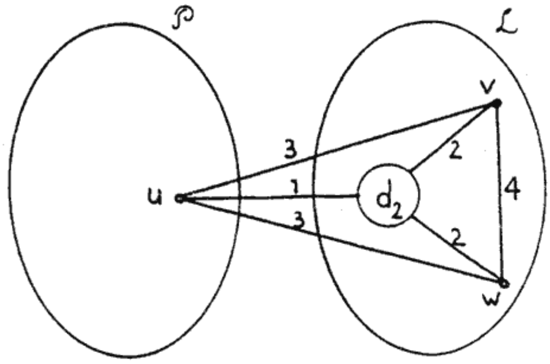


Conversely, each of these finite spaces satisfies (D2)

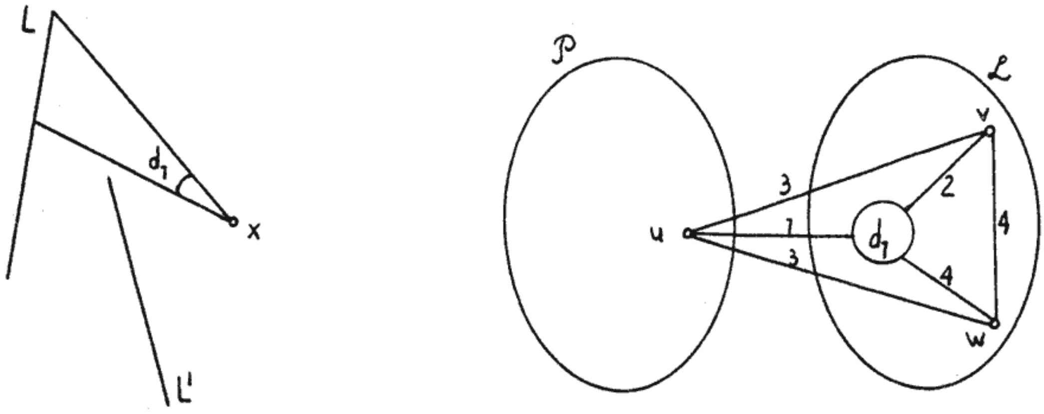
Note that condition (D2) can be viewed as a metrical condition on the incidence graph $\mathcal{G}$ of S. Indeed, remember that in the incidence graph of a linear space, any two points are at distance 2, two lines are at distance 2 or 4 according as they intersect or not, and a point and a line are at distance 1 or 3 according as they are incident or not. Therefore, condition (D2) may be translated in the following way :
there is a non-negative integer d_2 such that if u,v,w are any three vertices of $\mathcal{G}$ with distances d(u,v) = d(u,w) = 3 and d(v,w) = 4, then $\mathcal{G}$ contains exactly d_2 vertices t such that d(u,t) = 1, d(v,t) = d(w,t) = 2.
This leads naturally to the more general question : what happens if we choose other values for the distances in this condition ?
Among other things, we shall investigate the finite linear spaces satisfying one of the three conditions (D2), (D1), (D0), which are pictured below, first from a naive point of view, then in terms of the incidence graph.

(D2)

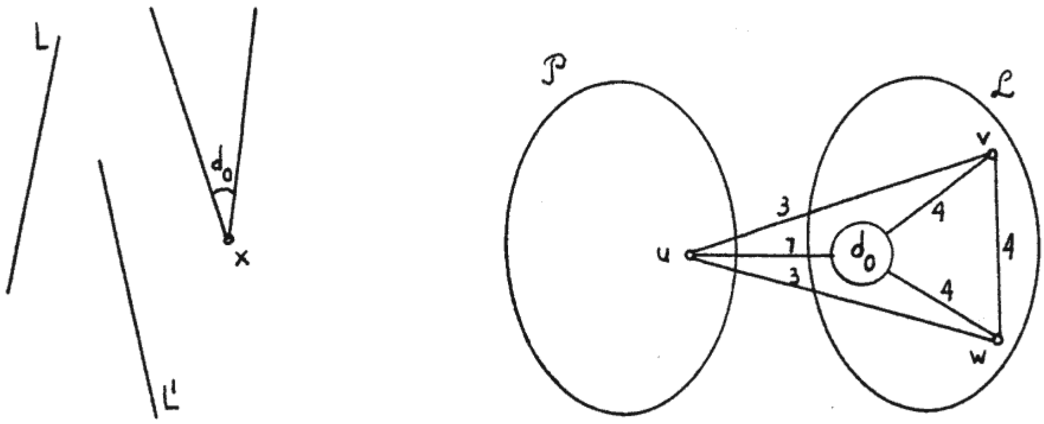




(D1)



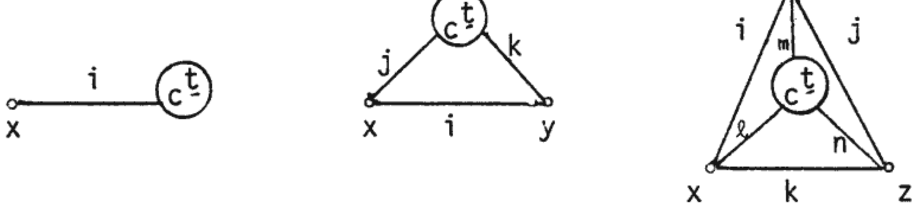
(D0)



The reason for the notations d_0, d_1 and d_2 is clear : the letter d reminds that the two lines L and L' are disjoint and the subscript reminds that we count certain lines intersecting 0, 1 or 2 of the lines L and L'.

2. GRAPH THEORETICAL BACKGROUND.

Actually, the above conditions form a part of the definition of a 3-metrically regular graph. Indeed, a connected graph is called *3-metrically regular* if G is metrically regular and if for any triple (x,y,z) of vertices such that d(x,y) = i, d(y,z) = j, d(z,x) = k, the number of vertices which are at distance ℓ from x, at distance m from y and at distance n from z depends only on the distances i,j,k,ℓ,m,n but not on the choice of triple (x,y,z)



These graphs have been studied quite a lot during the past few years. For instance, Cameron, Goethals and Seidel [16] have proved that if G is a connected 3-metrically regular graph of diameter 2, whose complement $\bar{G}$ is also connected, then G is the pentagon, or G is of pseudo or negative Latin square type, or G or $\bar{G}$ is a Smith graph (for more details, see [21]). On the other hand, Meredith [45] has proved that if G is connected 3-metrically regular graph of girth > 4, then G is a cycle (actually, the hypothesis of Meredith is stronger : he assumes that for any two isometric triples of vertices, there is an automorphism of G mapping the first onto the second; but his proof is essentially combinatorial).

Such metrical conditions are satisfied by point-, line- or incidence graphs of some classical geometries and have been used in certain characterization problems.

For instance, it follows immediately from theorems of Bose [5], Thas and Payne [48] that the point-graph of a generalized quadrangle of order (s,t) with $s > 1$ and $t > 1$ is 3-metrically regular if and only if $t = s^2$ if and only if every triad has exactly s+1 centers (i.e. for any triple (x,y,z) of pairwise non-collinear points, there are exactly s+1 points which are collinear with x,y and z). The point-graphs of generalized quadrangles with $s = 1$ or $t = 1$ are obviously 3-metrically regular.

Metrical conditions have also been used to characterize some classical generalized hexagons. Let us mention two examples. Thas [54] has proved that if S is a finite generalized hexagon of order (s,t) with $2 \leq t \leq s$, whose point-graph satisfies the following condition :

for any triple of vertices (x,y,z) with $d(x,y) = d(y,z) = 3$, $d(z,x) = 2$ (resp. $d(z,x) = 3$), there is at least one vertex v such that $d(y,v) = 1$, $d(x,v) = d(z,v) = 2$,

then $t = s$, s is a prime power, and S is isomorphic to the classical generalized hexagon H(s) associated with $G_2(s)$. Ronan [50] has characterized, among the finite generalized hexagons satisfying the regulus condition, those which are associated with $G_2(q)$., ${}^3D_4(q)$ and their duals, by means of the number n which counts, given any four vertices x,y,z,u with $d(x,y) = d(y,z) = 6$, $d(x,z) = d(u,y) = 4$, $d(u,x) = d(u,z) = 2$, the number (if it is distinct from t+1) of vertices v such that $d(x,v) = d(z,v) = 4$, $d(y,v) = 2$.

3. TERMINOLOGY AND NOTATIONS FOR LINEAR SPACES.

Let S be a finite linear space and $\mathcal{G}$ its incidence graph. We shall say that a triple (u,v,w) of vertices of $\mathcal{G}$ is of *type* $(\mathcal{S},i,j,k)$ (where $\mathcal{S}$ denotes either the point-set $\mathcal{P}$ or the line-set $\mathcal{L}$) if $u \in \mathcal{S}$, $d(u,v) = i$, $d(v,w) = j$ and $d(w,u) = k$. For a given type $(\mathcal{S},i,j,k)$ and a given triple of positive integers (ℓ,m,n), the *problem* $(\mathcal{S},i,j,k;\ell,m,n)$ consists in classifying the finite non-trivial linear spaces which satisfy the following condition : there is a constant c such that for any triple (u,v,w) of vertices of type $(\mathcal{S},i,j,k)$ in $\mathcal{G}$, the number of vertices t which are at distance ℓ from u, m from v and n from w is exactly c.

Obviously, certain choices of $\mathcal{S},i,j,k,\ell,m,n$ are absurd. An easy but rather tedious enumeration leads to 102 problems $(\mathcal{S},i,j,k;\ell,m,n)$ which have a sense (i.e. such that there exists a linear space whose incidence graph contains at least one 4-tuple (u,v,w,t) of the desired type). In the following sections, we shall investigate the most interesting of these problems namely $(\mathcal{P},3,4,3;1,2,2)$, $(\mathcal{P},3,4,3;1,2,4)$, $(\mathcal{P},3,4,3;1,4,4)$, $(\mathcal{P},1,2,1;3,4,4)$, $(\mathcal{P},3,2,3;1,4,4)$, $(\mathcal{P},3,2,3;1,4,2)$ and also a problem which is trivially equivalent to $(\mathcal{P},3,2,3;1,2,2)$.

Most of the remaining problems are easily solved and the answers are often the Steiner systems S(2,k,v), the projective planes or some "very small" linear spaces. However a few problems are still unsolved. For example, we have no other characterization of the finite linear spaces satisfying condition $(\mathcal{P},3,4,1;3,4,4)$ than saying that they are the finite linear spaces in which for any two disjoint lines L and L', the number of lines disjoint from $L \cup L'$ is a constant independent from L and L'. Note also that some of these problems may seem rather artificial : this is due to the fact that the distances between three vertices of $\mathcal{G}$ are not always sufficient to describe completely the corresponding geometrical configuration in S (for instance, three lines which are pairwise at distance 2 in $\mathcal{G}$ may be concurrent or form a triangle in S : this explains why the only solution of problem $(\mathcal{L},2,2,2;1,1,1)$ is the most trivial of all non-trivial linear spaces, namely the triangle S(2,2,3)).

Before starting the proofs of the main results, we briefly define some notations used in this chapter :

S will always denote a finite linear space of v points, with point-set $\mathcal{P}$ and line-set $\mathcal{L}$.

K is the set of line sizes in S

r_x is the degree of the point $x \in S$ (also denoted by r if all points of S have the same degree)

$r_L = \sum_{x \in L} r_x$ is the sum of the degrees of the points of L.

Usually, the size of a line A,B,C, ... will be denoted by a,b,c, ... respectively. A line G of S will be called *projective* if G intersects all the other lines of S. The total number of projective lines in S will be denoted by π and the number of projective lines containing a point x by ρ_x (or ρ if this number is independent of x).

A *bisecant* of two lines L and L' will be a line distinct from L and L' and intersecting $L \cup L'$ in exactly two points. A *trisecant* of three lines L, L', L" will be a line distinct from L, L', L" and intersecting $L \cup L' \cup L''$ in exactly three points.

For any triple (x,L,L') where L and L' are two disjoint lines and x is a point outside $L \cup L'$, we denote by $d_2(x,L,L')$ the number of lines through x which intersect both L and L', by $d_1(x,L,L')$ the number of lines through x which intersect L but not L' and by $d_0(x,L,L')$ the number of lines through x which are disjoint from L and L'. Then, the conditions (D2), (D1) and (D0) express that $d_2(x,L,L')$, $d_1(x,L,L')$ and $d_0(x,L,L')$ respectively are independent of the triple (x,L,L').

For any triple (x,L,L') where L and L' are two intersecting lines and x is a point outside $L \cup L'$, we denote by $i_2(x,L,L')$ the number of bisecants of L and L' through x, by $i_1(x,L,L')$ the number of lines through x which intersect L but not L' and by $i_0(x,L,L')$ the number of lines through x which are disjoint from L and L'. The conditions (I2), (I1) and (I0) express that $i_2(x,L,L')$, $i_1(x,L,L')$ and $i_0(x,L,L')$ respectively are independent of the triple (x,L,L').

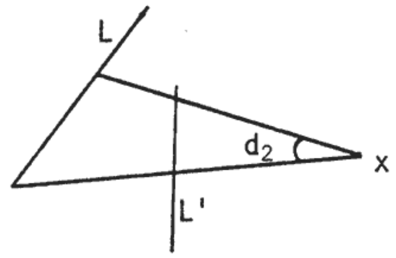
Note that in a Steiner system S(2,k,v), the three conditions (I0), (I1) and (I2) are equivalent. Indeed, for any two intersecting lines L and L', the degree of any point x outside L and L' is

$$\begin{aligned} r &= k + i_1(x,L,L') + i_0(x,L,L') \\ &= 2k - i_2(x,L,L') + i_0(x,L,L'), \end{aligned}$$

so that the constancy of one of the i_j's implies the constancy of the other two. This remark will be useful in the study of conditions (I0) and (I1). A similar argument shows that the conditions (D0), (D1) and (D2) are equivalent in a Steiner system S(2,k,v).

4. LINEAR SPACES SATISFYING CONDITION (D2)

We now prove Theorem 2 stated on pages II 1-2. Throughout this proof, S denotes a finite linear space satisfying condition (D2).



The proof is divided into two main parts : we first investigate the case where some additional regularity conditions are satisfied, then we handle the case in which S contains a projective line.

4.1. Some Additional Regularity Conditions.

Proposition 2.1. If $d_2 \leqslant 1$, *then* S *is a generalized projective space of dimension* d. *Moreover if* $d \geqslant 4$, $S = PG(d,1)$.

Proof. If $d_2 \leqslant 1$, condition (*) (hence also Pasch's axiom) is satisfied, so that S is a generalized projective space of dimension d.

Suppose that $d \geqslant 4$ and that there is a line L containing at least three distinct points p_1, q_1, q_2. There exist two disjoint lines L_1 and L_2 through q_1 and q_2, respectively. This implies $d_2 = 1$. On the other hand, since $d \geqslant 4$, there is a line L' disjoint from L and there is a point p outside the 3-dimensional subspace generated by L and L'. Clearly, there is no line through p intersecting L and L'. Hence $d_2 = 0$, a contradiction.

Thanks to Proposition 2.1, we may now assume that $d_2 \geqslant 2$, and also that there are two disjoint lines in S and that for any two disjoint lines there is a point outside their union (otherwise S would be a generalized projective space of dimension $\leqslant 3$).

First we consider the situation in which all lines of S have the same size.

Proposition 2.2. If all lines of S *have size* n, *then* S *is an affine plane of order* n.

Proof. Denote by L and L' two disjoint lines of S. Counting in two ways the number of flags (p,A) with $p \notin L \cup L'$ and $L \cap A \neq \emptyset \neq L' \cap A$, we get

$$(v - 2n)d_2 = n^2(n-2) ,$$

that is

$$(v-n)d_2 = n(n^2 - 2n + d_2) \qquad (1)$$

On the other hand, all points of S have the same degree r, with

$$v-1 = r(n-1) ,$$

or

$$v-n = (r-1)(n-1) . \qquad (2)$$

Equations (1) and (2) together imply that n-1 is a divisor of $n(n^2 - 2n + d_2)$, and so n-1 divides d_2-1. Using $2 \leq d_2 \leq n$, we conclude that $d_2 = n$. Therefore $v = n^2$ and S is an affine plane of order n.

The following Lemma is crucial for our purpose :

Lemma 2.1. Any two lines L_1 *and* L_2 *disjoint from a given line* L *have the same size.*

Proof. We count in two ways the number of trisecants of L, L_1 and L_2.
If L_1 and L_2 are disjoint, we get

$$|L_1|d_2 = |L_2|d_2 .$$

If L_1 and L_2 have a point in common, we get

$$(|L_1| - 1)d_2 = (|L_2| - 1)d_2 .$$

In both cases $|L_1| = |L_2|$.

Proposition 2.3. If S *contains two disjoint lines of different sizes, then* S *is the Fano quasi-plane.*

Proof. Let X and Y be two disjoint lines of different sizes x and y, respectively. We suppose $x < y$. Thanks to Lemma 2.1, any line of S intersects X or Y. Therefore, through any point p outside X and Y, there are d_2 bisecants of X and Y, $y-d_2$ lines of size y disjoint from X and $x-d_2$ lines of size x disjoint from Y. In particular, any point outside X and Y has degree $x+y-d_2$. Since $y > x \geq d_2$, we have $y > d_2$.

Step 1. If all lines disjoint from X have a point q in common, then S is the Fano quasi-plane.

Indeed, through any point not on X or Y there are exactly $y-d_2$ lines disjoint from X; since all these lines pass through $q \in Y$, it follows that $y = d_2+1$. This means conversely that any line through q which is incident with a point $p \neq q$ outside X, is disjoint from X. In other words, any line through q intersecting X is a line of size 2. But for any line X' of size 2, there exists a line Y' disjoint from X' and a point p' outside X' and Y'. Therefore $d_2 \leq 2$, hence $d_2 = 2$, $y = 3$, $x = 2$.

Since there are at least two lines through q disjoint from X, any point $p \neq q$ outside X has degree $x+y-d_2 = 3$. So, every line has at most three points. This implies $v \leq 7$. The assertion follows easily.

Now, let us assume that the lines disjoint from X have no point in common. We shall get a contradiction in three steps.

Step 2. There is a positive integer z such that any line intersecting X has size x or z with

$$z = 1 + (v-x)/y \tag{3}$$

and

$$y = z + (d_2-1)/(y-d_2) \ . \tag{4}$$

Indeed, since any line disjoint from X has size y, in our present situation, any point outside X has degree $x+y-d_2$. So, if we denote by d_X the number of lines disjoint from X, we have

$$d_X = (v-x)(y-d_2)/y \ .$$

Let Z be a line intersecting X. If Z is not a line of size x, then - in view of Lemma 2.1 - any line disjoint from X must intersect Z. Hence

$$(|Z| - 1)(y-d_2) = d_X, \text{ or } |Z| = 1 + (v-x)/y \ . \tag{5}$$

Using Lemma 2.1 again, we see that any two lines disjoint from X intersect. Counting in two ways the number of flags (p,L), where $L \neq Y$ is a line disjoint from X and $p \in Y$, we get by (5) :

$$y(y - d_2 - 1) = d_X - 1 = (z-1)(y-d_2) - 1.$$

Step 3. $d_2 = x$.

Indeed, assume on the contrary $d_2 < x$. Then any point outside X is on at least one line disjoint from Y.

We claim that the lines disjoint from Y have no point in common. (Assume that the lines disjoint from Y intersect in a point q. Since any point outside X and Y is on exactly $x-d_2$ lines disjoint from Y, we have $x = d_2+1$. This implies that any line through q and a point of Y is a line of size 2. Also, any such line X' is disjoint from at least one line disjoint from X. Using Lemma 2.1, we get $2 = |X'| = |X| = x$, and so $d_2 = x-1 = 1$, a contradiction).

