Sketch of the proof. Computing the critical point condition for $I(g)$ on $\mathcal{M}$ in general we find that it is

$$2(\nabla_m\nabla_i R_{jk}R^{km} + \nabla_m\nabla_j R_{ik}R^{km} - \nabla^m\nabla_m R_{jk}R^k{}_i - g_{ij}\nabla_m\nabla_l R^m{}_k R^{kl}$$

$$-2R_{im}R^m{}_k R^k{}_j + \frac{1}{2}R_{ij}) + \frac{1}{2}(\frac{4}{3}trQ^3 - R)g_{ij} = cg_{ij}.$$

Now since Q^2 is parallel and $Q^3 = \frac{1}{4}Q$ on the Abbena-Thurston manifold we see that this metric on the underlying manifold $M = G/\Gamma$ is a critical point of $I(g)$.

From the expression for Q it is clear that (M, g) is not Einstein nor is $QJ = JQ$. Thus this metric is not a critical point for $A(g) = \int_M R\, dV_g$ considered as a functional on $\mathcal{M}$ or on $\mathcal{A}$ or for $K(g) = \int_M R - R^*\, dV_g$ on $\mathcal{A}$. In particular it does not give a negative answer to the question of whether or not an almost Kähler manifold satisfying $QJ = JQ$ is Kählerian. On the other hand (M, g) is a critical point for K in a different context; C. M. Wood [25] showed that the Abbena-Thurston manifold is a critical point of K defined with respect to variations through almost complex structures J which preserve g. For this problem the critical point condition is

$$[J, \nabla^*\nabla J] = 0,$$

where $\nabla^*\nabla J$ is the rough Laplacian of the metric in question.

6. Problems involving other integrands

Finally we turn to a brief discussion of some related problems. In the Riemannian geometry of contact metric manifolds the tensor fields l and S defined by $lX = R(X,\xi)\xi$ and $S(X,Y) = R(X,Y)\xi$ play important roles. For example on a K-contact manifold l is the identity and on a Sasakian manifold $S(X,Y) = \eta(Y)X - \eta(X)Y$. More generally we have noted (equation (2.3)) that

$$\nabla_\xi h = \phi - \phi h^2 - \phi l.$$

Thus it seems reasonable to consider functionals defined by integrals such as $\int_M |l|^2\, dV_g$ and $\int_M |S|^2\, dV_g$. In the case of the first of these Mr. S. R. Deng computed the critical point condition of $\int_M |l|^2\, dV_g$ as a functional on $\mathcal{A}$ and noted the following.

Proposition 6.1 (Deng). *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then a K-contact metric is a critical point of the functional $\int_M |l|^2\, dV_g$ on $\mathcal{A}$. More generally if for a metric g, $\nabla_\xi h = 0$, then g is a critical point if and only if $h^3 - h = 0$.*

The original functionals $A(g)$, $B(g)$, $C(g)$, $D(g)$ on $\mathcal{M}$ have been study further in the context of contact geometry by Muto [17] and Yamaguchi and Chūman [26]; the general thrust of their work is to suppose that a critical point is a Sasakian metric. For example we have the following results.

Theorem 6.2 (Muto). *If a critical point g of $A(g)$, $B(g)$, $C(g)$ or $D(g)$ is a Sasakian metric, then its scalar curvature is constant.*

We remarked at the outset that Einstein metrics were critical points of $B(g)$. From Theorem 6.2 and the critical point condition for $B(g)$ we have an immediate converse in the case of a Sasakian metric.

Theorem 6.3 (Yamaguchi and Chūman). *In order for a Sasakian metric to be a critical point of $B(g)$ it is necessary and sufficient that it be an Einstein metric.*

The two papers [17] and [26] give many results and focus in particular on Sasakian submersions, discussing relations between $B(g)$, $C(g)$ and $D(g)$ defined relative to the bundle space and the base space of the submersion. There are other contexts where some of these functionals have been discussed, but further discussion would take us beyond the scope of these lectures.