metric. Now the *Webster scalar curvature* W on a 3-dimensional contact metric manifold is defined by

$$W = \frac{1}{8}(R + \frac{1}{4}|\tau|^2 + 2);$$

by virtue of (2.4) and $\frac{1}{4}|\tau|^2 = |h|^2$, W becomes

$$W = \frac{1}{8}(R - Ric(\xi) + 4).$$

Chern and Hamilton [11] studied the functional $E_W(g) = \int_M W\, dV_g$ for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M and proved the following Theorem.

Theorem 4.5 (Chern-Hamilton). *Let M be a compact 3-dimensional contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_W(g) = \int_M W\, dV_g$ if and only if g is K-contact.*

An alternate proof was given by D. Perrone [19]. In view of the work we have done so far we can prove this theorem as follows.

Proof of Theorem 4.5. Clearly it is enough to consider $\int_M R - Ric(\xi)\, dV_g$ and having computed the derivatives of each term separately we see that

$$\frac{d}{dt}\int_M R - Ric(\xi)\, dV_g\Big|_{t=0} = \int_M (-R^{ki} + h^i{}_m h^{mk} + R^k{}_{rs}{}^i \xi^r \xi^s - 2h^{ik}) D_{ik}\, dV_g.$$

Thus the critical point condition is

$$(Q\phi - \phi Q) - (l\phi - \phi l) - 4\phi h = -\eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi.$$

So far we have not used the fact that we are in dimension three and hence this is the critical point condition for the intergral of the generalized Tanaka-Webster scalar curvature as defined by Tanno [22]. Now in dimension 3 we can combine this condition with (3.6) to get $h = 0$.

5. The Abbena-Thurston manifold as a critical point

In 1976 W. Thurston [24] gave an example of a compact symplectic manifold with no Kähler structure. We will begin by discussing this manifold briefly and then turn to a natural Riemannian metric on this manifold introduced by E. Abbena [1]. For details of the topological obstructions to a Kähler structure we refer to [24] or [1] and simply remark here that the first Betti number of this manifold is 3 whereas the odd-dimensional Betti numbers of a compact Kähler are even.

Let G be the closed connected subgroup of $GL(4, \mathbf{C})$ defined by

$$\left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{pmatrix} \middle| \, a_{12}, a_{13}, a_{23}, a \in \mathbf{R} \right\},$$

i.e. G is the product of the Heisenberg group and S^1. Let Γ be the discrete subgroup of G with integer entries and $M = G/\Gamma$. Denote by x, y, z, t coordinates on G, say for $A \in G$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$, $t(A) = a$. If L_B is left translation by $B \in G$, $L_B^* dx = dx$, $L_B^* dy = dy$, $L_B^*(dz - xdy) = dz - xdy$, $L_B^* dt = dt$. In particular these forms are invariant under the action of Γ; let $\pi : G \longrightarrow M$, then there exist 1-forms α_1, α_2, α_3, α_4 on M such that $dx = \pi^*\alpha_1$, $dy = \pi^*\alpha_2$, $dz - xdy = \pi^*\alpha_3$, $dt = \pi^*\alpha_4$. Setting $\Omega = \alpha_4 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3$ we see that $\Omega \wedge \Omega \neq 0$ and $d\Omega = 0$ on M giving M a symplectic structure.

The vector fields

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \ \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \ \mathbf{e}_3 = \frac{\partial}{\partial z}, \ \mathbf{e}_4 = \frac{\partial}{\partial t}$$

are dual to dx, dy, $dz - xdy$, dt and are left invariant. Moreover $\{\mathbf{e}_i\}$ is orthonormal with respect to the left invariant metric on G given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2 + dt^2.$$

On M the corresponding metric is $g = \sum \alpha_i \otimes \alpha_i$. The Riemannian manifold (M, g) is refered to as the *Abbena-Thurston manifold.*

Moreover M carries an almost complex structure defined by

$$J\mathbf{e}_1 = \mathbf{e}_4, \ J\mathbf{e}_2 = -\mathbf{e}_3, \ J\mathbf{e}_3 = \mathbf{e}_2, \ J\mathbf{e}_4 = -\mathbf{e}_1.$$

Then noting that $\Omega(X, Y) = g(X, JY)$, we see that g is an associated metric.