Like in step 2 we see that any line intersecting Y has size y or z' with

$$z' = 1 + (v-y)/x \tag{6}$$

and

$$x = z' + (d_2-1)/(x-d_2) > z' .$$

The set K of line sizes of S is $\{x,y,z\} = \{x,y,z'\}$ with $z' < x < y$. Therefore $z = z'$, which yields, together with (3) and (6),

$$(v-x)/y = (v-y)/x ,$$

or

$$(y-x)(v-y-x) = 0 .$$

Since $x \neq y$, we have $v = x+y$, contradicting the fact that S contains a point belonging neither to X nor to Y.

Step 4. $d_2 \neq x$.

Indeed, assume on the contrary $d_2 = x$. This means that for any line L disjoint from X and any point p outside X and L, any line through p intersecting X has a point in common with L. In particular, any line Z intersecting X intersects any line disjoint from X as well. So, Z has size z by the argument of step 2. Therefore, the lines distinct from X have size z or y, according as they intersect X or not. Since any line $Z \neq X$ which intersects X has size z, we get

$$(v-x-y)x = (v-x-y)d_2 = x\,y\,(z-2) \qquad (7)$$

Moreover, for any point p on X,

$$(r_p - 1)(z-1) = v-x \qquad (8)$$

Equations (7) and (8) together imply $r_p = y+1$.

On the other hand, the degree of any point q not on X is $r_q = y$ since there is no line disjoint from both X and Y and since $d_2 = x$.

Next, we claim that there exist two disjoint lines intersecting X. (Assume that there were a line, say Z, which intersects X and all lines intersecting X. Counting the number of flags (p,L) with $p \notin X \cup Y$, $p \in Z$ and $L \cap X \neq \emptyset \neq L \cap Y$, we would get

$$(z-2)(d_2-1) = (x-1)(y-1) ,$$

that is $y = z-1 < z$, contradicting (4)).

Denote by Z and Z' two disjoint lines intersecting X. Lemma 2.1 states that any line L disjoint from $Z \cup Z'$ has size z. By (4), $z \neq y$, so L intersects X and therefore L meets Y. Counting in two ways the number of lines disjoint from $Z \cup Z'$, we get

$$(x-2)(y+1-2z+x) = (y-2)(y-2z+x)$$

hence

$$(y-x)(x+y-2z) = x-2 ,$$

therefore

$$y-x \mid x-2 .$$

But (4) implies that

$$y-x \mid x-1$$

so

$$y = x+1$$

and (4) yields $z = 2$.
Since any line intersecting X has size 2, no line through a point p outside $X \cup Y$ can intersect both X and Y, a contradiction.
By steps 1, 3 and 4, Proposition 2.3 is proved.

4.2. The case of projective lines.

In view of Propositions 2.2 and 2.3, we may suppose from now on that S contains lines of different sizes, and that any two disjoint lines have the same size.

Lemma 2.2. There is at least one projective line in S.

Proof. We assume that for any line L of S there is a line disjoint from L.

Let M be the maximal and m the minimal size of a line in S. Denote by X, X' (resp. Y, Y') two disjoint lines of size M (resp. m). Some obvious counting yields

$$M^2(m-2) \leq (v-2M)d_2 \tag{9}$$

and

$$(v-2m)d_2 \leq m^2(M-2) .$$

Together

$$M^2(m-2) + 2\, d_2(M-n) \leq m^2(M-2) ,$$

or

$$Mm(M-m) + 2d_2(M-m) \leq 2(M^2-m^2) = 2(M+m)(M-m).$$

Dividing by $M-m > 0$ gives

$$Mm + 2\, d_2 \leq 2(M+m) ,$$

therefore

$$0 \leq (M-2)(m-2) = Mm - 2(M+m) + 4 \leq Mm - 2(M+m) + 2d_2 \leq 0.$$

Hence $m = d_2 = 2$, and all the above inequalities are in fact equalities. In particular, equality holds in (9), so $v = 2M$, a contradiction.

With the following proposition, Theorem 2 is proved.

Proposition 2.4. S is a punctured projective plane or an affine plane with one point at infinity.

Proof. By Lemma 2.2, there exists a projective line G of size g. Let L and L' denote two disjoint lines, necessarily of the same size ℓ. Counting in two ways the number of flags (p,X) with $p \notin L \cup L'$, $p \in G$, where $X \neq G$ is a line intersecting L and L', we get

$$(g-2)(d_2-1) = (\ell-1)^2 . \tag{10}$$

This implies that all projective lines have the same size g and that all non-projective lines have the same size ℓ.

The proof of Proposition 2.4 will follow in a series of steps.

Step 1. There are at least two projective lines in S.
Indeed, assume that there is only one projective line G in S. Then any line through a point q outside G is a line of size ℓ and any line different from G through a point p on G is a line of size ℓ as well. Therefore

$$r_q(\ell-1) = v-1 = g-1+(r_p-1)(\ell-1).$$

Hence $\ell-1$ is a divisor of $g-1$, and so $\ell-1$ and $g-2$ are relatively prime. Now (10) implies that $(\ell-1)^2$ divides d_2-1, a contradiction.

Step 2. If all projective lines pass through a common point o, then S is an affine plane with one point o at infinity.
Indeed, since there is more than one projective line, any point $p \neq o$ has degree g. So, through any such point p there is the same number of projective lines; in particular, the set of projective lines is precisely the set of lines through o. Hence

$$v-1 = r_o(g-1) .$$

On the other hand, if p denotes a point different from o, we have

$$v-1 = g-1 + (r_p-1)(\ell-1) = g-1 + (g-1)(\ell-1) = (g-1)\ell$$

Together it follows that o has degree ℓ.

If L and L' are two disjoint lines, none of them passes through o. On the other hand, any of the ℓ lines through o is projective. Therefore $d_2 = \ell$. By (10), this implies $g = \ell+1$, hence $v = \ell^2+1$. Consider the incidence structure $S-\{o\}$, which consists of all points of S, except o. We have just seen that $S-\{o\}$ is a linear space with ℓ^2 points, in which any line has exactly ℓ points. Therefore, $S-\{o\}$ is an affine plane. Then, obviously, S itself is an affine plane with one point o at infinity.

Step 3. Suppose that for any point p of S there is a projective line not through p. Then S is a punctured projective plane.
Indeed, in the present situation, any point of S has degree g. Let us denote by π the total number of projective lines and by ρ the number of projective lines through a point. Clearly, the following equations hold :

$$\pi\, g = v\, \rho \tag{11}$$
$$v-1 = \rho(g-1) + (g-\rho)(\ell-1) = g(\ell-1) + \rho(g-\ell) \tag{12}$$
$$\pi-1 = g(\rho-1) \tag{13}$$

Equations (11) and (13) imply

$$v\, \rho = (g(\rho-1)+1)g \tag{14}$$

Using (12), we get

$$(\rho(g-\ell) - (g-1))(\rho-g) = 0$$

If $\rho=g$; S would be a projective plane, hence

$$\rho = (g-1)/(g-\ell) \tag{15}$$

Next, we claim that $g = \ell+1$. In order to prove this, denote by q and n the unique non-negative integers with

$$g = q\ell+n \quad \text{and} \quad 0 \leqslant n < \ell \;.$$

From (15) we deduce that $g-\ell$ divides $\ell-1$. Therefore

$$q\ell + n - \ell \mid \ell-1 \;,$$

in particular

$$q\ell + n-\ell \leqslant \ell-1 \;,$$

which implies $q=1$. So, n divides $\ell-1$. Denote by t the positive integer such that $nt = \ell-1$.

From (10) we infer that $g-2 = \ell+n-2 = n(t+1)-1$ divides $(\ell-1)^2 = n^2t^2$. But

$$n(t+1)-1 \mid (n(t+1)-1)n(t-1) = n^2t^2 - n^2 - n(t-1) \;,$$

therefore

$$n(t+1)-1 \mid n(n+t-1) ,$$

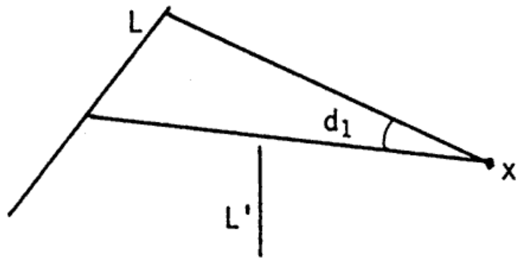
hence

$$n(t+1)-1 \mid n+t-1 ,$$

and so $n \leq 1$. Since $nt = \ell-1 \neq 0$, it follows that $n = 1$, i.e. $g = \ell+1$.

Now (10) implies $d_2 = \ell$, (15) yields $\rho=\ell$, and by (14) we have $v = \ell^2+\ell$. In particular, it follows that the lines of size ℓ form a "complete parallel class" of S. Introducing one new point which is incident precisely with the lines of size ℓ of S, it is easy to see that this new linear space is a projective plane of order ℓ. Thus S is a punctured projective plane.

5. LINEAR SPACES SATISFYING CONDITION (D1).



(D1) *there is a non-negative integer* d_1 *such that for any ordered pair of disjoint lines* L, L' *of* S *and any point* x *outside* L $\cup$ L', *there are exactly* d_1 *lines through* x *intersecting* L *but not* L'.

The finite linear spaces satisfying (D1) are classified in [27]:

Theorem 3. If S *is a finite non-trivial linear space satisfying condition* (D1), *then one of the following occurs :*

(i) S *is an affine plane, an affine plane with one point at infinity, a punctured projective plane or a (possibly degenerate) projective plane,*

(ii) S *is a 3-dimensional projective space* $PG(3,d_1)$,

(iii) S *is a 3-dimensional generalized projective space* $P(3, k+\ell)$,

(iv) S *is a degenerate projective space* $PG(d,1)$, $d \geq 2$.

Conversely, each of these finite spaces satisfies (D1).

Comparing Theorems 2 and 3, we observe that condition (D1) is stronger than condition (D2).

Let S denote a finite linear space satisfying (D1). The proof of Theorem 3 uses the following lemmas :

Lemma 3.1. If $d_1 = 0$, *then* S *is a semi-affine plane.*

Proof. Let L be a line and x a point outside L. If x is on two lines L' and L" both disjoint from L, then any point $y \neq x$ on L" is on at least one line (namely L") intersecting L' but not L, contradicting $d_1 = 0$. Therefore, for any line L of S, any point outside L is on at most one line disjoint from L. In other words, S is a semi-affine plane, and so, since S is finite, we know by (16) that S is an affine plane, an affine plane with one point at infinity, a punctured projective plane or a (possibly degenerate) projective plane.

Lemma 3.2. If S *is the union of two of its lines and if* $d_1 \geq 1$, *then* S *is either a degenerate projective plane or a generalized projective space* $P(3, k+\ell)$.

The proof is very easy and will be omitted.

Thanks to these lemmas, we may assume from now on that
(A) $d_1 \geq 1$ *and for any two lines of* S , *there is at least one point outside their union.*

Lemma 3.3. Any two disjoint lines have the same size.

Proof. Let L and L' be two disjoint lines. The degree of every point $x \notin L \cup L'$ is

$$\begin{aligned} r_x &= |L| + d_1 + d_0(x,L',L) \\ &= |L'| + d_1 + d_0(x,L,L'), \end{aligned}$$

and so $\qquad |L| = |L'|$.

Lemma 3.4. If S *contains non-projective lines of distinct sizes, then for every point* x *of* S *and for every size* ℓ *of a non-projective line, there are two disjoint lines of size* ℓ *not containing* x. *Moreover,* $\ell \geq 3\, d_1$.

Since ℓ is the size of a non-projective line, we conclude from Lemma 3.3 that S contains at least two disjoint lines of size ℓ, and so there is a non-projective line L of size ℓ not containing x. Suppose that all lines disjoint from L pass through x. Since L is non-projective, there is at least one line L' disjoint from L. Thanks to the assumption (A), we know that there is at least one point $y \notin L \cup L'$ and one line L" disjoint from L passing through y.

L' and L" have size ℓ by Lemma 3.3 and have the point x in common since we have assumed that all lines disjoint from L intersect in x.
Let $h \neq \ell$ be the size of a non-projective line. We conclude again from Lemma 3.3 that S contains at least two disjoint lines H_1 and H_2 of size h. If $x \notin H_1 \cup H_2$, then it follows from the assumption $d_1 \geq 1$ that x is on at least one line H_3 disjoint from H_1, and by Lemma 3.3, H_3 has size h. Therefore there is a line H of size h passing through x. If h = 2, let H' be a bisecant of L and L' disjoint from H. By lemma 3.3, H' has size h = 2, and so H' is disjoint from L". Hence, by Lemma 3.3 again, H' has size ℓ, contradicting $\ell \neq h$. Therefore h > 2 and H contains a point $y \notin L \cup L'$. Since $d_1 \geq 1$, we conclude that y is on at least one line disjoint from L and not containing x, contradicting the assumption that all lines disjoint from L contain x.

Therefore there exist two disjoint lines L_1, L_2 of size ℓ not containing x and two disjoint lines H_1, H_2 of size h not containing x. The point x is on at least 2 d_1 lines disjoint from H_1 or H_2 and so, by Lemma 3.3, x is on at least 2 d_1 lines of size h. Moreover, x is on exactly $\ell - d_1$ bisecants of L_1 and L_2, so that, by Lemma 3.3 again, x is on at most $\ell - d_1$ lines of size h. Therefore $3\,d_1 \leq \ell$.

Lemma 3.5. All non-projective lines have the same size.

Proof. Suppose that S contains non-projective lines of distinct sizes a and b, with a > b.

If S contains three pairwise disjoint lines A, A', A" of size a, and if B is a line of size b, then, by Lemma 3.3, every line of size a (in particular every line disjoint from A) intersects B. Therefore, counting in two ways the number of lines intersecting A' but disjoint from A and from $A' \cap B$, we get

$$(b-2)d_1 = \sum_{x \in A' - (A' \cap B)} (r_x - a - 1) ,$$

where

$$r_x = a + d_1 + d_0(x,A,A") \geq a + d_1 + 1$$

and so

$$(b-2)d_1 \geq (a-1)d_1$$

and, since $d_1 > 0$,

$b \geq a+1$, contradicting the assumption $a > b$.

Therefore S does not contain three pairwise disjoint lines of size a, and so for any triple (x,A,A') where A,A' are two disjoint lines of size a and $x \notin A \cup A'$, $d_0(x,A,A') = 0$. By Lemma 3.4, we conclude that every point x of S has degree

$$r = r_x = a + d_1 \ . \qquad (1)$$

Let B and B' be two disjoint lines of size b and let A be a line of size a. We know by Lemma 3.3 that every line of size b (in particular every line disjoint from B) intersects A. Therefore, counting in two ways the number of lines intersecting B' but disjoint from B and from $B' \cap A$, we get

$$(a-2)d_1 = \sum_{x \in B'-(B' \cap A)} (r_x - b-1) \ ,$$

and so, by (1),

$$(a-2)d_1 = (b-1)(a + d_1 - b-1)$$

or equivalently

$$(a-b-1)(d_1 - b+1) = 0 \ .$$

Since by Lemma 3.4, $b \geq 3\ d_1$ and since $d_1 \geq 1$, we have $d_1 - b+1 \neq 0$. Therefore $a = b+1$. Let A' be a line disjoint from A. Counting in two ways the number of lines intersecting A' but disjoint from A and $A' \cap B$, we get

$$(b-2)d_1 = (a-1)(r-a-1) \ ,$$

and so, using (1) and $a = b+1$, we conclude that $b = 2\ d_1$, contradicting Lemma 3.4.

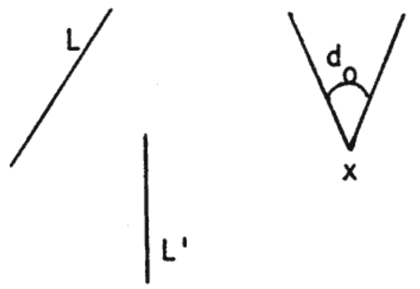
Proof of Theorem 3.

Let L, L' be two disjoint lines and let x be a point outside $L \cup L'$. Counting the lines containing x and intersecting L, we get

$$|L| = d_2(x,L,L') + d_1 \ ,$$

from which it follows, by Lemma 3.5, that $d_2(x,L,L')$ is independent of the triple (x,L,L'). In other words, condition (D2) is satisfied. Theorem 3 follows now easily from Lemmas 3.1, 3.2 and Theorem 2.

6. LINEAR SPACES SATISFYING CONDITION (D0).



(D0) *there is a non-negative integer* d_0 *such that for any two disjoint lines* L,L' *of* S *and any point* x *outside* $L \cup L'$, *there are exactly* d_0 *lines through* x *disjoint from* $L \cup L'$.

The finite linear spaces satisfying (D0) with $d_0 > 0$ are classified in

> *Theorem 4* [24]. *If* S *is a finite non-trivial linear space satisfying condition* (D0) *with* $d_0 > 0$, *then one of the following occurs :*
>
> (i) S *is an affino-projective plane (but not an affine plane with one point at infinity),*
>
> (ii) S *is an affine plane of order* ≥ 3 *from which either one point or one line has been removed,*
>
> (iii) S *is a 3-dimensional projective space* PG(3,q),
>
> (iv) S *is a generalized projective space* P(2, k+1), P(3, k+k) *or* PG(d,1) *with* $d \geq 2$
>
> *Conversely, each of these finite spaces satisfies* (D0) *with* $d_0 > 0$.

We assume here that the parameter d_0 is non-zero while in (D2) and (D1) we have also considered the case where the parameter was zero. We have no classification of the finite linear spaces which do not contain three pairwise disjoint lines.

Proof of Theorem 4.

Let S be a finite non-trivial linear space satisfying condition (D0) with $d_0 > 0$. During the proof, we always assume that S is not the union of two of its lines, because otherwise, as it is easily seen, S would be a degenerate projective plane P(2, k+1) or a generalized projective space P(3, k+k) consisting of two lines of the same size k.

We divide the proof into three cases, according as $d_0 = 1$, or $d_0 > 1$ and S contains a projective line, or $d_0 > 1$ and S contains no projective line.

6.1. The case $d_0 = 1$.

Proposition 4.1. If $d_0 = 1$, *then one of the following occurs*

- S *is a generalized projective space of 6 points, either* P(3, 3+3) *or* PG(5,1)
- S *is an affino-projective plane (but not an affine plane with one point at infinity)*
- S *is an affine plane from which either one point or one line has been removed.*

Proof. If for any line L of S and any point $p \notin L$, p is on at most one line disjoint from L, then S is a finite semi-affine plane, and so, by (I6), S is a projective plane, a punctured projective plane, an affine plane or an affine plane with one point a infinity (the last case is easily ruled out).

Assume now that there are two intersecting lines L_1 and L_2, both disjoint from a line L of S. Since $d_0 = 1$, the line L_1 (resp. L_2) determines a partition Δ_1 (resp. Δ_2) of the points of S-L into lines. On the other hand, any two lines $L_1' \in \Delta_1$ and $L_2' \in \Delta_2$ must intersect, otherwise a point of L_2' would be on at least two lines disjoint from $L \cup L_1'$. Therefore all lines of Δ_1 (resp. Δ_2) have the same size $|L_1|$ (resp. $|L_2|$). Moreover any line $L' \notin \Delta_1 \cup \Delta_2$ distinct from L is disjoint from at most one line of Δ_i $(i = 1,2)$: indeed if on the contrary L' is disjoint from two lines L_i' and L_i'' of Δ_i, then any point of L' inside S-L is on at least two lines disjoint from $L_i' \cup L_i''$, contradicting $d_0 = 1$. Therefore $|L_1| = |L_2|$ or $|L_1| = |L_2| \pm 1$.

