2. THE MAXIMUM NUMBER OF POINTS ON AN ALGEBRAIC CURVE

Let \( C \) be an algebraic curve defined over \( GF(q) \) of genus \( g \), and let \( N \) be the number of points, rational over \( GF(q) \), on a non-singular model of \( C \). Define \( N_q(g) = \max N \), where \( C \) varies over all curves of genus \( g \). We recall the following bounds.

(i) Hasse-Weil: \[ N_q(q) \leq q+1+2gq^{1/2} \]
(ii) Serre: \[ N_q(g) \leq q+1+g[2q^{1/2}] \]
(iii) Ihara: \[ N_q(g) \leq q+1-\frac{1}{2}g+(2q+1/8)g^2+(q^2-q)/2 \]
(iv) Manin: \[ N_2(q) \leq 2g - \sigma(g) \text{ as } g \to \infty \]
\[ N_3(g) \leq 3g + \sigma(g) \text{ as } g \to \infty \]
(v) Drinfeld-Vladut: \[ N_q(g) \leq g(q^{1/2}-1)+\sigma(g) \text{ as } g \to \infty \]

For a summary of results on \( N_q(g) \) and references, see [9] Appendix IV.

The estimates (i) and (ii) are good for \( g \leq \frac{1}{2}(q-q^{1/2}) \), but not for \( g > \frac{1}{2}(q-q^{1/2}) \).

One of the aims of these notes is to describe improvements to (i), (ii), (iii). First, it is elementary that (ii) is sometimes better than (i) and never worse.

Let \( m = [2q^{1/2}] \). Then \( 2q^{1/2} = m+\epsilon \), where \( 0 \leq \epsilon < 1 \). So \[ [2q^{1/2}] = [g(m+\epsilon)] = [gm+g\epsilon] = gm+[g\epsilon]. \]

3. THE DEDUCTION OF SERRE'S AND IHARA'S RESULTS FROM THE RIEMANN HYPOTHESIS.

(a) Serre's result