The curvature of g was computed by E. Abbena in [1]. With respect to the basis $\{\mathbf{e}_i\}$ the non-zero components of the curvature tensor are

$$R_{1221} = \frac{3}{4}, \ R_{2332} = -\frac{1}{4}, \ R_{1331} = -\frac{1}{4}.$$

Thus the Ricci operator Q is given by the matrix

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we note that Q^2 is parallel with respect to the Levi-Civita connection of g but that Q is not parallel.

The following observation stems from conversations between Won-Tae Oh and myself.

Proposition 5.1 (Blair-Oh). *The Abbena-Thurston manifold is a critical point of the functional*

$$I(g) = \int_M (\frac{4}{3} trQ^3 - R)\, dV_g$$

on $\mathcal{M}$.

Sketch of the proof. Computing the critical point condition for $I(g)$ on $\mathcal{M}$ in general we find that it is

$$2(\nabla_m\nabla_i R_{jk}R^{km} + \nabla_m\nabla_j R_{ik}R^{km} - \nabla^m\nabla_m R_{jk}R^k{}_i - g_{ij}\nabla_m\nabla_l R^m{}_k R^{kl}$$

$$-2R_{im}R^m{}_k R^k{}_j + \frac{1}{2}R_{ij}) + \frac{1}{2}(\frac{4}{3}trQ^3 - R)g_{ij} = cg_{ij}.$$

Now since Q^2 is parallel and $Q^3 = \frac{1}{4}Q$ on the Abbena-Thurston manifold we see that this metric on the underlying manifold $M = G/\Gamma$ is a critical point of $I(g)$.

From the expression for Q it is clear that (M, g) is not Einstein nor is $QJ = JQ$. Thus this metric is not a critical point for $A(g) = \int_M R\,dV_g$ considered as a functional on $\mathcal{M}$ or on $\mathcal{A}$ or for $K(g) = \int_M R - R^*\,dV_g$ on $\mathcal{A}$. In particular it does not give a negative answer to the question of whether or not an almost Kähler manifold satisfying $QJ = JQ$ is Kählerian. On the other hand (M, g) is a critical point for K in a different context; C. M. Wood [25] showed that the Abbena-Thurston manifold is a critical point of K defined with respect to variations through almost complex structures J which preserve g. For this problem the critical point condition is

$$[J, \nabla^*\nabla J] = 0,$$

where $\nabla^*\nabla J$ is the rough Laplacian of the metric in question.

6. Problems involving other integrands

Finally we turn to a brief discussion of some related problems. In the Riemannian geometry of contact metric manifolds the tensor fields l and S defined by $lX = R(X,\xi)\xi$ and $S(X,Y) = R(X,Y)\xi$ play important roles. For example on a K-contact manifold l is the identity and on a Sasakian manifold $S(X,Y) = \eta(Y)X - \eta(X)Y$. More generally we have noted (equation (2.3)) that

$$\nabla_\xi h = \phi - \phi h^2 - \phi l.$$

Thus it seems reasonable to consider functionals defined by integrals such as $\int_M |l|^2\,dV_g$ and $\int_M |S|^2\,dV_g$. In the case of the first of these Mr. S. R. Deng computed the critical point condition of $\int_M |l|^2\,dV_g$ as a functional on $\mathcal{A}$ and noted the following.

Proposition 6.1 (Deng). *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then a K-contact metric is a critical point of the functional $\int_M |l|^2\,dV_g$ on $\mathcal{A}$. More generally if for a metric g, $\nabla_\xi h = 0$, then g is a critical point if and only if $h^3 - h = 0$.*

The original functionals $A(g)$, $B(g)$, $C(g)$, $D(g)$ on $\mathcal{M}$ have been study further in the context of contact geometry by Muto [17] and Yamaguchi and Chūman [26]; the general thrust of their work is to suppose that a critical point is a Sasakian metric. For example we have the following results.