1°) Consider first the case $|L_1| = n$, $|L_2| = n+1$. In this case, S-L is a set of n(n+1) points partitioned by Δ_1 into n+1 lines of n points and by Δ_2 into n lines of n+1 points. Any line $L' \notin \Delta_1 \cup \Delta_2$ $(L' \neq L)$ intersects at least n lines of Δ_1 and at most n lines of Δ_2, therefore L' contains exactly n points of S-L and is disjoint from exactly one line $L_1' \in \Delta_1$. If L' is disjoint from L, then any point of L' is on at least two lines disjoint from $L \cup L_1'$, a contradiction. Therefore any line $L' \notin \Delta_1 \cup \Delta_2$ $(L' \neq L)$ intersects L and has size n+1.

Let p be a point of S-L. Counting in two different ways the number of pairs (q, L') where L' is the line $\langle p,q\rangle$, we get

$$n(n+1) - 1 + |L| = (n-1) + n + |L|n$$

and so $|L| = n$. Thus the lines of size n partition S and all other lines have size n+1. Completing the lines of size n with one new point x, we define a linear space $S \cup \{x\}$ of $(n+1)^2$ points in which all lines have size n+1, that is an affine plane of order n+1. Therefore S is an affine plane from which one point has been removed.

2°) Consider now the case $|L_1| = |L_2| = n$. Then S-L has n^2 points and the lines intersecting L have size n or n+1. Let p be a point of L. Counting in two ways the number of pairs (q,L') where $\langle p,q\rangle = L' \neq L$, we get

$$n^2 = r_{p,n}(n-1) + r_{p,n+1}\, n \qquad (1)$$

where $r_{p,n}$ (resp. $r_{p,n+1}$) denotes the number of lines of size n (resp. n+1) intersecting L in p. It follows that n divides $r_{p,n}$ and $r_{p,n} \leqslant 2n$.

If there is a point $p \in L$ such that $r_{p,n} = 2n$, then (1) implies that n=2 and $r_{p,n+1} = 0$. Thus S-L consists of 4 points x,y,z and t. If there is a line L' distinct from L, of size greater than 2 (hence of size 3), say $L' = \{x,y,q\}$, then $\langle z,t\rangle = \{z,t,q\}$ (otherwise there would be no line through p disjoint from the two disjoint lines $\langle x,y\rangle$ and $\langle z,t\rangle$), and so S is the union of the three lines $\langle p,q\rangle$, $\{x,y,q\}$ and $\{z,t,q\}$, but there is no line through q disjoint from the disjoint lines $\langle x,z\rangle$ and $\langle p,t\rangle$, a contradiction. We conclude that all lines of S distinct from L have size 2. Therefore, since $d_0 = 1$, S is the generalized projective space PG(5,1) in which all lines have size 2.

If $r_{p,n} = 0$ for every point $p \in L$, then the linear space of n^2 points induced on S-L has only lines of size n, and so it is an affine plane of order n. Therefore S is an affine plane A completed with at most n-1 points at infinity, since A has n+1 directions of lines and since the lines of at least two directions Δ_1 and Δ_2 are disjoint from L. In other words, S is an affino-projective plane of order n which is not a semi-affine plane.

We may now assume that $r_{x,n} = 0$ or n for every point x of L and that $r_{p,n} = n$ for some point $p \in L$. Hence, thanks to (1), $r_{p,n+1} = 1$.

We first examine the case n = 2. Then S consists of the points of L, together with four additional points x,y,z,t. We know that the point p is on exactly one line of size 3 distinct from L, say {p,x,y}. If L contains a point $q \neq p$ not belonging to the line $\langle z,t\rangle$, then there is no line through q disjoint from $\{z,t\} \cup \{p,x,y\}$, a contradiction. Therefore S is the generalized projective space P(3, 3+3) consisting of two disjoint lines of size 3.

Finally, suppose that $n \geqslant 3$. Consider the linear space induced on S-L. The lines of S disjoint from L, together with the restrictions to S-L of the lines of S intersecting L in a point y for which $r_{y,n} = 0$, determine k partitions of S-L into n lines of size n ($k \geqslant 2$). On the other hand, the restrictions to S-L of the lines intersecting L in a point x for which $r_{x,n} = n$ determine h partitions of S-L into n lines of size n-1 and one line of size n

($h \geq 1$ is the number of points $x \in L$ for which $r_{x,n} = n$). Counting in two ways the number of ordered pairs of distinct points in the linear space S-L, we get

$$n^2(n^2-1) = k\, n^2(n-1) + h\, n(n-1)(n-2) + h\, n(n-1)$$

that is $n(n+1) = k\, n + h(n-1)$.

Therefore n divides h and, since $h \geq 1$, we have $n \leq h$, which, together with $k \geq 2$, implies

$$n(n+1) \geq 2\, n + n(n-1).$$

This inequality being in fact an equality, we conclude that k=2 and h=n. In other words, $r_{x,n} \neq 0$ for every point x of L and $|L| = h = n$.

Now we construct from S a bigger linear space in the following way : we add a new point to L, as well as to each line of Δ_1, these n+1 new points forming a new line N. We also add the new point of L to all lines of Δ_2. Finally, to each line L' of size n intersecting L, we add the new point of the unique line of Δ_1 disjoint from L'. The space S U N constructed in this way is a linear space. Indeed, if two lines of S of size n intersecting L in a given point x are both disjoint from the same line $L_1' \in \Delta_1$, then one of the lines through x must contain at least two points of L_1', contradicting the fact that S is a linear space. On the other hand, if two lines L', L" intersecting L in distinct points x', x" are both disjoint from a line $L_1' \in \Delta_1$ and intersect in a point z of S, then there is no line through x" disjoint from $L' \cup L_1'$, contradicting the hypothesis.

Since S U N is a linear space of $(n+1)^2$ points in which all lines have size n+1, it is an affine plane of order n+1. Therefore S is an affine plane from which one line has been removed.

6.2. The case $d_o > 1$ with a projective line.

From now on, we always assume $d_o > 1$.

Lemma 4.1.

(i) *If* A *and* B *are two disjoint lines and if* C *is a line disjoint from* A U B, *then*

$$(a-b)(c+d_o) = r_A - r_B$$

(ii) *If* A *and* B *are two intersecting lines and if* C *is a line disjoint from* A U B, *then*

$$(a-b)(c+d_o+1) = r_A - r_B$$

Proof. If L and L' are two disjoint lines and if p is a point outside $L \cup L'$, we have

$$r_p = |L| + |L'| - d_2(p,L,L') + d_o \qquad (2)$$

Counting in two ways the number of trisecants of A, B and C, we get

(i) if A and B are disjoint

$$\sum_{x\in A} d_2(x,B,C) = \sum_{y\in B} d_2(y,A,C)$$

that is, using (2),

$$a(b+c+d_o) - r_A = b(a+c+d_o) - r_B$$

or $\quad (a-b)(c+d_o) = r_A - r_B$

(ii) if A and B intersect in z

$$\sum_{\substack{x\in A\\ x\neq z}} d_2(x,B,C) = \sum_{\substack{y\in B\\ y\neq z}} d_2(y,A,C)$$

that is, using (2),

$$(a-1)(b+c+d_o) - r_A = (b-1)(a+c+d_o) - r_B \quad \text{or}$$

$$(a-b)(c+d_o+1) = r_A - r_B \ .$$

Corollary 4.1. If A *and* B *are two lines of different sizes, then all lines disjoint from* $A \cup B$ *have the same size.*

Corollary 4.2. If the lines A *and* B *have the same size and if there is a line disjoint from* $A \cup B$*, then* $r_A = r_B$.

Lemma 4.2. If two disjoint lines A *and* B *have different sizes, then all lines disjoint from* $A \dot{\cup} B$ *have the same size, equal either to* a *or to* b.

Proof. Suppose that there is a line C disjoint from $A \cup B$, of size $c \neq a,b$. By Corollary 4.1, all lines disjoint from $A \cup B$ have size c and they intersect each other (otherwise A and B would have the same size). So there are $c(s-1)+1$ such lines. Counting in two ways the number of pairs (p,L) where L is a line disjoint from $A \cup B$ and $p \in L$ yields

$$(v-a-b)d_o = (c(d_o-1)+1)c$$

that is

$$(v-a-b-c^2)d_o = -c^2+c \qquad (3)$$

By the same argument, all lines disjoint from A U C have b points and intersect each other, so that

$$(v-a-b^2-c)d_o = -b^2 + b \qquad (4)$$

Subtracting (4) from (3), we get

$$(b^2 - c^2 + c - b)d_o = b^2 - c^2 + c - b .$$

Since $d_o > 1$, we must have

$$(b-c)(b+c) = b-c .$$

But $b \neq c$, and so $b+c = 1$, a contradiction. Therefore any line disjoint from $A \cup B$ has size a, or b, and Corollary 4.1 ends the proof.

Lemma 4.3. If S *contains a projective line* G *of size* g *and if* S *is not a degenerate projective plane, then* $g \geq 4$ *is the size of any projective line of* S, *as well as the degree of any point outside a projective line. Moreover, the size of a non-projective line is less than* $g-1$.

Proof. Consider two disjoint lines A and B. Since $d_o \geq 1$, there is a point p outside $A \cup B \cup G$. The degree of p is equal to g because p is outside the projective line G and it is greater than a+1 since there are at least $d_o > 1$ lines disjoint from A through the point p outside $A \cup B$. So $g > a+1$ and in particular $g \geq 4$. Moreover, any projective line G' different from G has size g : indeed, the degree of any point outside $G \cup G'$ is $|G| = |G'| = g$.

Proposition 4.2. If S *contains a projective line, then* S *is a projective plane.*

Proof. It suffices to prove that any two lines intersect. Assume on the contrary that S contains two disjoint lines A and B. Let x and y be the points of intersection of A and B with the projective line G. Let C be a line disjoint from $A \cup B$, intersecting G in a point z. Counting in two ways the number of pairs (p,L) where $p \in L \cap G$ and $L \cap (A \cup C) = \emptyset$, we get

$$(g-2)d_o = |\mathcal{L}| - (r_x+r_z-1) - (a+c-2)(g-2) + (a-1)(c-1) \qquad (5)$$

where $|\mathcal{L}|$ denotes the total number of lines in S.
Considering the disjoint lines B and C, we have similarly

$$(g-2)d_o = |\mathcal{L}| - (r_y+r_z-1) - (b+c-2)(g-2) + (b-1)(c-1) \qquad (6)$$

Subtracting (6) from (5), we have

$$r_x - r_y = (b-a)(g-1-c) \qquad (7)$$

On the other hand, Lemma 4.1 gives

$$(a-b)(c+d_o) = (a-1)g + r_x - (b-1)g - r_y$$

or $r_x - r_y = (a-b)(c+d_o-g)$ (8)

Subtracting (8) from (7) yields

$$(b-a)(d_o-1) = 0$$

which implies b=a since $d_o > 1$, moreover $r_x = r_y$ thanks to (8). Thus any two disjoint lines have necessarily the same size and all points of G have the same degree r.

If n denotes the common size of two disjoint lines,

$$(g-2)s = |\mathcal{L}| - (2r-1) - (2n-2)(g-2) + (n-1)^2 \tag{9}$$

Solving for n, we get

$$n = g-1 \pm \sqrt{\delta}$$

where δ is the discriminant of equation (9).
Since $n < g-1$ by Lemma 4.3, n is uniquely determined, and so all non-projective lines have the same size n.

α) Suppose first that G is the only projective line in S. The total number of points in S is easily seen to be

$$g + (r-1)(n-1) = 1 + g(n-1)$$

(count the points on the lines passing through a point of G, or through a point outside G). It follows that n-1 divides g-1.

Given two disjoint lines A and B and a point p of G outside A ∪ B, the number of bisecants of A and B through p does not depend on $p \in G$ and is equal to $t = d_2(p,A,B) = 2n + d_o - r \leq n$. Counting in two ways the number of pairs (p,L) where $p \in G \cap L$, $p \notin A \cup B$ and $L \cap A \neq \phi \neq L \cap B$, we get

$$(n-1)^2 = (g-2)(t-1)$$

Since n-1 and g-2 are relatively prime, $(n-1)^2$ divides t-1, in contradiction with $t \leq n$.

β) Suppose now that there are at least two projective lines in S. If all projective lines have a point p in common, then, since $d_o > 0$, there is at least one non-projective line L through p. Let G be one of the projective lines. Consider two points x and y distinct from p and lying respectively on L and G. Counting in two ways the number of pairs (q,L') where $q \neq x$ and $\langle q,x\rangle = L'$ (resp. $q \neq y$ and $\langle q,y\rangle = L'$), we get

$$v-1 = (n-1) + (g-1)(n-1) = g-1 + (g-1)(n-1)$$

and so $n = g$, contradicting $n < g-1$. Therefore the projective lines of S have no point in common. This implies that all points of S have the same degree g.

Thus for any two disjoint lines A and B and for any point p outside $A \cup B$, the number of bisecants of A and B passing through p is $d_2(p,A,B) = 2n + d_0 - g$, which is independent of the triple (p,A,B), and so S satisfies condition (D2). Proposition 4.2 follows now easily from Theorem 2.

6.3. The case $d_0 > 1$ with no projective line.

Lemma 4.4. If there is no projective line in S, *then two disjoint lines have always the same size.*

Proof. Suppose on the contrary that A and B are two disjoint lines with different sizes a and b. By Lemma 4.2, we know that all lines disjoint from $A \cup B$ have the same size, equal either to a or to b. It is no loss of generality to assume that this size is b. Then Lemma 4.1 yields

$$(a-b)(b+d_0) = r_A - r_B \tag{14}$$

Now we shall give a proof in three steps :

Step 1. All lines disjoint from A have size b.

Suppose on the contrary that there is a line C disjoint from A and of size $c \neq b$. Lemma 4.2 implies that C intersects B, as well as any line disjoint from $A \cup B$. Thus, counting in two ways the number of pairs (p,L) where $p \in L$ and L is disjoint from $A \cup B$, we get

$$(c-1)d_0\, b = (v-a-b)d_0$$

that is

$$cb = v-a \tag{15}$$

Therefore c is uniquely determined and the only possible sizes for the lines disjoint from A are b and c. This implies that the size of any line disjoint from $A \cup C$ is either b or c. We show that b is impossible. Indeed, if $a \neq c$, this is obvious by Lemma 4.2; if $a = c$, suppose on the contrary that there is a line B' of size b disjoint from $A \cup C$. Then Lemma 4.1 gives

$$(a-b)(a+d_0) = r_A - r_{B'}$$

but Corollary 4.2 implies that

$$r_B = r_{B'}$$

and so

$$(a-b)(a+d_o) = r_A - r_B$$

contradicting (14).

Since any line disjoint from $A \cup C$ has size c, by Lemma 4.1,

$$(a-c)(c+d_o) = r_A - r_C \tag{16}$$

On the other hand, since A is disjoint from the two intersecting lines B and C, Lemma 4.1 yields

$$(b-c)(a+d_o+1) = r_B - r_C \tag{17}$$

Subtracting (17) from (16) and using (14), we get

$$(a-c)(c+d_o) - (b-c)(a+d_o+1) = (a-b)(b+d_o)$$

that is

$$(b-c)(b+c-1-2a) = 0$$

and, since $b \neq c$

$$2a = b+c-1 \tag{18}$$

Now (15) becomes

$$v = 2\,ab - b^2 + a + b \tag{19}$$

or $\quad v = 2\,ac - c^2 + a + c \qquad (20)$

Let D be a line of size d intersecting A. We shall prove that $d = a$. If on the contrary $d \neq a$ and if there is a line B' of size b disjoint from $A \cup D$, then, by Corollary 4.1, D intersects all lines of size c disjoint from A. Counting in two ways the number of pairs (p,L) where L is disjoint from $A \cup C$ and $p \in L$, we get

$$(d-2)d_o\, c = (v-a-c)d_o$$

Therefore

$$d = (v-a+c)/c = b + 1 \quad \text{thanks to (15).}$$

According to Lemma 4.2, the size of all lines disjoint from $B' \cup D$ is either b or $d = b+1$. In particular, there are lines disjoint from $B' \cup D$ and intersecting A, which have necessarily size a or b+1 because they intersect A and we have just seen that if $d \neq a$ is the size of a line intersecting A, then $d = b+1$.

Therefore, since $a \neq b$, all lines disjoint from $B' \cup D$ have size $b+1$. On the other hand, A being disjoint from the lines B and B' of the same size b, Corollary 4.2. yields

$$r_B = r_{B'}$$

which, together with (14) and Lemma 4.1, implies that all lines disjoint from $A \cup B'$ have size b. Therefore A intersects any line disjoint from $B' \cup D$. Counting in two ways the number of flags (p,L) where L is a line disjoint from $B' \cup D$, we get

$$(a-1)d_o\,(b+1) = (v - 2b - 1)d_o$$

and, using (19),

$$b(a-b) = 0, \text{ a contradiction.}$$

This implies that if $d \neq a$, D intersects all lines of size b disjoint from A. In particular, D intersects B and all lines disjoint from $A \cup B$. Then, counting in two ways the number of flags (p,L) where L is a line disjoint from $A \cup B$, we get

$$(d-2)d_o\,b = (v-a-b)d_o$$

Therefore

$$d = (v-a+b)/b = c+1 \quad \text{thanks to (15).}$$

We have seen that if a line of size $d \neq a$ intersects A, C and all lines disjoint from $A \cup C$, then $d = b+1$. Since $b \neq c$, we conclude that there is a line C' of size c disjoint from A and D. By Lemma 4.2, the size of all lines disjoint from $D \cup C'$ is either c or $d = c+1$. But some of these lines meet A and therefore have size a or $c+1$. If $a \neq c$, then all lines disjoint from $D \cup C'$ have size $c+1$, and so cannot be disjoint from $A \cup C'$. Therefore all these lines intersect A. Counting in two ways the number of flags (p,L) where L is disjoint from $D \cup C'$, we get

$$(a-1)d_o\,(c+1) = (v-2c-1)d_o$$

and, using (20),

$$c(a-c) = 0, \text{ a contradiction.}$$

Therefore $a = c$ and, by (18),

$$b = a+1 = c+1 \tag{21}$$

It follows that a and b are the only line sizes in S. For any point p of S, we denote by α_p the number of lines of size a through p. Counting in two ways the number of pairs (q,L) where $q \neq p$ and $\langle q,p\rangle = L$, we get

$$v-1 = \alpha_p\,(a-1) + (r_p - \alpha_p)(b-1)$$

which, using (19) and (21), can be written

$$a(a+2) - 1 = \alpha_p\,(a-1) + (r_p - \alpha_p)a$$

or $$a(a+2-r_p) = 1 - \alpha_p \qquad (22)$$

Therefore a divides $\alpha_p - 1$. Let m be the positive integer such that

$$m\,a = \alpha_p - 1 \qquad (23)$$

(22) yields

$$r_p = a + 2 + m \qquad (24)$$

Consider a line B' of size $b = a+1$ not passing through p and note that p is on at least d_o lines disjoint from B'. We have

$$r_p = a+2+m \geq b+d_o = a+1+d_o \geq a+3 \qquad (25)$$

and so $m \geq 1$. (23) and (24) imply that the number of lines of size b through p is

$$r_p - \alpha_p = 1+m+a(1-m) \geq 0$$

Therefore if $a \geq 4$, then $m = 1$ and, using (25) and (24), we get

$$r_p = a+3 = b+d_o$$

for any point p of S. We conclude that any line intersecting A meets B. Therefore, if $p \in A$, then A is the only line disjoint from B through p, contradicting $d_o \geq 2$.

If $a = 2$, then $b = 3$ and (19) gives $v = 8$. Similarly, if $a = 3$, then $b = 4$ and $v = 15$. But these values of v are incompatible with the fact that through any point $p \notin A \cup B$, there are at least two lines of size b disjoint from $A \cup B$.

Thus we have proved that all lines meeting A have a points. Let A' be a line meeting A. If A' is disjoint from a line B' of size b (necessarily disjoint from A), Corollary 4.2 gives

$$r_A = r_{A'}$$

since B' is disjoint from $A \cup A'$, and

$$r_B = r_{B'}$$

since A is disjoint from $B \cup B'$.

