7. THE CANONICAL SERIES

Let $\mathscr{C}$ be an irreducible curve in $PG(2,\bar{K})$ where $\bar{K}$ is the algebraic closure of K and let X be a non-singular model of $\mathscr{C}$ with $\Psi : X \to \mathscr{C}$ birational. Points of X are places or branches of $\mathscr{C}$. A place Q is centred at P if $Q\Psi = P$. Let $r_Q = m_P(\mathscr{C})$, the multiplicity of $\mathscr{C}$ at P, where $\mathscr{C}$ has only ordinary singular points. If $\mathscr{C}' = V(G)$ is any other plane curve such that div(G)-E is effective, where $E = \sum_{Q \in X} (r_Q - 1)Q$, then $\mathscr{C}'$ is an adjoint of $\mathscr{C}$; essentially, $\mathscr{C}'$ passes m-1 times through any point of $\mathscr{C}$ of multiplicity m. If $\deg\mathscr{C} = d$ and $\deg\mathscr{C}' = d-3$, then $\mathscr{C}'$ is a special adjoint of $\mathscr{C}$. In this case, div(G) - E is a canonical divisor. The canonical series, consisting of all canonical divisors, is therefore cut out by all the special adjoints of $\mathscr{C}$. The series is a γ^{g-1}_{2g-2} of (projective) dimension g-1 and order 2g-2. For example,

$$\mathscr{C}^6 = V(z^2xy(x-y)(x+y)+x^6+y^6)$$

is a sextic with an ordinary quadruple point at P(0,0,1) and no other singularity. So

$$g = \tfrac{1}{2}(6-1)(6-2) - \tfrac{1}{2}\,4(4-1) = 4 .$$

The special adjoints are cubics with a triple point at P(0,0,1), that is triples of lines through the point. A special adjoint has equation $V((x-\lambda_1 y)(x-\lambda_2 y)(x-\lambda_3 y))$ and has freedom 3. It meets $\mathscr{C}^6$ in 6.3-4.3=6 points other than P(0,0,1). Hence the special adjoints cut out a γ^3_6, as expected.

The Riemann-Roch theorem says that if W is a canonical divisor on X and D is any divisor, then

$$\ell(D) = \deg D + 1 - g + \ell(W-D).$$

8. THE OSCULATING HYPERPLANE OF A CURVE

Let X be an irreducible, non-singular, projective, algebraic curve of genus g defined over K but viewed as the set of points defined over $\bar{K}$, and let $f : X \to \mathscr{C} \subset PG(n,\bar{K})$ be a suitable rational map. Then $\mathscr{C}$ is viewed as the set of branches of X.

Assume that $\mathscr{C}$ is not contained in a hyperplane. The degree d of $\mathscr{C}$ is the number of points of intersection of $\mathscr{C}$ with a generic hyperplane. For any hyperplane H, if n_p is the intersection multiplicity of H and $\mathscr{C}$ at P, then

$$H\cdot\mathscr{C} = \sum_{P\in\mathscr{C}} n_P\, P$$

is a divisor of degree $d = \Sigma n_P$. Also

$$\mathscr{D} = \{H.\mathscr{C} \mid H \text{ a hyperplane}\}$$

is a linear system. In this case, $D \sim D'$ for any D, D' in $\mathscr{D}$. Hence $\mathscr{D}$ is contained in the complete linear system $|D| = \{D' \mid D' \sim D\}$, where D is some element of $\mathscr{D}$.

A complete linear system defines an embedding $f : X \to \mathscr{C}$ given by

$$f(Q) = P(f_o(Q),\ldots,f_n(Q))$$

where $\{f_o,\ldots,f_n\}$ is a basis of

$$L(D) = \{g\in\bar{K}(X) \mid \operatorname{div}(g) + D \geq 0\}.$$