Then Lemma 4.1 and (14) imply that all lines disjoint from $A' \cup B'$ have size b, contradicting the fact that some of these lines meet A.

Therefore A' meets B and all lines disjoint from $A \cup B$. Counting the number of flags (p,L) where L is disjoint from $A \cup B$, we get

$$(a-2)d_o\, b = (v-a-b)d_o$$

and so, using (19), $a = b-2$
and, using (18), $c = b-3$

In particular, $a \neq c$. Now we claim that A' meets C and all lines disjoint from $A \cup C$ (indeed, suppose, on the contrary, that one of these lines, say C', is disjoint from A'. Corollary 4.2 and relation (16) imply that all lines disjoint from $A' \cup C'$ have size c, contradicting the fact that some of these lines meet A and that $a \neq c$). Counting in two ways the number of pairs (p,L) where L is disjoint from $A \cup C$ and $p \in L$, we get

$$(a-2)d_o\, c = (v-a-c)d_o$$

and so, using (20), $a = c-2$
which, together with (21) implies $b = c$, a contradiction.

This ends the proof of step 1. From now on, we may assume that
(*) in any pair of disjoint lines of different sizes, one of the lines has the property that all lines disjoint from it have the same size.

Step 2. a and b are the only two line sizes in S.

It suffices to prove that any line D intersecting A has size a or b. Suppose on the contrary that the size d of D is different from a and b. We shall prove that D intersects all lines disjoint from A. Indeed, if there is a line B' disjoint from $A \cup D$, then the size of all lines disjoint from $D \cup B'$ is either b or d. Assume first that this size is b. Then Lemma 4.1 yields

$$(d-b)(b+d_o) = r_D - r_B$$

Subtracting this from (14), we get

$$(a-d)(b+d_o) = r_A - r_D$$

On the other hand, Lemma 4.1 gives

$$(a-d)(b+d_o+1) = r_A - r_D$$

since B' is disjoint from the two intersecting lines A and D. These last two relations are contradictory. Therefore all lines disjoint from $D \cup B'$ have size d, and so they intersect A. Counting in two ways the number of flags (p,L) where L is disjoint from D U B', we get

$$(a-1)d_o\ d = (v-b-d)d_o\ .$$

Note that all lines disjoint from A U B' intersect D; then a similar counting argument yields

$$(d-1)d_o\ b = (v-a-b)d_o$$

Subtracting these two relations and simplifying by d_o, we get

$$d(a-b) = a-b$$

Hence

d=1, a contradiction.

Thus D intersects any line disjoint from A. In particular, D intersects B and all lines disjoint form A U B. Counting in two ways the number of flags (p,L) where the line L is disjoint from A U B, we get

$$(d-2)\ b = v-a-b$$

which shows that d is uniquely determined. Therefore there are exactly three line sizes in S, namely a, b and d.

Since D meets all lines disjoint from A, any line C disjoint from D intersects A. We shall prove that C intersects also B. This has already been proved if C is of size d. Suppose now that C has size a and is disjoint from B. Then Corollary 4.2 yields

$$r_A = r_C$$

Together with (14), this implies

$$(a-b)(b+s) = r_C - r_B$$

which means, by Lemma 4.1, that all lines disjoint from C U B have size b. Thanks to assumption (*), we conclude that all lines disjoint from C have size b, contradicting the fact that D and C are disjoint.

Thus if C is disjoint from B, C has size b. Since there is a line disjoint from B U C, Corollary 4.2 yields

$$r_B = r_C\ .$$

Applying Lemma 4.1 to the line B disjoint from the two intersecting lines A and C, we get

$$(a-b)(b+d_o+1) = r_A - r_C$$

These two relations contradict (14), and so any line disjoint from D intersects both A and B. Let C be such a line. Counting in two ways the number of lines disjoint from C ∪ D, we get

$$(a-2)d_o = (b-2)d_o$$

Hence a = b, a contradiction. Therefore any line intersecting A has size a or b.

Step 3. Let C be a line intersecting both A and B. Any line D disjoint from C and intersecting A ∪ B meets A and B.
In order to prove this, we shall consider several cases, according to the sizes of C and D.

(i) If D has size a, then we already know that D intersects A.

(ii) If C has size a and if D, intersecting A, has size b, then D meets B. Indeed, suppose on the contrary that D and B are disjoint. Lemma 4.1, applied to the line B disjoint from the two intersecting lines A and D, yields

$$(a-b)(b+d_o+1) = r_A - r_D$$

and Corollary 4.2 implies that

$$r_D = r_B$$

but these two relations contradict (14).

(iii) If C has size a and if D, intersecting B, has size b, then D meets A. Indeed, suppose on the contrary that D and A are disjoint. Corollary 4.2 yields $r_A = r_C$ and $r_B = r_D$.
Let E be a line disjoint from C and D, through a point $p \in A$. These last two relations, together with (14), imply that E has size b. Moreover, by Corollary 4.2,

$$r_E = r_D = r_B \ .$$

Lemma 4.1 applied to the line D disjoint from C ∪ E (resp. A ∪ E) yields

$$(a-b)(b+d_o) = r_C - r_E = r_A - r_B$$

and $$(a-b)(b+d_o+1) = r_A - r_E = r_A - r_B$$

These two relations contradict each other.

(iv) If C has size a and if D, intersecting A, has size a, then D meets B. Indeed, if on the contrary B is disjoint from D, Corollary 4.2 yields

$$r_A = r_D$$

which, toegether with (14) and Lemma 4.1 applied to the disjoint lines B and D, implies that all lines disjoint from B $\cup$ D have size b. Therefore, thanks to assumption (*), we conclude that any line disjoint from D has size b, contradicting the fact that C is disjoint from D.

(v) If C and D have size b and if D intersects B, then D meets A. Indeed, if on the contrary A and D are disjoint, Corollary 4.2 yields

$$r_B = r_D$$

and

$$r_C = r_D$$

Lemma 4.1 applied to the line D disjoint from the two intersecting lines C and A yields

$$(a-b)(b+d_o+1) = r_A - r_C = r_A - r_B$$

in contradiction with (14).

(vi) If C and D have size b and if D meets A, then D meets B. Indeed, if on the contrary D and B are disjoint, then

$$r_B = r_D$$

and, using Lemma 4.1, we get

$$(a-b)(b+d_o+1) = r_A - r_D .$$

These two relations contradict (14).

(vii) Finally, if C has size b and if D has size a and intersects A, then D intersects B. Indeed, suppose on the contrary that B and D are disjoint, and let B' be a line intersecting D and disjoint from A $\cup$ B. Corollary 4.2 yields

$$r_A = r_D$$

By Lemma 4.1, we get

$$(a-b)(b+d_o) = r_A - r_{B'}$$

since B is disjoint from the two disjoint lines A and B', and

$$(a-b)(b+d_o+1) = r_D - r_{B'} = r_A - r_{B'}$$

since B is disjoint from the two intersecting lines D and B'. These two relations contradict each other.

Now, take a line D disjoint from C and intersecting A $\cup$ B. Thanks to step 3, counting in two ways the number of lines disjoint from C $\cup$ D and intersecting A ∪ B, we have

$$(a-2)d_o = (b-2)d_o$$

which implies a = b, a contradiction. This proves Lemma 4.4.

Lemma 4.5. If there is no projective line in S *and if any two disjoint lines of* S *have the same size, then all lines of* S *have the same size.*

Proof. Suppose on the contrary that there are at least two distinct line sizes n and ℓ. Consider two disjoint lines A and B of size n and let L be a line of size ℓ. L and all lines disjoint from L intersect A and B. Counting in two ways the number of bisecants of A and B passing neither through $p \in A \cap L$ nor through $q \in B \cap L$, we get

$$(n-1)^2 = \sum_{\substack{x \in L \\ x \neq p,q}} [d_2(x,A,B)-1] + \delta$$

where δ denotes the number of lines disjoint from L.
Since

$$d_2(x,A,B) = 2n + d_o - r_x$$

we get

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_p + r_q \tag{26}$$

Through any point $y \neq p,q$ on L, consider a line C disjoint from A ∪ B. The same counting argument applied to the pairs of lines {A,C} and {B,C} instead of {A,B} yields

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_p + r_y \tag{27}$$

and

$$(n-1)^2 = (\ell-2)(2n+d_o-1) + \delta - r_L + r_q + r_y \tag{28}$$

(26), (27) and (28) imply that $r_p = r_q = r_y$, and so all points of L have the same degree. One proves in the same way that all points of any line of size n have the same degree. Finally, since any line of size n intersects L, all points of S have the same degree r.

Counting in two ways the number of pairs (y,C) where C is disjoint from A ∪ B and $y \in C \cap L$, we get

$$(\ell-2)d_o = |\mathcal{L}| - (2r-1) - 2(n-1)(r-2) + (n-1)^2$$

Similarly,

$$(n-2)d_0 = |\mathcal{L}| - (2r-1) - 2(\ell-1)(r-2) + (\ell-1)^2$$

Subtracting these two relations, we have

$$(\ell-n)d_0 = 2(\ell-n)(r-2) + n^2 - \ell^2 + 2(\ell-n)$$

Therefore, since $\ell \neq n$,

$$2r = \ell + n + d_0 + 2 \ . \qquad (29)$$

Suppose $\ell > n$ and let L' be a line disjoint from L. The number of lines intersecting L but not L', and passing through a point $p \notin L \cup L'$, is

$$r - \ell - d_0 \geq 0$$

Multiplying by 2 and using (29), we get

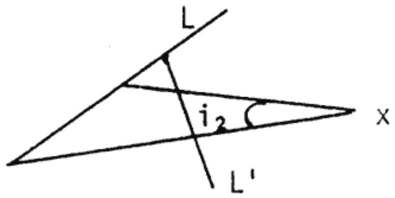
$$2 - d_0 \geq \ell - n$$

in contradiction with $d_0 \geq 2$ and $\ell \geq n+1$.

Proposition 4.3. If S *contains no projective line, then* S *is an affine plane, a projective space of dimension 3 or a generalized projective space in which all lines have size 2.*

Proof. Thanks to Lemmas 4.4 and 4.5, we know that all lines of S have the same size n, and so that all points have the same degree r. Thus, for any two disjoint lines L, L' and any point p outside $L \cup L'$, $d_2(p,L,L') = 2n + d_0 - r$ is independent of the choice of p, L and L'. Therefore by Theorem 2 (more precisely by Propositions 2.1 and 2.2), we have the desired conclusion.

7. LINEAR SPACES SATISFYING CONDITION (I2).



(I2) *there is a non-negative integer* i_2 *such that for any two intersecting lines* L, L' *and any point* x *outside* $L \cup L'$, *there are exactly* i_2 *bi-secants of* L *and* L' *passing through* x.

The only finite linear spaces which are known to satisfy condition (I2) are the (possibly degenerate) projective planes, the affine planes and the Steiner systems S(2,2,v). We will prove that other examples (if any) should be

Steiner systems $S(2,k,v)$ satisfying some rather restrictive arithmetical conditions on i_2 and k.

Theorem 5 [20]. *If* S *is a finite non-trivial linear space satisfying condition* (I2), *then*

(i) S *is a degenerate projective plane* (and $i_2 = 1$),

or (ii) S *is a Steiner system* $S(2,k,v)$ *with* $i_2 < k < v$, *such that*

(1) $i_2 | (k-1)(k-2)$

(2) $k | [(k-1)(k-2)/i_2+2][(k-1)(k-2)/i_2+1]$

(3) $i_2(2k-2-i_2) | k(k-1)^2(k-2)$

(4) *if* $(k-1)(k-2)/i_2$ *is odd, then* $k-1-i_2$ *is a square*
if $(k-1)(k-2)/i_2$ *is even, then the Diophantine equation*

$$(k-1-i_2)x^2 + (-1)^{(k-1)(k-2)/2i_2}\, i_2\, y^2 = z^2$$

has a solution in integers x,y,z *not all zero.*
Moreover $i_2 = k-1$ *iff* S *is a projective plane of order* $k-1$

$i_2 = k-2$ *iff* S *is an affine plane of order* k.

Note that this theorem gives a partial answer to an open problem mentioned by Cameron in [15, p.54].

Proof. The proof of (3) (resp. (4)) is based on the construction of certain partial geometries (resp. symmetric 2-designs) associated with S, whence all other statements follow directly from the linear structure of S. Thus we shall divide the proof into three parts :

7.1. The linear space S.

First of all, note that if $i_2 = 0$, any bisecant of two intersecting lines has size 2, and so all lines of S have size 2. Therefore, from now on, we shall suppose $i_2 \geq 1$.

It is easy to check that if S is the union of two intersecting lines, then condition (I2) is satisfied iff S is a degenerate projective plane. Hence we may assume, in what follows, that any point of S is on at least three lines.

Let L and L' be two lines intersecting in x. Consider a third line L" passing through x. Counting in two different ways the number of trisecants of L, L' and L", we get

$$(|L| - 1)i_2 = (|L'| - 1).i_2$$

that is $|L| = |L'|$.
Therefore any two intersecting lines have the same size, and so all lines of S have the same size, which we denote by k. Counting in two different ways the number of pairs (p, L_1) where $p \notin L \cup L'$ and L_1 is a bisecant of L and L' passing through p, we get

$$(v - 2k + 1)i_2 = (k-1)^2(k-2) \qquad (5)$$

If r denotes the degree of a point of S, we have

$$v-k = (r-1)(k-1) \qquad (6)$$

(5) and (6) yield

$$(r-2)i_2 = (k-1)(k-2) \qquad (7)$$

which implies $i_2|(k-1)(k-2)$ (1).
On the other hand, the number of lines of S is vr/k, which must be an integer. Thanks to (6), we deduce that

$$k|r(r-1) \qquad (8)$$

which, using (7), gives

$$k|[(k-1)(k-2)/i_2 + 2][(k-1)(k-2)/i_2+1] \qquad (2)$$

For $i_2 = k-2$ (resp. $k-1$), (5) yields $v = k^2$ (resp. k^2-k+1) and so S is an affine (resp. projective) plane.
This ends the first part of the proof.

Note that (2) implies

$$k|2(i_2+1)(i_2+2) \qquad (9)$$

Thanks to (1), (2), (3) it is easily shown that $k \neq 2(i_2+1)(i_2+2)$ and that $k = (i_2+1)(i_2+2)$ is admissible only for $i_2 = 2$. So (9) implies that $k \leq 2(i_2+1)(i_2+2)/3$ for $i_2 > 2$ and we conclude that $\sqrt{k} < i_2$ for any $i_2 > 3$.

7.2. Partial geometries and association schemes.

Given a line L of S, the point set S-L, provided with the restrictions to S-L of the lines of S intersecting L, forms a partial geometry with para-

meters $(R,K,T) = (k, k-1, i_2)$ having $V = v-k = (k-1)^2(k-2)/i_2+k-1$ points and $B = k(k-1)(k-2)/i_2+k$ lines.
The point graph (resp. line graph) of a partial geometry is defined by calling two points (resp. two lines) adjacent iff they are collinear (resp. concurrent). The line graph is also the point graph of the dual partial geometry. Thus four strongly regular graphs are associated with every partial geometry, namely the point graph G_p, the line graph G_L and their complements $\bar{G}_p$ and $\bar{G}_L$.
In our problem, the parameters of G_p are

$$v_p = V = (k-1)^2(k-2)/i_2 + k - 1 \qquad k_p = k(k-2)$$

$$\lambda_p = (k-1)(i_2-1) + k - 3 \qquad \mu_p = ki_2$$

Besides the trivial eigenvalue k_p, the other eigenvalues of G_p are

$$r_p = k - i_2 - 2 \quad \text{with multiplicity } f_p = (k-2)(k-1)^2k/i_2(2k - 2 - i_2)$$

and

$$s_p = -k \qquad \text{with multiplicity } g_p = v_p - 1 - f_p$$

For the line graph G_L,

$$v_L = k(k-1)(k-2)/i_2+k \qquad k_L = (k-1)^2$$

$$\lambda_L = (k-2)i_2 \qquad \mu_L = (k-1)i_2$$

$$r_L = k - i_2 - 1 \qquad f_L = f_p$$

$$s_L = -(k-1) \qquad g_L = v_L-1-f_L = k-1 + \frac{k(k-1)(k-2)(k-1-i_2)}{i_2(2k-2-i_2)}$$

Multiplicities of eigenvalues being integers, we get

$$i_2(2k-2-i_2) \mid (k-2)(k-1)^2 \qquad (3)$$

The other known necessary conditions for the existence of a strongly regular graph, namely the Krein condition (I10), the absolute bound (I11), the μ-bound (I12) and the claw-bound (I13), are tedious but easy to check : they give nothing more than the previous conditions.

The line graph G_L may also be viewed as a 2-class association scheme Σ_2 if we say that two lines are first associates when they are distinct and adjacent (i.e. when the corresponding lines of S form a triangle with L) and second associates otherwise. Actually we can define a 3-class association scheme Σ_3 by subdividing the second class of Σ_2. The vertices of Σ_3 are the lines of S intersecting L, two vertices are first associates if the corresponding lines intersect in a point outside L,

, second associates if the corresponding lines intersect in a point of L and third associates if the corresponding lines are disjoint. Indeed, the number n_i of i-th associates of a vertex x is independent of x , and for any two i-th associates x and y, the number p^i_{jk} of vertices which are j-th associates of x and k-th associates of y does not depend on the pair x,y.

Consider the $v_L \times v_L$ association matrices $A_j = (a^j_{xy})$ with entries $a^j_{xy} = 1$ if the vertices x and y are j-th associates and $a^j_{xy} = 0$ otherwise. We denote by λ_{jk} the (not necessarily distinct) eigenvalues of A_j and by μ_k the multiplicity of λ_{jk} (it can be shown that μ_k does not depend on j). The parameters of Σ_3 are

$$
\begin{array}{lll}
n_1 = (k-1)^2 , & n_2 = (k-1)(k-2)/i_2 , & n_3 = (k-1)(k-2)(k-1-i_2)/i_2 \\
p^1_{11} = (k-2)i_2, & p^1_{12} = k-2 , & p^1_{13} = (k-2)(k-1-i_2) \\
p^1_{22} = 0 , & p^1_{23} = (k-2)(k-1-i_2)/i_2, & p^1_{33} = (k-2)(k-1-i_2)(k-2-i_2)/i_2 \\
p^2_{11} = (k-1)i_2 , & p^2_{12} = 0 , & p^2_{13} = (k-1)(k-1-i_2) \\
p^2_{22} = (k-1)(k-2)/i_2-1, & p^2_{23} = 0 , & p^2_{33} = (k-1)(k-2-i_2)(k-1-i_2)/i_2 \\
p^3_{11} = (k-1)i_2, & p^3_{12} = k-1 , & p^3_{13} = (k-1)(k-2-i_2) \\
p^3_{22} = 0 , & p^3_{23} = (k-1)(k-2-i_2)/i_2, & p^3_{33} = (k-1)[(k-1-i_2)(k-3-i_2)+i_2+1 \\
 & & \qquad /i_2-1 \\
\lambda_{11} = k-1-i_2 , & \lambda_{12} = -(k-1) , & \lambda_{13} = -(k-1) \\
\lambda_{21} = -1, & \lambda_{22} = -1 , & \lambda_{23} = (k-1)(k-2)/i_2 \\
\lambda_{31} = -(k-1-i_2), & \lambda_{32} = k-1 , & \lambda_{33} = -(k-2)(k-1-i_2)/i_2
\end{array}
$$

$$\mu_1 = (k-2)(k-1)^2k/i_2(2k-2-i_2), \quad \mu_2 = k(k-1)(k-2)(k-1-i_2)/i_2(2k-2-i_2), \quad \mu_3 = k-1$$

Condition (3) and $1 + \mu_1 + \mu_2 + \mu_3 = v_L$ imply that all multiplicities are integers. Moreover it is not difficult to check that the Krein condition as well as the condition given by Mathon (I14) are satisfied for all pairs (i_2,k) satisfying (1), (2), (3) with $i_2 \leq k-3$.

7.3. Symmetric 2-designs.

If $i_2 \leq k-2$, a non-trivial symmetric 2-design D(p,q) can be associated with any pair (p,q) of distinct points of S as follows : the points of D(p,q) are the lines through p distinct from the line <p,q>, the blocks of D(p,q) are the lines through q distinct from <p,q>, a point and a block being incident iff the corresponding lines intersect. Each point of $D(p,q)$ is on $k-1$ blocks and each block contains $k-1$ points, any two points are on i_2 blocks and any two blocks have i_2 points in common. Thus $D(p,q)$ is an $S_{i_2}(2,k-1,(k-1)(k-2)/i_2+1)$. The Bruck-Chowla-Ryser theorem (I3) gives a necessary condition for the existence of such a design, namely (4). This ends the proof of Theorem 5.

Unfortunately, there remain infinitely many pairs of parameters (i_2,k) with $i_2 \leqslant k-3$ satisfying the conditions (1), (2), (3) and (4) : for instance, all pairs (i_2,k) with $i_2 = 9^n$ $(n \geqslant 1)$ and $k = (i_2+1)(i_2+2)/2$. Indeed, it is easy to check that these pairs satisfy the first three conditions. We shall prove that the fourth condition is also satisfied by using the following

Theorem [37], [40]. The equation

$$b\, x^2 + c\, y^2 = z^2 \qquad (10)$$

has solutions in integers x,y,z not all zero if and only if for every prime p as well as for $p = \infty$, the Hilbert norm-residue symbol $(b,c)_p$ is equal to $+1$.

The symbol $(b,c)_p$ is defined to be $+1$ or -1 according as the congruence

$$b\, x^2 + c\, y^2 \equiv z^2 \pmod{p^m}$$

does have solutions in integers x,y,z, not all multiples of p, for every power p^m of the prime p, or not, and $(b,c)_\infty = +1$ or -1 according as (10) does or does not have solutions in real numbers $x,y,z \neq 0$. Thus $(b,c)_\infty = +1$ unless both b and c are negative. We shall use the following properties of the Hilbert norm-residue symbol :

(P1) $\prod_p (b,c)_p = 1$ (the product being over all primes p, including $p = \infty$)

(P2) $(b,c^2)_p = 1$,

(P3) if p is an odd prime and $b,c \not\equiv 0 \pmod p$, then $(b,c)_p = 1$,

(P4) if p is an odd prime and if $c_1 \equiv c_2 \not\equiv 0 \pmod p$, then $(b,c_1)_p = (b,c_2)_p$.

If $i_2 = 9^n$ $(n \geqslant 1)$ and $k = (i_2+1)(i_2+2)/2$, then $k \equiv (9^n+2)(9^n+1)/2 \equiv 3 \pmod 4$, and so $(k-1)(k-2) \equiv 2 \pmod 4$. Since i_2 is odd and divides $k-1$, we conclude that $(k-1)(k-2)/2\, i_2$ is an odd integer and (4) becomes

$$\frac{9^n+1}{2}\, 9^n\, x^2 - 9^n\, y^2 = z^2$$

has a solution in integers x,y,z not all zero, or equivalently

$$\frac{9^n+1}{2} x^2 - y^2 = m^2 \qquad (11)$$

has a solution in integers x,y,m not all zero. Since (11) has solutions in real numbers $x,y,m \neq 0$, thanks to the above theorem, we have only to prove that $(\frac{9^n+1}{2}, -1)_p = +1$ for every prime p. Moreover thanks to (P1) and to $(\frac{9^n+1}{2}, -1)_\infty = +1$, it is sufficient to check this for the odd primes. Thus, let p be any odd prime.

If $p \nmid \frac{9^n+1}{2}$, then $(\frac{9^n+1}{2}, -1)_p = 1$ by (P3). If on the contrary $p \mid \frac{9^n+1}{2}$, then $9^n \equiv -1 \pmod p$ and we deduce from (P4) and (P2) that

$$(\frac{9^n+1}{2}, -1)_p = (\frac{9^n+1}{2}, 9^n)_p = 1 .$$

Therefore condition (4) is satisfied for the above values of i_2 and k.

For $k \leq 100$ there are only six pairs (i_2,k) with $i_2 \leq k-3$ satisfying conditions (1) to (4), namely (2,12), (24,65), (20,66), (3,10), (7,36) and (9,55) (for the last three, $k = (i_2+1)(i_2+2)/2$). The existence of a symmetric 2-design $S_{i_2}(2, k-1, (k-1)(k-2)/i_2+1)$ is known [57] for only two of these pairs, namely (2,12) and (3,10).

Note that if S satisfies one further hypothesis, the symmetric 2-designs D(p,q) are extendable and (I4) rules out all pairs (i_2,k) with $i_2 \leq k-3$, except one. Given a point p and a line L not through p, let $\mathcal{L}(p,L)$ denote the set of lines passing through p and intersecting L and let S(p,L) denote the set of points distinct from p and belonging to a line of $\mathcal{L}(p,L)$. Note that if $i_2 < k-1$, two lines L, L' such that $\mathcal{L}(p,L) = \mathcal{L}(p,L')$ are necessarily disjoint.

Proposition 5.1. Let S *be a finite linear space satisfying condition* (I2) *with* $1 \leq i_2 \leq k-3$. *If there is a point* $p \in S$ *such that for any line* L *not through* p, *any point of* S(p,L) *is on a line* L' *such that* $\mathcal{L}(p,L) = \mathcal{L}(p,L')$, *then* $i_2 = 2$ *and* k = 12.

Proof. Consider the design D(p) whose points are the lines through p, whose blocks are the distinct sets $\mathcal{L}(p,L)$, incidence of points and blocks being given by set inclusion. Thus any block has exactly k points and the total number of points is $r = (k-1)(k-2)/i_2+2$. By hypothesis, given a line L not through p, any point of S(p,L) is on one and only one line L' such that $\mathcal{L}(p,L) = \mathcal{L}(p,L')$ (only one because we have seen before that such lines are necessarily disjoint).

It follows that any three points of D(p) are in exactly i_2 blocks and that the total number of blocks is r(r-1)/k. Therefore, D(p) is a symmetric 3-design $S_{i_2}(3,k,(k-1)(k-2)/i_2+2)$. Note that for any point q distinct from p, the 2-design D(p,q) is isomorphic to the derived design of D(p) relative to <p,q>. Theorem 5 and the theorem of Cameron (I4) listing the admissible parameters of symmetric 3-designs imply immediately that $i_2 = 2$ and $k = 12$. The pair $(i_2,k) = (2,12)$ corresponds to a hypothetical linear space S(2, 12, 628), the existence of which is unsettled.

Finally, let us mention the following

Proposition 5.2. The only finite linear spaces S(2,k,v) *with* k *a prime power that satisfy condition* (I2) *are projective or affine planes.*

Proof. Let S be a finite linear space S(2,k,v) satisfying condition (I2) which is neither a projective plane nor an affine plane, so that

$$k \geq i_2 + 3 \qquad (12)$$

Let $k = p^n$ with p a prime number. (2) implies that

$$k \mid 2(i_2+1)(i_2+2) .$$

Since i_2+1 and i_2+2 are relatively prime, we conclude that

$$p^n \mid 2(i_2+1) \quad \text{or} \quad p^n \mid 2(i_2+2) .$$

Therefore, by (12),

$$p^n = 2(i_2+1) \quad \text{or} \quad p^n = 2(i_2+2) ,$$

and so $p^n = 2^n$ with $n \geq 2$.

If $p^n = 2(i_2+2)$, then (1) becomes

$$2^{n-2} -1 \mid (2^n-1)(2^{n-1} -1)$$

and so

$$2^{n-2} -1 \mid 2^{(n,n-2)} -1 = 1 \text{ or } 3.$$

Therefore $(i_2,k) = (2,8)$ or $(6, 16)$, contradicting (2).

If $k = 2(i_2+1)$, then (3) becomes

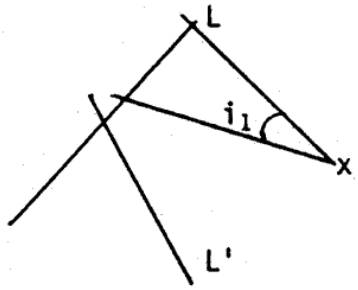
$$2(2^n -1) - (2^{n-1} -1) \mid 2^{n+1}(2^n -1)(2^{n-1} -1) .$$

Since 2^{n+1} and 2^n-1 are each relatively prime with $2(2^n - 1)-(2^{n-1} -1)$, this implies that

$$2(2^n -1)-(2^{n-1} -1) \mid 2^{n-1} - 1 > 0$$

and so $2(2^n -1) \leq 2(2^{n-1} -1)$, a contradiction.

8. LINEAR SPACES SATISFYING CONDITION (I1).



(I1) *there is a non-negative integer* i_1 *such that for any ordered pair of intersecting lines* L, L' *and for any point* x *outside* $L \cup L'$, *there are exactly* i_1 *lines through* x *which intersect* L *but not* L'.

The study of finite linear spaces satisfying condition (I1) reduces exactly to that of finite linear spaces satisfying condition (I2), thanks to the following result :

Theorem 6[26] *The finite non-trivial linear spaces satisfying* (I1) *are the finite degenerate projective planes and the Steiner systems* $S(2,k,v)$ $(k < v)$ *satisfying condition* (I2).

We have seen in Theorem 5 that the finite non-trivial linear spaces satisfying condition (I2) are necessarily Steiner systems $S(2,k,v)$ or degenerate projective planes, so that conditions (I1) and (I2) are equivalent.

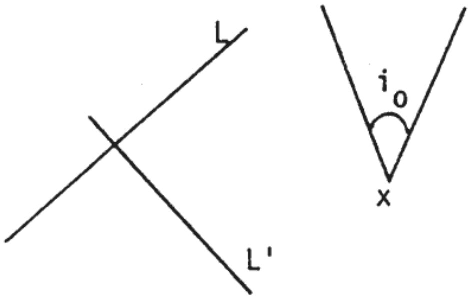
Proof of Theorem 6.

Let S be a finite non-trivial linear space satisfying condition (I1). It is easily seen that if S is the union of two intersecting lines, then S is a degenerate projective plane. Suppose now that S is not the union of two intersecting lines. Then, for any two intersecting lines L and L', the degree of any point x outside $L \cup L'$ is

$$r_x = |L| + i_1 + i_0(x,L,L') = |L'| + i_1 + i_0(x,L,L'),$$

and so L and L' have the same size. Therefore all lines of S have the same size. Since conditions (I1) and (I2) are equivalent for Steiner systems S(2,k,v), the theorem is proved.

9. LINEAR SPACES SATISFYING CONDITION (I0).



(I0) *there is a non-negative integer* i_0 *such that for any two intersecting lines* L, L' *of* S *and for any point* x *outside* L ∪ L', *there are exactly* i_0 *lines through* x *which are disjoint from* L ∪ L'.

The study of finite linear spaces satisfying condition (I0) reduces essentially to that of finite linear spaces satisfying condition (I2), thanks to the following

Theorem 7 [26]. *If* S *is a finite non-trivial linear space satisfying condition* (I0), *then one of the following occurs*

(i) S *is a punctured projective plane or an affine plane with one point at infinity,*

(ii) S *is a degenerate projective plane or a Steiner system* S(2,k,v) (k<v) *satisfying condition* (I2)

Conversely, each of these finite spaces satisfies condition (I0).

In particular, the only known finite linear spaces satisfying condition (I0) are, besides the trivial examples P(2, k+1) and PG(d,1), the finite semi-affine planes (in which i_0 is always zero).

Proof. Let S be a finite non-trivial linear space satisfying condition (I0). Note first that if S is the union of two intersecting lines, then S is either a degenerate projective plane or AG(2,2) with one point at infinity. On the other hand, as mentioned in section 3, conditions (I0) and (I2) are equivalent for Steiner systems $S(2,k,v)$, so that we may assume from now on that S contains two lines of distinct sizes and that every point of S has degree ≥ 3. The following lemmas will show that, under these assumptions, $i_o = 0$; this will make the proof of Theorem 7 easier.

Lemma 7.1. If S *contains a point* x *of degree 3, then* $i_o = 0$.

Proof. Let A,B,C be the three lines passing through x. If $i_o \neq 0$, there would be a line D disjoint from $B \cup C$ through a point y of A distinct from x, so that $r_x \geq 4$, a contradiction.

In the following lemmas, we assume that every point of S has degree ≥ 4.

Lemma 7.2. If A,B,C *are three pairwise intersecting lines of size* a,b,c *respectively, then*

$$r_A - r_B = (a-b)(c+i_o) + r_{A\cap C} - r_{B\cap C}$$

In particular, if A,B,C *are concurrent, then*

$$r_A - r_B = (a-b)(c+i_o)$$

Proof. Counting in two ways the number of bisecants of A and B which are disjoint from C, we get

$$\sum_{\substack{y\in A\\ y\notin B\cup C}} i_1(y,B,C) = \sum_{\substack{z\in B\\ z\notin A\cup C}} i_1(z,A,C)$$

from which we immediately deduce the desired formulas by using

$$i_1(y,B,C) = r_y - c - i_o$$

and

$$i_1(z,A,C) = r_z - c - i_o .$$

Corollary 7.2.

(i) *If* A *and* B *are two intersecting lines of the same size, then* $r_A = r_B$.

(ii) *For any point* x *of* S, *all lines passing through* x, *except possibly one, have the same size.*

(iii) *If three points* x,y,z *are such that the lines* $A = \langle x,z\rangle$ *and* $B = \langle y,z\rangle$ *are distinct and have the same size, then* $r_x = r_y$.

Proof. Since we have assumed that all points of S have degree ≥ 4, there is a line C concurrent with the two intersecting lines A and B. (i) follows immediately from Lemma 7.2 applied to A, B and C.

In order to prove (ii), suppose that A and B are two lines of distinct sizes a and b passing through x (which, by hypothesis, has degree ≥ 4). If C and C' are two lines passing through x and distinct from A and B, it follows immediately from Lemma 7.2 that C and C' have the same size c. On the other hand, if $c \neq b$, then the same argument applied to C' and A which are distinct from the two lines B and C of distinct sizes shows that $c = a$, and so (ii) is proved.

In order to prove (iii), note first that $r_A = r_B$ by (i). Then Lemma 7.2 applied to the three lines A, B and $C = \langle x,y \rangle$ yields $r_{A\cap C} - r_{B\cap C} = 0$, that is $r_x = r_y$.

Lemma 7.3. Let p *be a point of* S. *If* B *is the only line of size* b *passing through* p, *then*

(i) *any two points outside* B *have the same degree,*
(ii) *any two lines distinct from* B *and concurrent with* B *have the same size.*

Proof. Let x and y be two points outside B such that x, y and p are not collinear. By Corollary 7.2 (ii) all lines distinct from B through p have the same size a, and so we deduce from Corollary 7.2. (iii) that x and y have the same degree. This implies that all points outside B have the same degree r.

Now let C and D be two lines distinct from B and intersecting in a point $z \in B$. If $z = p$, we already know that C and D have the same size. If $z \neq p$, let A (resp. A') be a line through p, distinct from B and intersecting C (resp. D). Then, by Lemma 7.2 and Corollary 7.2 (i), we get

$$r_A - r_B = (a-b)(c+i_0) + r - r_z$$

and

$$r_A - r_B = (a-b)(d+i_0) + r - r_z \ .$$

Since $a \neq b$, these equalities imply $c = d$.

Lemma 7.4. If every point of S *has degree* ≥ 4, *then* $i_0 = 0$

Proof. As we have seen at the beginning, we may assume that S contains two lines A and B having distinct sizes a and b respectively and intersecting in a point p.

Moreover, by Corollary 7.2 (ii), we may also assume that all lines distinct from B and containing the point p have size a.

Suppose first that every line intersecting B and not passing through p has size $\neq a$. Then, if $b > 2$, any point $x \notin B$ is on at least two lines of size $\neq a$. Therefore, by Corollary 7.2 (ii), $A(x) = \langle x,p\rangle$ is the only line through x having size a. From Lemma 7.3 (ii), we deduce that any two lines distinct from A(x) and concurrent with A(x) have the same size. Since p has degree ≥ 4 and $p \in A(x)$, this contradicts the fact that B is the only line of size b containing p. This shows at the same time that $b = 2$ (let $B = \{p,q\}$) and that any point $x \notin B$ is on at least two lines of size a. Since by hypothesis, the size c(x) of the line $C(x) = \langle x,q\rangle$ is distinct from a, we deduce from Corollary 7.2 (ii) that C(x) is the only line of size c(x) containing x. Therefore, by Lemma 7.3 (i), any point $y \notin C(x)$ has degree

$$r_y = r_p .$$

Since B is the only line of size $\neq a$ containing p, we have

$$v-1 = (r_p-1)(a-1) + (b-1) ,$$

and since C(y) is the only line of size $\neq a$ containing y, we have

$$v-1 = (r_y-1)(a-1) + (c(y)-1) .$$

Since $b = 2$, these three equations imply that

$$c(y) = b = 2 ,$$

from which we deduce that all lines containing q have size 2, all other lines of S having size a. Therefore $S - \{q\}$ is a Steiner system $S(2,a,v-1)$ with point degree $r' = (v-2)/(a-1)$.
Let $C \neq B$ be a line of S passing through q. If u is a point of A outside $B \cup C$, then $i_0(u,B,C) = r'-1$; but if v is a point outside $A \cup B \cup C$, then $i_0(v,B,C) = r'-2$, and so condition (I0) is not satisfied, a contradiction.

Suppose now that there is a line intersecting B, not passing through p and having size a. Then, by Lemma 7.3 (ii), there is a bisecant C of A and B having size a. Lemma 7.2 applied to the triple (A,B,C) yields

$$r_A - r_B = (a-b)(a+i_0) + r - r_{B\cap C} \qquad (1)$$

where $r = r_{A\cap C}$ is, by Lemma 7.3 (i), the common degree of the points outside B. On the other hand, let A' be a line distinct from A and B and passing through $A \cap B$. Lemma 7.2 applied to the triple (A,B,A') yields

$$r_A - r_B = (a-b)(a+i_o) \qquad (2)$$

(1) and (2) together give

$$r = r_{B\cap C} \qquad (3)$$

Since A' and B are concurrent with A and have different sizes, Lemma 7.3 (ii) implies that every point $x \in A$ distinct from p is on at least r-1 lines of size a, so that

$$v-1 = (r-1)(a-1) + c-1 \qquad (4)$$

where $c = a$ or is the size of the unique line of size $\neq a$ passing through x. On the other hand, we know by Lemma 7.3 (ii) that all lines distinct from B and passing through $B \cap C$ have size a, so that

$$v-1 = (r_{B\cap C} - 1)(a-1) + b-1 \qquad (5)$$

(3), (4) and (5) together imply $b = c$. Therefore $a \neq c$ and (4) shows that every point outside B is on exactly one line of size b, all the other lines having size a. Either the lines of size b are pairwise disjoint or they are concurrent in a point $y \in B$. We shall successively consider these two possibilities.

If the lines of size b are pairwise disjoint, then by Lemma 7.3 (i), all points of S have the same degree r.Lemma 7.2 applied to the triple (A,B,A') yields

$$r(a-b) = (a-b)(a+i_o) ,$$

and so, since $a \neq b$, we get $r = a + i_o$, which means that for any point x outside A, every line through x which is disjoint from A is also disjoint from every line intersecting A and not passing through x. Therefore every line through x intersects A and $i_o = 0$.

If the lines of size b are concurrent in a point $y \in B$, then, by Lemma 7.3 (i) all points distinct from y have the same degree r. Lemma 7.2, applied to the triple (A,B,B'), where $B' \neq B$ is a line of size b intersecting A, yields

$$a\,r - (b-1)r - r_y = (a-b)(b+i_o) + r - r_y ,$$

and so, since $a \neq b$, we get $r = b + i_o$.
Thus, every line through a point outside B intersects B and $i_o = 0$.

We will now end easily the proof of Theorem 7. Indeed we have seen in Lemmas 7.1 and 7.4 that $i_o = 0$, which means that S contains no line disjoint from two intersecting lines. Therefore, for any line L, every point outside L is on at most one line disjoint from L, in other words, S is a semi-affine plane. Since we know (I6) that the finite semi-affine planes are, besides the finite affine planes and the (possibly degenerate) finite projective planes, the finite punctured projective planes and the finite affine planes with one point at infinity, and since each of these planes satisfies condition (I0), Theorem 7 is proved.

10. LINEAR SPACES SATISFYING CONDITION (ID).

We shall end this chapter with the study of finite linear spaces satisfying condition (ID), that is ($\mathcal{P}$, 1,2,1; 3,4,4) according to our conventions in section 2. Actually (ID) can be expressed in a simpler way (in terms of lines only) :

(ID) *there is a non-negative integer* δ *such that for any two intersecting lines* L *and* L', *there are exactly* δ *lines disjoint from* $L \cup L'$.

Using the work done in section 9, we shall prove the following

> *Theorem* 8[26]. *The finite non-trivial linear spaces satisfying condition* (ID) *are exactly the finite semi-affine planes and the Steiner systems* S(2,k,v) *with* $k < v$.

Proof. Let A and B be two intersecting lines of a finite non-trivial linear space S, and let $\delta(A,B)$ denote the number of lines disjoint from both A and B. The proof is based on the following counting argument

$$|\mathcal{L}| = (r_A - a + 1) + (r_B - b + 1) - (r_{A\cap B} + (a-1)(b-1)) + \delta(A,B) \qquad (1)$$

where the first (resp. second) term on the right hand side counts the number of lines having a non-empty intersection with A (resp. B) and the third term counts the number of lines having a non-empty intersection with both A and B.

We immediately deduce from (1) that all Steiner systems S(2,k,v) satisfy condition (ID), since all terms different from $\delta(A,B)$ are independent of the choice of the two intersecting lines A and B. Moreover, it is obvious from their definition that the semi-affine planes are exactly the non-trivial linear spaces satisfying condition (ID) with $\delta = 0$. Therefore, it remains only to prove that if S is a finite non-trivial linear space satisfying condition (ID) and containing two lines of different sizes, then necessarily $\delta = 0$. We shall prove this in the following lemmas, which are similar to those of section 9.

Lemma 8.1. If S *contains a point* x *of degree* ≤ 3, *then* $\delta = 0$.

Proof. Let A and B be two lines intersecting in x. If $\delta \neq 0$, then there is a line C disjoint from A ∪ B, so that $r_x \geq 4$, a contradiction.
In the following lemmas, we assume that every point of S has degree ≥ 4.

Lemma 8.2. If C *intersects* A *and* B, *then*

$$r_A - r_B = (a-b)c + r_{A\cap C} - r_{B\cap C}$$

In particular, if A, B, C *are concurrent, then*

$$r_A - r_B = (a-b)c.$$

Proof. Since A and C intersect, (1) yields

$$|\mathcal{L}| = (r_A-a+1) + (r_C-c+1) - (r_{A\cap C} + (a-1)(c-1)) + \delta \tag{2}$$

Since B and C intersect, we have similarly

$$|\mathcal{L}| = (r_B-b+1) + (r_C-c+1) - (r_{B\cap C} + (b-1)(c-1)) + \delta \tag{3}$$

By subtracting (2) from (3), the lemma is proved.

Corollary 8.2. Identical to Corollary 7.2, both in statement and proof (it suffices to replace "Lemma 7.2" by "Lemma 8.2").

Lemma 8.3. Identical to Lemma 7.3, both in statement and proof.

Lemma 8.4. If all points of S *have degree* ≥ 4, *then* $\delta = 0$.

Proof. Since it is very similar to that of Lemma 7.4, we shall only indicate what has to be changed. At the end of the second paragraph, we conclude that $S' = S - \{q\}$ is a Steiner system $S(2,a,v-1)$ with point degree $r' = (v-2)/(a-1)$. Then $\delta(B,C)$ is the number of lines in S' which are disjoint from A ∩ B and A ∩ C, and $\delta(B,A)$ is the number of lines in S' which are disjoint from A. Therefore, if $b' = (v-1)(v-2)/a(a-1)$ denotes the total number of lines in S', we have

$$\delta(B,C) = b' - 2r' + 1$$

and

$$\delta(B,A) = b' - a\, r' + a-1$$

Since $\delta(B,C) = \delta(B,A) = \delta$ and $r' > 1$, we conclude that $a = 2$, so that all lines of S have the same size, a contradiction.

The third paragraph of the proof of Lemma 7.4 remains valid here, if we replace "Lemma 7.2" by "Lemma 8.2" and if we delete "i_0" in (1) and (2).

In the fourth paragraph, we suppose that the lines of size b are pairwise disjoint, so that, by Lemma 8.3 (i), all points of S have the same degree r. Then, Lemma 8.2 applied to the triple (A,B,A') yields

$$r(a-b) = (a-b)a \ ,$$

and so, since $a \neq b$, we get $r = a$. This means that, for any point x outside A, every line through x intersects A. Therefore $\delta = 0$.

Finally, in the fifth paragraph, we suppose that the lines of size b are concurrent in a point $y \in B$, and so, by Lemma 8.3 (i), all points distinct from y have the same degree r. Lemma 8.2, applied to the triple (A,B,B'), where $B' \neq B$ is a line of size b intersecting A, yields

$$a\, r - (b-1)r - r_y = (a-b)b + r - r_y \ .$$

Therefore, since $a \neq b$, we get $r = b$, which means that for any point x outside B, all lines through x intersect B, and so $\delta = 0$. This ends the proof.

CHAPTER III. SOME REGULARITY CONDITIONS IN FINITE PLANAR SPACES.

1. INTRODUCTION.

The generalized projective spaces of dimension > 2 (endowed with all their planes) may be defined as the non-trivial planar spaces S satisfying the following condition :

(*) for every pair of planes Π and Π' intersecting in a line and for every point $x \notin \Pi \cup \Pi'$ such that there is a line through x intersecting Π and Π' in two distinct points, every line through x intersecting Π intersects Π'.

Indeed, let L and L' be two lines intersecting in some point x and let A and A' be two lines not passing through x and intersecting each of the lines L and L'. Suppose that A and A' are disjoint. Since S is a non-trivial planar space, there is a point y outside $\langle L,L'\rangle$. The planes $\Pi = \langle y,L\rangle$ and $\alpha = \langle y,A\rangle$ intersect in a line. The line L' intersects the planes Π and α in two distinct points and contains a third point $x' \in L' \cap A'$. Therefore, by condition (*), every line passing through x' and intersecting Π intersects α. In particular, the line A' which intersects $L \subset \Pi$ intersects α. Hence A', which is contained in $\langle L,L'\rangle$, intersects $A = \alpha \cap \langle L,L'\rangle$, a contradiction. Hence Pasch's axiom is satisfied.

In particular, the 3-dimensional generalized projective spaces are the non-trivial planar spaces satisfying

(I) for every pair of planes Π and Π' intersecting in a line, every line intersecting Π intersects Π'.

Indeed, any non-trivial planar space S satisfying condition (I) satisfies also condition (*), and so is a generalized projective space; moreover, since S is necessarily the smallest linear subspace containing two planes intersecting in a line, S is 3-dimensional.

Note that the condition obtained from (I) by deleting the words "intersecting in a line", though apparently stronger than (I) is equivalent to (I).

Two problems arise now in a natural way : is it possible to classify the non-trivial planar spaces which satisfy the condition obtained from (I) by replacing "intersecting in a line" by "intersecting in a point" (resp. by "having an empty intersection") ? This is the subject of the following two theorems, concerning finite planar spaces.

points would allow us to rule out the rather uninteresting cases (c) and (d), and to shorten the proof a little bit.

Proof of the theorem

The proof is divided into a series of lemmas. The planar space PG(4,1) obviously satisfies the hypotheses and we shall always assume in what follows that $S \neq PG(4,1)$

Lemma 9.1. If each of the two planes Π' *and* Π'' *intersects a third plane* Π *in exactly one point, then* $\Pi \cap \Pi' = \Pi \cap \Pi'' = \Pi' \cap \Pi''$.

Proof. Suppose first that $\Pi \cap \Pi' = \{x'\}$ and $\Pi \cap \Pi'' = \{x''\}$ where the points x' and x'' are distinct. By condition (II), every line of Π' passing through x' intersects Π'' in a point, and so $\Pi' \cap \Pi''$ is a line L. Condition (II), applied to the pair of planes $\{\Pi,\Pi'\}$ (resp. $\{\Pi,\Pi''\}$), shows that any line of Π'' (resp. Π') intersecting L passes through x'' (resp. x'), which implies that $\Pi' = L \cup \{x'\}$ and $\Pi'' = L \cup \{x''\}$. If there is a point $x \notin \Pi \cup L$, the line $\langle x,x''\rangle$ must intersect $\Pi' = L \cup \{x'\}$, a contradiction. Therefore $S = \Pi \cup L$.

Let y be a point of L and let A be a line of Π passing through x'' and distinct from $\langle x',x''\rangle$. Since $\Pi \cap \Pi'' = \{x''\}$, the lines A and L are not coplanar and the plane $\alpha = \langle y,A\rangle$ intersects Π' in the point y only. By condition (II), any line of Π intersecting A (hence α) must intersect Π', and so must contain x'; it follows that $\Pi = A \cup \{x'\}$. Similarly, $\Pi = B \cup \{x''\}$ for any line B of Π passing through x' and distinct from $\langle x',x''\rangle$. Therefore Π contains only three points x, x' and x''. If L has at least three points y, y' and y'', then the line $\{x',y''\}$ intersects the plane $\{x'', x', y'\}$ but not the plane $\{x'', x, y\}$, and condition (II) is not satisfied. Therefore L has size 2 and $S = PG(4,1)$ contradicting the initial assumption.

This proves that $x' = x''$. By condition (II), any line of Π' intersecting $\Pi' \cap \Pi''$ must intersect Π, which implies that $\Pi' \cap \Pi'' = \{x'\}$.

A maximal set of planes having the property that any two of them intersect in the point x only will be called a *direction of planes with top* x. It follows from Lemma 9.1. that any plane Π belongs to at most one direction, denoted by dir Π. The top of dir Π will also be called the top of Π and a top in S.

Corollary 9.1. If dir Π *contains at least three planes with top* x*, then all the lines passing through* x *and belonging to a plane of dir* Π *have the same size.*

Proof. If Π, Π' and Π'' are three distinct planes of dir Π and if L (resp. L') is a line of Π (resp. Π') passing through x, Lemma 9.1. implies that the plane $\langle L,L' \rangle$ intersects Π'' in a line L''. By condition (II), any line intersecting L'' and L (resp. L'' and L') in two distinct points must intersect L' (resp. L), and so L and L' have the same size. The corollary follows easily.

Lemma 9.2. For any point x *of* S*, the residue* S_x *of* x *is one of the following*

(i) *a projective plane (possibly degenerate)*

(ii) *a punctured projective plane*

(iii) *an affine plane with one point at infinity*

(iv) *an affine plane.*

Proof. Two planes of S intersect in x (and in x only) iff the corresponding lines of S_x are disjoint. Therefore Lemma 9.1. implies that if L and L' are two disjoint lines in S_x, any line of S_x intersecting L in one point must also intersect L' in one point. In other words, the linear space S_x is a semi-affine plane. Since S is assumed to be finite, S_x is finite and we know by (I 6) that S_x is either an affine plane, or an affine plane with one point at infinity, or a punctured projective plane, or a (possibly degenerate) projective plane.

The finiteness assumption is essential here : indeed, Dembowski has constructed infinite semi-affine planes which are not of the four types described above [30] .

Note that S_x is always an affino-projective plane, except if S_x is a degenerate projective plane. Note also that S_x is a (possibly degenerate) projective plane iff x is not a top in S.

Corollary 9.2. If S_x *is an affino-projective plane of order* k*, then* x *has degree* k *in every plane with top* x.

Proof. It suffices to observe that a plane with top x corresponds to a line of S_x having at least one disjoint line in S_x, that is a line of size k in S_x.

Lemma 9.3. If S *contains a point* x *such that* S_x *is a degenerate projective plane, then* S *is of type* (c).

Proof. The hypothesis implies that S is the union of a plane Π and of a line A intersecting Π in x. Let z be a point on A, distinct from x. Since $S = \Pi \cup A$, every line passing through z intersects Π. Therefore the plane Π is isomorphic to S_z and, by Lemma 9.2 , Π is a semi-affine plane.

Suppose that there are two points z and z', distinct from x, on the line A. The plane Π contains two intersecting lines L and L' not passing through x

(except if Π is a degenerate projective plane in which all lines through x have size 2, but in this case S is a 3-dimensional generalized projective space and condition (II') is not satisfied). If Π contains either a point $y \notin L \cup L' \cup \{x\}$ or a line L" intersecting L' but not L, then the planes $\langle L,z \rangle$ and $\langle L',z' \rangle$ intersect in the point $L \cap L'$ only, and either the line $\langle y,z' \rangle$ or the line L" intersects $\langle L',z' \rangle$ but not $\langle L,z \rangle$, in contradiction with (II). Therefore the semi-affine plane Π has no such point y and no such line L", and so Π is necessarily a degenerate projective plane with 4 points, in which x is of degree 2. Denote by B the line of size 3 in Π and by x' the point of degree 3 in Π. Then $S = \langle A,B \rangle \cup \{x'\}$ and $S_{x'}$ is isomorphic to $\langle A,B \rangle$. It follows that $\langle A,B \rangle$ is an affine plane of order 2 with the point x at infinity, and so S is of type (c).

Therefore we may assume that A is a line $\{x,z\}$ of size 2. Then $S = \Pi \cup \{z\}$ and all lines through z have size 2. If Π is a (possibly degenerate) projective plane, then S is a 3-dimensional generalized projective space and condition (II') is not satisfied. Therefore the semi-affine plane Π is either a punctured projective plane, or an affine plane with one point at infinity, or an affine plane, and the Lemma is proved.

From now on, we shall always assume that there is no point $x \in S$ such that S_x is a degenerate projective plane.

Lemma 9.4. If S *contains a point* x *such that* S_x *is an affine plane of order* k *with one point at infinity, then* S *is obtained from* PG(3,k) *by deleting an affino-projective plane which is neither projective nor punctured projective.*

Proof. Denote by L_∞ the line of S corresponding to the point at infinity of S_x, by y any point of L_∞ distinct from x, and by Π any plane passing through x and not containing L_∞. S is the union of L_∞ and of all planes of dir Π. Therefore any line through y intersects at least one (hence every) plane of dir Π, and so we define an isomorphism between S_y and Π by mapping any line passing through y onto its point of intersection with Π. Therefore Π is a semi-affine plane (distinct from a degenerate projective plane). By Corollary 9.2 , x has degree k in Π, and so either Π has order k-1 or Π is an affine plane of order k with the point x at infinity.

If Π is a projective plane of order k-1, then all lines of S distinct from L_∞ and passing through x have size k. Let Π_∞ be a plane of S containing L_∞ and let Π_∞^* denote the linear space induced on $\Pi_\infty - (L_\infty - \{x\})$ by the linear structure of Π_∞. Since Π_∞ intersects every plane of dir Π in a line through x and since every line of Π_∞ not passing through x intersects each of the k planes of dir Π in a point, all the lines of the linear space Π_∞^* have size k. The degree of x in Π_∞^* is $k = |\text{dir } \Pi|$, and so Π_∞^* is a projective plane. On the other hand, the lines of Π_∞ passing through y induce pairwise disjoint lines in Π_∞^*, a contradiction.

If the semi-affine plane Π has order k-1 and is not a projective plane, then Π contains a line L of size k-1 not passing through x. Let $\Pi_1 = \langle L,y\rangle$ where $y \neq x$ is a point of L_∞. The intersections of Π_1 with the planes of dir Π form a partition Δ_1 of $\Pi_1 - \{y\}$ into k lines of size $k-1 = |L|$. On the other hand, the lines of Π_1 passing through y define a partition Δ_2 of $\Pi_1 - \{y\}$ into k-1 lines of size k. Let $L' \notin \Delta_1 \cup \Delta_2$ be a line of Π_1. By condition (II), L' intersects each of the lines of Δ_1, which is impossible since $L' \notin \Delta_2$.

Therefore every plane Π containing x but not L_∞ is an affine plane of order k with the point x at infinity. Since any line of S distinct from L_∞ is either contained in some plane of dir Π or intersects every plane of dir Π in a point, the lines of S distinct from L_∞ have size k+1 or k according as they intersect L_∞ or not. Moreover, the planes of S containing x have exactly k^2 points outside L_∞ and the planes not containing x intersect the planes of dir Π in k pairwise disjoint lines of size k. Therefore, in the planar space of k^3 points induced on $S-L_\infty$, all lines have k points and all planes have k^2 points. In other words, $S-L_\infty$ is a planar space of k^3 points in which all planes are affine planes of order k. If k=2, $S-L_\infty$ is the unique Steiner system S(3,4,8), that is the affine space AG(3,2). If k=3, $S-L_\infty$ is the unique Hall triple system of 27 points [36], that is the affine space AG(3,3). If $k \geq 4$, then by (I5), $S-L_\infty$ is the affine space AG(3,k).

It follows that S is obtained from an affine space AG(3,k) by adding a line at infinity L_∞ to a direction of parallel planes. Using the classical process of completion by points at infinity we conclude easily that S is obtained by deleting from PG(3,k) an affino-projective plane (which is neither projective nor punctured projective since L_∞ contains at least 2 points).

Corollary 9.4. (i) *If* S *contains a point* x *such that* S_x *is an affine plane with one point at infinity, then for any top* y *in* S, S_y *is also an affine plane with one point at infinity.*

(ii) *If* S *contains a point* x *such that* S_x *is an affine plane, then* x *is the only top in* S.

Proof. (i) is an immediate consequence of Lemma 9.4. In order to prove (ii), suppose on the contrary that there is a top $y \neq x$ in S. By (i), S_y is not an affine plane with one point at infinity, and so, by Lemma 9.2., S_y is either an affine plane or a punctured projective plane. In both cases, the line $\langle x,y \rangle$ is contained in a plane Π with top y. On the other hand, there is in S_x a line disjoint from the line Π_x of S_x corresponding to Π, and so Π is a plane with top x. Therefore Π has two distinct tops x and y, in contradiction with Lemma 9.1.

Lemma 9.5. If S *contains a point* x *such that* S_x *is a punctured projective plane of order* k *or an affine plane of order* k, *then every plane* Π *with top* x *is an affine plane of order* k *with the point* x *at infinity.*

Proof. Let $\Pi' \neq \Pi$ be a plane of dir Π and let $y \neq x$ be a point of Π'. By condition (II), all the lines passing through y and disjoint from Π are included in Π'. Therefore if we map each line of S passing through y and intersecting Π onto its point of intersection with Π, we define an isomorphism between Π and the linear space induced by S_y on $S_y - (\Pi'_y - L_y)$ where Π'_y is the line of S_y corresponding to the plane Π' and L_y is the point of S_y corresponding to the line $L = \langle x,y \rangle$. Thanks to Corollary 9.4., we know that S_y is either a projective plane or a punctured projective plane. If $|\text{dir } \Pi| > 2$, then all lines of Π passing through x have the same size by Corollary 9.1. If $|\text{dir } \Pi| = 2$, then S_x must be an affine plane of order 2 and Corollary 9.4. implies that S_y is a projective plane. Therefore, in any case, Π is an affine plane with the point x at infinity and, by Corollary 9.2., the order of Π is k.

Lemma 9.6. If S *contains a point* x *such that* S_x *is an affine plane of order* k, *then* S *is obtained from* PG(3,k) *by deleting a punctured projective plane of order* k.

Proof. By Lemma 9.5. and condition (II), the lines of S have k+1 or k points according as they contain x or not, and the planes of S have k^2+1 or k^2 points according as they contain x or not. Therefore $S - \{x\}$ is a planar space of k^3 points in which all lines have k points and all planes have k^2 points. By the same arguments as in the proof of Lemma 9.4., we conclude that $S - \{x\}$ is an affine space AG(3,k) and that S is obtained from the projective space PG(3,k) by deleting a punctured projective plane of order k.

affine planes of order 3 or planes of 13 points consisting of three concurrent lines of size 5 (all the other lines having size 3). We denote by S'_H the linear space obtained from S_H by replacing every line of size 5 by 10 lines of size 2. Let F_9 be the affine plane induced on S'_H by a plane intersecting S'_H in 9 points and let x be a point of $S'_H - F_9$ which is on at least one line of size 3 intersecting F_9. The smallest linear subspace of S'_H containing F_9 and x has exactly 18 points and is a Fischer space, denoted by F_{18}. S_{18} is the planar space induced by PG(3,4) on the set of points of F_{18}.

In order to define S_{36}, let H' be a Hermitian quadric in PG(3,4) having exactly one singular point s (for instance, the quadric of equation $x\bar{y} + \bar{x}y + z\bar{z} = 0$). The planar space induced on H' by PG(3,4) has lines of size 3 or 5, and its planes are either affine planes of order 3 not passing through s or planes of 13 points consisting of 3 lines of size 5 concurrent in s (all the other lines having size 3). S_{36} is the planar space induced by PG(3,4) on $H' - \{s\}$ and F_{36} is obtained from S_{36} by replacing every line of size 4 by 6 lines of size 2.

We still need a notation for five small spaces satisfying (III) and (III'). The space K_7^7 is obtained from PG(2,2) by taking as points the points of PG(2,2), as lines the pairs of points and as planes the lines of PG(2,2) and their complements. The planar space of 6 points in which all lines have size 2 and which contains 0, 1, 2 or 3 planes of 4 points (all the other planes having 3 points) will be denoted by K_6^o, K_6^1, K_6^2 and K_6^3. It is easy to check that these spaces are uniquely determined by the above properties.

Statement of the theorem.

We shall first prove two fundamental lemmas. In what follows, S denotes always a finite planar space satisfying (III) and (III').

Lemma 10.1. For any plane Π *of* S *and for any point* $x \in S - \Pi$, *there is at most one plane passing through* x *and disjoint from* Π.

Proof. Suppose on the contrary that x is on two distinct planes Π' and Π'' disjoint from Π. Let L be a line passing through x, contained in Π' but not in Π''. Since L intersects Π'', condition (III) implies that L must intersect Π, a contradiction.

A maximal set of pairwise disjoint planes will be called a *direction* of planes, provided there are at least two planes in it. By Lemma 10.1, a plane Π either intersects any other plane or is in exactly one direction, denoted by *dir* Π.

We can now state our main result :

Theorem 10 [25] *If* S *is a finite planar space such that*
(III) *for any two disjoint planes* Π *and* Π', *every line intersecting* Π *intersects* Π'
and (III') *there are at least two disjoint planes*
then one of the following occurs :
(a) S *is obtained from* PG(3,k) *by deleting a line,*
(b) S *is obtained from* PG(3,k) *by deleting an affino-projective (but not affine) plane of order* k,
(c) $S = S_{36}$,
(d) $S = S_{18}$,
(e) S *is a space* S_7 *of 7 points lying on 3 concurrent lines of size 3, all the other lines having size 2, in which the planes either have only 3 points or are unions of two lines of size 3,*
(f) $S = K_7^7, K_6^0, K_6^1, K_6^2$ or K_6^3 ,
(g) S *has only one direction of planes and* $S-S^*$ *contains at least four non-coplanar points.*

We do not know whether there is a finite planar space of type (g).

The proof will be divided into three main parts : we shall first handle some small exceptional spaces (types (e) and (f)), then we shall classify the spaces having at least two directions of planes (types (b) and (c)) and finally we shall examine the spaces having exactly one direction of planes (types (a), (d) and (g)).

3.1. Small exceptional spaces

Lemma 10.3. If S *contains two disjoint planes* Π *and* Π' *such that* $S = \Pi \cup \Pi'$, *then* S *is the union of any two disjoint planes (in particular, every direction has exactly two planes).*

Proof. Suppose on the contrary that Π_1 and Π_1' are two disjoint planes of S such that there is a point $x \notin \Pi_1 \cup \Pi_1'$. We may assume without loss of generality that $x \in \Pi$. Then for any point $y \in \Pi_1 \cap \Pi'$, the line $\langle x,y \rangle$ has size 2, and so $\langle x,y \rangle$ is disjoint from Π_1', in contradiction with (III).

Proposition 10.1. If S *contains two disjoint planes* Π *and* Π' *such that* $S = \Pi \cup \Pi'$, *then* S *is the affine space* AG(3,2), $K_7^7, K_6^0, K_6^1, K_6^2$ or K_6^3 .

1.3) Suppose finally that $\Pi_1 \cap \Pi_2$ is a point t and that $\Pi_1' \cap \Pi_2'$ is a point z. Then Π_1, Π_1', Π_2, Π_2' are degenerate projective planes and $\Pi_1 \cap \Pi_2' = X$, $\Pi_1' \cap \Pi_2 = Y$ are lines. Suppose that X has at least three points x,x',x'' and let y,y' be two points of Y. If there is a transversal plane $\Pi = \{x,y,z\}$ not containing t, then the planes $<x',y',t>$ and $<x'',y',t>$ must intersect Π by Lemma 10.3, and so $\{x',y',z,t\}$ and $\{x'',y',z,t\}$ are distinct planes, a contradiction since they have three non-collinear points in common. Therefore, any transversal plane contains t and $<x,y,z> = <x,z,t> = <x,z,y'>$, a contradiction. This shows that X (and similarly Y) has size 2. Hence S has exactly 6 points and all lines have size 2. Since the union of two disjoint planes contains at least 6 points, S will automatically be the union of any two disjoint planes. It is a trivial exercise to check that there are exactly 4 non-isomorphic planar spaces of 6 points in which all lines have size 2 and which satisfy conditions (III) and (III') (they have respectively 0, 1, 2 or 3 planes of 4 points).

2) In order to complete the proof, it remains to show that the case where S has only one pair of disjoint planes Π and Π' such that $S = \Pi \cup \Pi'$ leads to a contradiction.

Suppose first that Π and Π' are two projective planes (possibly degenerate). If there is a plane intersecting Π in a line A and Π' in a line A', let $a \in \Pi - A$ and $a' \in \Pi' - A'$. The planes $<A,a'> = A \cup \{a'\}$ and $<A',a> = A' \cup \{a\}$ are disjoint, a contradiction. Hence there is no plane intersecting both Π and Π' in a line. Let L (resp. L') be a line of Π (resp. Π') and let $x \in \Pi - L$, $x' \in \Pi' - L'$. The planes $<L,x'> = L \cup \{x'\}$ and $<L',x> = L' \cup \{x\}$ are disjoint, a contradiction.

Therefore we may assume that Π contains two disjoint lines A and B. If there is a plane α containing A and intersecting Π' in only one point x', then every plane $\beta \neq \Pi$ containing B must intersect α, and so must contain x', a contradiction because two such planes β_1 and β_2 would have three non-collinear points in common. Therefore the planes $\alpha_1,\ldots,\alpha_n \neq \Pi$ containing A intersect Π' in lines $A_1', \ldots, A_n'$ partitioning Π'. Similarly, the planes $\beta_1, \ldots, \beta_m \neq \Pi$ containing B intersect Π' in lines $B_1', \ldots, B_m'$ partitioning Π'. Since Π and Π' are the only two disjoint planes of S, any line A_i' intersects each line B_j', and so there is no line in Π' which is coplanar with A and also coplanar with B. For the same reason, there is no line in Π which is coplanar with A_1' and also coplanar with A_2', a contradiction since A is coplanar with A_1' and with A_2'.

From now on, we shall assume that S is not the union of two disjoint planes.

Proposition 10.2. If there is a plane $\Pi \notin \text{dir } \Pi_1$ *intersecting* Π_1 *in only one point* x*, then* $S = S_7$.

Proof. Let $\Pi_1' \neq \Pi_1$ be any other plane of dir Π_1. By Lemma 10.2, Π is contained in $\Pi_1 \cup \Pi_1'$ (which implies that dir $\Pi_1 = \{\Pi_1, \Pi_1'\}$), and so Π is a degenerate projective plane. Let y be a point outside $\Pi_1 \cup \Pi_1'$ (hence outside Π). By condition (III), the line $\langle x,y\rangle$ intersects Π_1' in a point $x' \notin L' = \Pi \cap \Pi_1'$, and, by Lemma 10.2, the plane $\langle L',y\rangle$ intersects Π_1 in a line L not containing x. Thus the plane $\Pi' = \langle x',L\rangle$ is disjoint from Π and is isomorphic to Π by Corollary 10.2.1. If there is a line L" disjoint from $\Pi \cup \Pi'$, then Lemma 10.2 implies that the planes through L" which are not disjoint from Π intersect Π in disjoint lines, which is impossible since Π is a degenerate projective plane. Therefore all lines passing through y intersect Π or Π', and so, by (III), intersect Π and Π'.

If L contains at least three points u, v, w, let $u' = L' \cap \langle u,y\rangle$, $v' = L' \cap \langle v,y\rangle$, $w' = L' \cap \langle w,y\rangle$. The planes $\alpha = \langle x,u,w'\rangle$ and $\alpha' = \langle x',u',v\rangle$ are two disjoint planes of 3 points and the line $\langle v,v'\rangle$ intersects α' but not α, so that condition (III) is not satisfied. Therefore the lines L and L' have size 2 and y is on exactly 3 lines.

If the line $\langle x,y\rangle$ contains a fourth point x", then the planes $\langle L',x''\rangle$ and $\langle L',x\rangle = \Pi$ have a line in common and are both disjoint from $\Pi' = \langle L,x'\rangle$, contradicting Lemma 10.1. The same argument shows that the 3 lines passing through y have size 3, and so $S = S_7$.

From now on, we shall assume that $S \neq S_7$ so that, by Proposition 10.2, any plane not belonging to a direction dir Π intersects all planes of dir Π in a line.

3.2. Suppose that there are at least two directions of planes dir Π_1 and dir Π_2.

By Lemma 10.2 and Corollary 10.2.2, the set of lines $\Pi_1^i \cap \Pi_2^j$, where $\Pi_1^i \in \text{dir } \Pi_1$ and $\Pi_2^j \in \text{dir } \Pi_2$, is a partition of S^* and will be denoted by $\delta(\Pi_1 \cap \Pi_2)$.

Lemma 10.4. If a plane Π *intersects a line of* $\delta(\Pi_1 \cap \Pi_2)$ *in a single point, then* Π *intersects every line of* $\delta(\Pi_1 \cap \Pi_2)$ *in a single point and* $\Pi^* = \Pi \cap S^*$ *is an affine plane of order* $k = |\text{dir } \Pi_1| = |\text{dir } \Pi_2|$.

Proof. The intersections of the planes of dir Π_1 (resp. dir Π_2) with Π^* define a partition δ_1 (resp. δ_2) of Π^* into lines of S. Note that $\delta_1 \neq \delta_2$, otherwise

Π^* would not intersect a line of $\delta(\Pi_1 \cap \Pi_2)$ in a single point. If L^* is any line of Π^* not in δ_1 (resp. not in δ_2), condition (III) implies that L^* intersects every line of δ_1 (resp. δ_2). There is a line of δ_1 which is not in δ_2; since this line must intersect every line of δ_2, δ_1 and δ_2 have no line in common. Therefore, if L_i is any line of δ_i ($i = 1,2$) and if L^* is any line of Π^* not in $\delta_1 \cup \delta_2$, we have

$$|L_1| = |\delta_2| = |L^*| = |\delta_1| = |L_2|$$

and so all lines of Π^* have the same size

$$k = |\delta_1| = |\text{dir } \Pi_1| = |\delta_2| = |\text{dir } \Pi_2|$$

Moreover,

$$|\Pi^*| = |\delta_1| \cdot |L_1| = k^2$$

and so Π^* is an affine plane of order k.

Since every plane of dir Π_1 is partitioned into $k = |\text{dir } \Pi_2|$ lines of $\delta(\Pi_1 \cap \Pi_2)$, since Π^* intersects such a plane in a line of size k and since Π^* contains no line of $\delta(\Pi_1 \cap \Pi_2)$ (because $\delta_1 \cap \delta_2 = \emptyset$), we conclude that Π^* intersects every line of $\delta(\Pi_1 \cap \Pi_2)$ in a single point.

The planes of S (or S^*) intersecting every line of $\delta(\Pi_1 \cap \Pi_2)$ will be called *transversal* and those containing a line of $\delta(\Pi_1 \cap \Pi_2)$ will be called *non-transversal*. Note that any plane of S is either transversal or non-transversal. For any triple of non-coplanar lines $L, L', L'' \in \delta(\Pi_1 \cap \Pi_2)$, the product $|L|.|L'|.|L''|$ counts the total number of transversal planes in S. It follows that all lines of $\delta(\Pi_1 \cap \Pi_2)$ have the same size ℓ. Since Π_1 (resp. Π_2) is partitioned into k lines of size ℓ by its intersections with the planes of dir Π_2 (resp. dir Π_1), we have

$$|\Pi_1| = |\Pi_2| = k\ell \qquad (1)$$

and $|S^*| = |\text{dir } \Pi_1| \cdot |\Pi_1| = k^2\ell$

Lemma 10.5. $S - S^*$ *is a linear subspace of* S, *and any transversal plane* Π *of* S *has at most* $k-1$ *points outside* S^*.

Proof. By condition (III), any line intersecting a plane of dir Π_1 intersects every plane of dir Π_1. It follows that, for any point $x \in \Pi - \Pi^*$, the set of all lines passing through x and intersecting Π^* determines a partition of Π^* into lines, i.e. a parallel class in the affine plane Π^*. Therefore, if x and

y are any two points of $S-S^*$, the line $\langle x,y\rangle$ must be disjoint from S^*. This proves that $S-S^*$ is a linear subspace of S. Since there are $k+1$ parallel classes in Π^* and since at least two (induced by dir Π_1 and dir Π_2) are also classes of pairwise disjoint lines in Π, there are at most $k-1$ points in $\Pi-\Pi^*$.

Note that the planes of S are not necessarily the smallest linear subspaces containing three non-collinear points. On the contrary, if x, y, z are non-collinear points in $S-S^*$, it follows from Lemma 10.2 that the plane $\langle x,y,z\rangle$ of S intersects S^* in a plane of S^*.

Proposition 10.3. If $\ell=k$, *then* S *is obtained from a 3-dimensional projective space* PG(3,k) *by deleting an affino-projective (but not affine) plane of order* k.

Proof. If $\ell=k$, then S^* is a planar space of k^3 points in which all lines have k points and all planes have k^2 points, hence S^* is the 3-dimensional affine space AG(3,k). Indeed, by (I 5) if $k = 2$, S^* is the unique Steiner system S(3,4,8), that is the affine space AG(3,2); if $k = 3$, S^* is the unique Hall triple system of 27 points, that is the affine space AG(3,3); if $k \geq 4$, S^* is the affine space AG(3,k).

On the other hand, if the linear subspace $S - S^*$ contains three non-collinear points x, y, z, then the planes containing $\langle x,y\rangle$ and those containing $\langle x,z\rangle$ induce two distinct partitions of the affine space S^* into classes of parallel planes, but these partitions have the plane $\langle x,y,z\rangle \cap S^*$ in common, a contradiction. Therefore $S - S^*$ is either empty, or a point, or a line of size at most $k-1$, and the lemma is proved.

From now on, we shall assume $\ell \neq k$, so that any transversal plane Π intersects all the other planes of S (otherwise Π would belong to a direction dir Π of planes of S with $|\text{dir } \Pi| = \ell$ and, by applying to dir Π and dir Π_1 the arguments used in the proof of Lemma 10.4, we would get $\ell = k$).

Lemma 10.6. For every transversal plane Π, *the number of planes of* S *whose intersection with* Π *is disjoint from* S^* *is a constant* c *independent from* Π, *and*

$$c = (\ell-1)[\ell^2 + \ell + 1 - (k^2 + k) - k^2(\ell-k)]$$

$$= (\ell-1)[b' + v'(\ell+1-k) - \sum_{x\in\Pi-\Pi^*} r'_x]$$

where v' *(resp.* b'*) denotes the number of points (resp. the number of lines) of the linear subspace* $\Pi-\Pi^*$ *and* r'_x *denotes the degree of* x *in* $\Pi-\Pi^*$.

Proof. Since every non-transversal plane intersects Π^* in a line, any plane of S whose intersection with Π is disjoint from S^* is necessarily a transversal plane. The total number of transversal planes distinct from Π is ℓ^3-1. The number of transversal planes intersecting Π^* in one line (resp. one point) is $(k^2+k)(\ell-1)$ (resp. $k^2(\ell-1)(\ell-k)$), since any line (resp. any point) of Π^* is in exactly $\ell-1$ (resp. ℓ^2-1) transversal planes distinct from Π. Hence the number of planes of S intersecting Π outside S^* is equal to

$$c = (\ell-1)[(\ell^2 + \ell + 1) - (k^2 + k) - k^2(\ell-k)] \quad ,$$

which is clearly independent of the choice of the transversal plane Π.

Let L be a line of $\Pi-\Pi^*$ (if there is one). Since the planes containing L intersect Π_1 in pairwise disjoint lines, the number of planes intersecting Π in L is $\ell-1$. Therefore the number of planes intersecting Π in a line outside Π^* is equal to $(\ell-1)b'$.

Now let x be a point of $\Pi-\Pi^*$ (if there is one). Any plane of S disjoint from Π^* intersects Π_1 in a line disjoint from $\Pi \cap \Pi_1$ and, for any line A of Π_1 disjoint from $\Pi \cap \Pi_1$, the plane $\langle A,x\rangle$ is disjoint from Π^* (otherwise it would intersect Π in a line intersecting $\Pi \cap \Pi_1$, a contradiction). Hence the number of planes through x which are disjoint from Π^* is equal to the number of lines of Π_1 which are disjoint from $\Pi \cap \Pi_1$, that is $(\ell-1)(\ell+1-k)$. Therefore the number of planes whose intersection with Π is the point x is equal to $(\ell-1)(\ell+1-k-r'_x)$. It follows that

$$c = (\ell-1)(b' + v'(\ell+1-k) - \sum_{x\in\Pi-\Pi^*} r'_x)$$

Corollary 10.6. If some transversal plane Π *contains at least one line of* $S - S^*$, *then every transversal plane of* S *contains at least one line of* $S - S^*$.

Proof. Let $x, y \in \Pi-\Pi^*$. We have seen that the number of planes of S passing through x (resp. y) and disjoint from Π^* is equal to $(\ell-1)(\ell+1-k)$, and that the number of planes of S intersecting Π in the line $\langle x,y\rangle$ is $\ell-1$. Therefore

$$c \geq 2(\ell-1)(\ell+1-k) - (\ell-1) > (\ell+1)(\ell+1-k)$$

since $\ell > k$. The existence of a transversal plane of S contained in S^* or having a single point outside S^* would imply $c = 0$ or $c = (\ell+1)(\ell+1-k)$, a contradiction.

Lemma 10.7. Any non-transversal plane Π_i^* *of* S *belongs to a unique partition* dir Π_i^* *of* S^* *into non-transversal planes. The set of all planes of* S *whose intersections with* S^* *are the planes of* dir Π_i^* *will be called the pseudo-direction* dir* Π_i.

Proof. Let Π^* be a transversal plane of S^*. If $\Pi_i^* \in$ dir Π_1, the lemma is obvious. If $\Pi_i^* \notin$ dir Π_1, Π_i^* is partitioned by its intersections with the planes of dir Π_1 into k lines belonging to $\delta(\Pi_1 \cap \Pi_2)$. Since all these lines intersect Π^*, $\Pi_i^* \cap \Pi^*$ is a line of Π^*. On the other hand, every line of Π^* is in a unique plane containing a line of $\delta(\Pi_1 \cap \Pi_2)$, that is in a unique non-transversal plane. Therefore Π_i^* belongs to a partition of S^* into k pairwise disjoint non-transversal planes, each of which contains one of the k parallels to $\Pi_i^* \cap \Pi^*$ in the affine plane Π^*. Such a partition is clearly unique.

Proposition 10.1. If $\ell \neq k$, *then* $S = S^* = S_{36}$.

Proof. Suppose on the contrary that $S - S^*$ is non-empty. A point $x \in S - S^*$ cannot belong to two non-transversal planes Π_i and Π_j whose pseudo-directions are distinct, because $\Pi_i \cap \Pi_j$ is a line of $\delta(\Pi_1 \cap \Pi_2)$ included in S^* . On the other hand, any line through x intersecting S^* is contained in a unique non-transversal plane. Therefore x belongs to the planes of exactly one pseudo-direction dir* Π_i. We shall say that x and dir* Π_i are *associated*. Obviously, all the points of $S - S^*$ associated with a given pseudo-direction are collinear.

Suppose first that all points of $S - S^*$ are associated with the same pseudo-direction dir* Π_i. Then the points of $S - S^*$ are collinear (this includes the case where $|S - S^*| = 1$), and so there is a transversal plane Π having exactly one point outside S^*. Hence, by Lemma 10.6, $c = (\ell-1)(\ell+1-k)$. Since $k \neq \ell+1$ (because $c \neq 0$), there is a line A of Π_1 disjoint from $\Pi \cap \Pi_1$. The number of transversal planes through A is $|\Pi_1|/k = \ell$ and the number of transversal planes through A intersecting Π in a line is k-1. Therefore, since any two transversal planes have a non-empty intersection, the number of transversal planes through A intersecting Π in exactly one point is $\ell-(k-1)$. By counting in two ways the number of pairs (y,α) where y is a point of Π and $\alpha = \langle y,A\rangle$ we get

$$k^2 + 1 = k + (k-1)k + \ell - k + 1$$

which implies $\ell = k$, a contradiction.

This proves that $S - S^*$ contains two points associated with distinct pseudo-directions. If all the points of a line L_i of $S - S^*$ are associated with a pseudo-direction dir* Π_i, let x be a point of $S - S^*$ associated with another pseudo-direction dir$^* \Pi_j$. Then L_i is the intersection of the plane $<x,L_i>$ with each plane of dir$^* \Pi_i$. Since the planes of dir$^* \Pi_i$ induce a partition of S^*, the plane $<x,L_i>$ must be disjoint from S^* , a contradiction. Therefore any pseudo-direction of planes is associated with at most one point outside S^*. Since there are k+1 pseudo-directions and since dir$^* \Pi_1$ and dir Π_2 are not associated with any point of $S - S^*$, we have $|S - S^*| \leq k-1$. Since $S - S^*$ contains at least one line (all of whose points are associated with distinct pseudo-directions), there is a transversal plane having at least one line outside S^*. Therefore, by Corollary 10.6, every transversal plane has at least one line outside S^*. It follows that the number n of pairs (Π,L), where Π is a transversal plane and L a line of $\Pi-\Pi^*$, is not less than the number ℓ^3 of transversal planes and is equal to ℓ times the number of lines in $S - S^*$, that is

$$\ell^3 \leq n \leq \ell(k-1)(k-2)/2 \qquad (3)$$

On the other hand, the degree $\ell+1$ of a point in Π_1 cannot be less than the size k of some of the lines of Π_1, and so $\ell^2 \geq (k-1)^2$, contradicting (3).

We have proved that $S = S^*$. Therefore c = 0 and, using Lemma 10.6, we get

$$\ell = (k^2 - 1 \pm (k-1)\sqrt{k^2 - 2k - 3})/2$$

Obviously, k = 2 is excluded and, for k > 3

$$(k-2)^2 < k^2 - 2k - 3 < (k-1)^2$$

shows that $k^2 - 2k - 3$ is not a perfect square. Therefore k = 3 and $\ell = 4$.

Thus every transversal plane is an affine plane of order 3 and every non-transversal plane consists of 3 pairwise disjoint lines of size 4, all the other lines having size 3. If we replace each line of size 4 by 6 lines of size 2, we get a linear space F of 36 points consisting of lines of sizes 2 and 3 and in which the smallest linear subspaces containing three non-collinear points are degenerate projective planes of 3 points, punctured projective planes of order 2 or affine planes of order 3, and so F is a Fischer space of 36 points. Let dir $\Pi_1 = \{\Pi_1, \Pi_1', \Pi_1''\}$ and let F_6 be a linear subspace of 6 points of F contained in the plane Π_1 of S. If $x \in \Pi_1'$, x is joined to every point of F_6 by a line of size 3, and the smallest linear subspace of F containing x and F_6 has obviously at least 6 points in each of the planes Π_1, Π_1' and Π_1''. Buekenhout [10] has proved that a Fischer space having at least 18 points and generated by a plane α isomorphic to F_6 and by a point joined by a line of size 3 to at least one point of α is necessarily either F_{18} or F_{36}. Since x is joined to every point of F_6 by a line of size 3, a situation which does not occur in F_{18}, F is isomorphic to F_{36}. There is a unique way to construct 9 mutually disjoint lines of size 4 from the lines of size 2 of F_{36} and to provide this new linear space with planes isomorphic to those of S. The planar space S_{36} constructed in this way from F_{36} has the required properties.

3.3. Suppose that S contains only one direction Δ of planes

The planes of Δ will be called *Δ-planes*, the lines contained in a Δ-plane will be called *Δ-lines*, and the lines intersecting all Δ-planes will be called *transversal lines* (by condition (III), a line intersecting a Δ-plane must intersect all Δ-planes).

Lemma 10.8. Every Δ-line L *contained in a Δ-plane* Π *belongs to at least one partition of* Π *into lines which are coplanar with the same line of a Δ-plane* $\Pi' \neq \Pi$.

Proof. Let $\Pi' \neq \Pi$ be a Δ-plane and let L' be a line of Π' coplanar with L. The set of intersections of Π with the planes passing through L' (and distinct from Π') is clearly a partition of Π into lines, and L belongs to this partition.

Since we have assumed that S is not the union of two disjoint planes, all Δ-planes are isomorphic by Corollary 10.2.1. Let v' denote the number of points of any Δ-plane.

Lemma 10.9. If $S = S^*$, *then*

(i) *all transversal lines have size* $\ell = |\Delta| \geq 3$

(ii) *any two coplanar Δ-lines contained in two distinct Δ-planes have the same size*

(iii) *the number* p_L *of planes containing a Δ-line* L *is* $1 + v'/|L|$.

Proof. $S = S^*$ is partitioned by the Δ-planes. Moreover, $|\Delta| \geq 3$ because S is not the union of two disjoint planes. This proves (i).
(ii) is a consequence of (i) and of Lemma 10.2.
Let Π be a Δ-plane not containing L. The planes not belonging to Δ and containing L intersect Π in lines of size $|L|$ by (ii). This proves (iii).

Lemma 10.10. If $S = S^*$, *then any two disjoint Δ-lines contained in the same Δ-plane* Π *have the same size.*

Proof. Suppose on the contrary that Π contains two disjoint lines A of size a and B of size b with a > b. Let α be a plane intersecting Π in A. Any line C of α distinct from A and coplanar with B is disjoint from A, hence C is a Δ-line and, by Lemma 10.9, C has size a (because C is coplanar with A) and C has size b (because C is coplanar with B), a contradiction. Therefore A is the only line of α coplanar with B, and so any plane containing B and distinct from Π intersects α in exactly one point. This, together with Lemma 10.9, implies that

$$p_B - 1 = v'/b = a(\ell-1) \qquad (4)$$

By Lemmas 10.8 and 10.9, Π contains at least one line B' disjoint from B and having size b. Let $\beta \neq \Pi$ be a plane containing B and let n be the number of lines of β which are distinct from B and coplanar with B' (such a line being necessarily a Δ-line, $0 \leqslant n \leqslant \ell-1$). Moreover, since a plane containing B' must intersect β in a line or in a point, we have

$$p_{B'} - 1 = v'/b = n + b(\ell-1-n) \qquad (5)$$

(4) and (5) yield

$$a(\ell-1) + n(b-1) = b(\ell-1) \quad \text{where } n \geqslant 0 ,$$

contradicting the assumption a > b.

Proposition 10.5. If $S = S^*$ *contains two* Δ*-lines of different sizes, then* $S = S_{18}$.

Proof. Let a > b be two sizes of Δ-lines and let Π be a Δ-plane. Since all Δ-planes are isomorphic, Π contains a line of size a and a line of size b. By Lemmas 10.8 and 10.9, there is a partition of Π into lines of size a and a partition of Π into lines of size b. Since a > b, any line L of Π which does not belong to any of these two partitions is necessarily disjoint from at least one line of size b in the second partition, and so L has size b by Lemma 10.10. Moreover, by Lemma 10.10 again, in the plane Π, every line of size b must intersect every line of size a. Therefore Π contains exactly v' = ab points, and every point of Π is on exactly one Δ-line of size a, on exactly a Δ-lines of size b and has degree r' = a+1 in Π. It follows that the Δ-lines of size a partition S and are pairwise coplanar.

Let A be a line of size a in Π. By Lemma 10.9, the number p_A of planes containing A is given by

$$p_A = 1 + v'/a = 1+b$$

Let L be a transversal line disjoint from A. Each of the ℓ points of L is on a unique Δ-line of size a and the union of these ℓ lines is a plane λ. Since Δ is the only direction of planes in S, every plane containing A intersects the plane λ and this intersection is necessarily a Δ-line of size a. Thus every plane containing A intersects the line L, and so

$$p_A = 1 + v'/a = 1+b = \ell \qquad (6)$$

Let x be a point of Π and let α (resp. β) be a plane containing x and intersecting Π in a line A (resp. B) of size a (resp. b). We shall count the number $n(x,\alpha)$ (resp. $n(x,\beta)$) of planes intersecting α (resp. β) in the point x only. Let $\Pi' \neq \Pi$ be a Δ-plane and let $A' = \Pi' \cap \alpha$, $B' = \Pi' \cap \beta$. Since any plane intersecting α in the point x only intersects Π' in a line disjoint from A' and since all lines of Π' which are disjoint from A' have size a and are coplanar with A, we have

$$n(x,\alpha) = 0 \qquad (7)$$

On the other hand, the number of planes (distinct from Π) containing B is $v'/b = a$, the number of planes intersecting β in a transversal line passing through x is $|B'|(r'-1) = ba$, and the total number planes (distinct from Π) passing through x is equal to the number a^2+b of lines in Π'. Therefore

$$n(x,\beta) = a^2 + b - ba - a \qquad (8)$$

Since any plane of S belongs to Δ or is α (resp. β) or intersects α (resp. β) in a Δ-line or intersects α (resp. β) in a transversal line or intersects α (resp. β) in a single point, the total number p of planes of S is given respectively by

$$\begin{aligned} p &= |\Delta| + 1 + |\Delta|(v'/a-1) + a^2(r'-1) + |\alpha| n(x,\alpha) \\ &= |\Delta| + 1 + |\Delta|(v'/b-1) + b^2(r'-1) + |\beta| n(x,\beta) \end{aligned}$$

from which it follows, by (7) and (8), that

$$\ell(b-1) + a^3 = \ell(a-1) + b^2 a + b\ell(a^2 + b - ab - a)$$

Using (6), we get, after simplification by $a-b \neq 0$ and $a-1 \neq 0$,

$$b^2 = a+1 \qquad (9)$$

Let B" be a line of Π' disjoint from B'. The number of planes containing B" is

$$1 + v'/b = 1+a = m + b(\ell-m) \qquad (10)$$

where $1 \leqslant m \leqslant \ell$ denotes the number of lines of β which are coplanar with B"
By (6), (9) and (10), we get

$$m(b-1) = b$$

and so $m = b = 2$, $a = \ell = 3$ and $|S| = 18$.
Therefore every transversal line has size 3, every Δ-plane is the union of two disjoint lines of size 3 and the planes not belonging to Δ are punctured projective planes of order 2 or affine planes of order 3 according as their Δ-lines have size 2 or 3. This implies that the linear space S is a Fischer space of 18 points. Moreover, it is easy to check that the smallest linear subspace of S containing a punctured projective plane π of order 2 and a point $x \notin \pi$ joined to a point of π by a line of size 3 is S itself. Buekenhout [10] has proved that a Fischer space of 18 points having this property is necessarily isomorphic to F_{18}. Moreover, there is a unique way to provide F_{18} with planes isomorphic to those of S. The planar space S_{18} constructed in this way from F_{18} has the required properties.

Proposition 10.6. If $S = S^*$ *and if all* Δ*-lines have the same size* a, *then* S *is obtained from* PG(3,a) *by deleting a line.*

Proof. Any Δ-plane π is a Steiner system S(2,a,v'). Thus, if we denote by b' the number of lines of π and by r' the degree of any point in π, we have

$$v' = r'(a-1) + 1 \qquad (11)$$

and

$$b' = v'r'/a \qquad (12)$$

Let α be a plane not belonging to Δ and let $x \in \alpha$. Counting in the same way as in Lemma 10.14 the number $n(x,\alpha)$ of planes of S intersecting α in the point x only, we have

$$n(x,\alpha) = b' - a(r'-1) - v'/a \qquad (13)$$

and so the total number p of planes of S is

$$p = \ell + 1 + \ell(v'/a - 1) + a^2(r'-1) + a\ell n(x,\alpha) \qquad (14)$$

On the other hand, every plane not belonging to Δ intersects π in a line and every line of π is contained in exactly v'/a planes not belonging to Δ, so that

$$p = \ell + b'v'/a \qquad (15)$$

Let A' be a line of Π disjoint from $A = \Pi \cap \alpha$. The number of planes containing A' is given by

$$1 + v'/a = n + a(\ell-n) \qquad (16)$$

where $1 \leqslant n \leqslant \ell$ is the number of lines of α which are coplanar with A'.
Using (11), (16) becomes

$$r' = (\ell a^2 - a - 1)/(a-1) - na \qquad (17)$$

which implies $a-1 \mid \ell-2 > 0$
and so $a-1 \leqslant \ell-2$.
On the other hand, the degree a+1 of a point in α cannot be less than the size ℓ of a transversal line, and so

$$a+1 \geqslant \ell$$

These two inequalities imply that

$$\ell = a+1 \qquad (18)$$

and (17) becomes

$$r' = a^2 - (n-2)a+1 \qquad (19)$$

From (13), (14), (15), (11), (12) and (18), we deduce, after some straightforward computation,

$$(r'-1)(r'-a-1)(r' - a^2 - a) = 0$$

and so
$r' = 1$, $a+1$ or a^2+a
$r' = 1$ is clearly impossible and $r' = a^2+a$ contradicts (19). Therefore $r' = a+1$ and the Δ-planes are affine planes of order a. The planes not belonging to Δ have exactly $\ell a = a(a+1)$ points and are punctured projective planes of order a. It is now a simple matter to deduce that S is obtained from PG(3,a) by deleting one line.

Proposition 10.7. If $S \neq S^*$, *then* $S - S^*$ *contains at least four non-coplanar points.*

Proof. Suppose on the contrary that $S - S^*$ is contained in a plane α.

Consider first the case where $|\Delta| = 2$. Let Π and Π' be the two Δ-planes. Since any plane which is not in Δ intersects both Π and Π' in a line, $A = \Pi \cap \alpha$ and $A' = \Pi' \cap \alpha$ are two Δ-lines. The planes (distinct from Π) containing A determine a partition of Π' into lines. If this partition contains two lines A" and A"' distinct from A', then A ∪ A" and A ∪ A"' are two planes of S because every point of $S - S^*$ is in the plane α. Let $x \in \Pi - A$. The plane $\langle x, A''' \rangle$ must intersect the plane $A \cup A''$, but this is impossible since A and A''' are coplanar. Therefore $\Pi' = A' \cup B'$, where B' is a line of Π' disjoint from A' and $A \cup B'$ is a plane of S. Similarly $\Pi = A \cup B$, where B is a line of Π disjoint from A and $A' \cup B$ is a plane of S. The planes $A \cup B'$ and $A' \cup B$ are disjoint, contradicting the assumption that Π and Π' are the only disjoint planes in S.

Suppose now that $|\Delta| = \ell \geq 3$. Let x_1, x_2 be any two points of a Δ-plane Π. If Π' and Π'' are two Δ-planes distinct from Π and if x_1'' is a point of Π'', the lines passing through x_1'' and intersecting Π determine, by Corollary 10.2.1, an isomorphism φ_1 from Π onto Π'. Let $x_1' = \varphi_1(x_1)$ and let x_2'' be the point of intersection of the line $\langle x_2, x_1' \rangle$ with the plane Π''. The lines passing through x_2'' and intersecting Π' determine an isomorphism φ_2 from Π' onto Π. Since $\varphi_2 \circ \varphi_1$ is an automorphism of Π mapping x_1 on x_2, all points of Π have the same degree r'.

Let $A = \Pi \cap \alpha$ and let β be a plane containing A, distinct from Π and α. Lemma 10.2 implies that any two coplanar Δ-lines contained in two distinct Δ-planes have the same size. Therefore, for any point $x \in \alpha^* = \alpha \cap S^*$ and for any point $y \in \beta$,

$$n(x,\alpha) = b' - a(r'-1) - v'/a = n(y,\beta) \qquad (20)$$

where $a = |A|$ and v' (resp. b') is the number of points (resp. of lines) in a Δ-plane. Counting in two ways the number p of planes in S, we get

$$\begin{aligned} p &= \ell+1 + \ell(v'/a-1) + a^2(r'-1) + |\beta| n(y,\beta) \\ &= \ell+1 + \ell(v'/a-1) + a^2(r'-1) + |\alpha^*| n(x,\alpha) + \sum_{z \in \alpha-\alpha^*} n(z,\alpha) + \sum_{L \subset \alpha-\alpha^*} (p_L - 1) \end{aligned}$$

where p_L denotes the number of planes containing the line L. Using (20) and the fact that $|\alpha^*| = |\beta| = \ell a$, this implies

$$\sum_{z \subset \alpha-\alpha^*} n(z,\alpha) + \sum_{L \subset \alpha-\alpha^*} (p_L - 1) = 0$$

Since $n(z,\alpha) \geq 0$ and $p_L - 1 \geq 1$ for every line $L \subset \alpha-\alpha^*$, we conclude that $n(z,\alpha) = 0$ for every point $z \in \alpha-\alpha^*$ and that there is no line contained in $\alpha-\alpha^*$. On the other hand, by Lemma 10.8, there is a line B of Π disjoint from A. If $z \in \alpha-\alpha^*$, the plane $\langle B, z \rangle$ is disjoint from A, thus also from α^*. Therefore, either $\langle B, z \rangle$ intersects α in the point z only and $n(z,\alpha) \neq 0$, or $\langle B, z \rangle$ intersects α in a line contained in $\alpha-\alpha^*$. In both cases, we have a contradiction.

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