For contact manifolds of general odd dimension, if g is critical for both A and K, $h = 0$ so ξ is Killing. To see this note that the commutativities of Theorems 3.5 and 3.6 together imply that $\phi h = h\phi$ but $\phi h = -h\phi$ and hence $h = 0$ easily follows. As with the almost Kähler case, the question of whether or not a K-contact structure satisfying $Q\phi = \phi Q$ is Sasakian would seem to be difficult.

4. Integral of the Ricci curvature in the direction of the characteristic vector field

We devote this section to a discussion of a particular functional defined on the set of metrics associated to a contact structure. The main theorem is the following [6].

Theorem 4.1 (Blair). *Let M be a compact regular contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M Ric(\xi)\, dV_g$ if and only if g is K-contact.*

One might conjecture this without the regularity, however we have the following counterexample: The standard contact metric structure on the tangent sphere bundle of a compact surface of constant curvature -1 is a critical point of L but is not K-contact. It is a result of Y. Tashiro [23] that the standard contact metric structure on the tangent sphere bundle of a Riemannian manifold is K-contact if and only if the base manifold is of constant curvature +1. Also recall the result of [4] that the standard contact structure of the tangent sphere bundle of a compact Riemannian manifold of non-positive constant curvature is not regular. Our second result is the following theorem [7].

Theorem 4.2 (Blair). *Let T_1M be the tangent sphere bundle of a compact Riemannian manifold (M, G) and $\mathcal{A}$ the set of all Riemannian metrics associated to its standard contact structure. Then the standard associated metric is a critical point of the functional $L(g)$ if and only if (M, G) is of constant curvature +1 or -1.*

Recall that by a K-contact structure we mean a contact metric structure for which ξ is Killing and that this is the case if and only if $h = 0$. Recall also equation (2.4), viz.

$$Ric(\xi) = 2n - trh^2.$$

Thus K-contact metrics when they occur are maxima for the function $L(g)$ on $\mathcal{A}$. Also the critical point question for $L(g)$ is the same as that for $\int_M |h|^2\, dV_g$ or $\int_M |\tau|^2\, dV_g$ where $\tau(X, Y) = (\mathcal{L}_\xi g)(X, Y) = 2g(X, h\phi Y)$. This last integral was studied by Chern and Hamilton [11] for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M (there was an error in their calculation of the critical point condition as was pointed out by Tanno[22]).

Proof of Theorem 4.1. As with our other critical point problems, the first step is to compute $\frac{dL}{dt}$ at $t = 0$ for a path $g(t) \in \mathcal{A}$

$$\left.\frac{dL}{dt}\right|_{t=0} = \int_M (-h^i{}_m h^{mk} - R^k{}_{rs}{}^i \xi^r \xi^s + 2h^{ik}) D_{ik}\, dV_g.$$

Thus if $g(0)$ is a critical point, Lemma 3.2 gives

$$R(X,\xi)\xi = -\phi^2 X - h^2 X + 2hX \tag{4.1}$$

as the critical point condition. Using equation (2.3) this becomes

$$(\nabla_\xi h)X = -2\phi h X. \tag{4.2}$$

From this we see that the eigenvalues of h are constant along the integral curves of ξ and that for an eigenvalue $\lambda \neq 0$ and unit eigenvector X, $g(\nabla_\xi X, \phi X) = -1$.

If now M is a regular contact manifold, then M is a principal circle bundle with ξ tangent to the fibres; locally M is $\mathcal{U} \times S^1$ where $\mathcal{U}$ is a neighborhood on the base manifold. Since $h\phi + \phi h = 0$, we may choose an orthonormal ϕ-basis of eigenvectors of h at some point of $\mathcal{U} \times S^1$ say, X_{2i-1}, $X_{2i} = \phi X_{2i-1}$, ξ. Since the eigenvalues are constant along the fibre, we can continue this basis along the fibre with at worst a change of orientation of some of the eigenspaces when we return to the starting point. Thus if Y is a vector field along the fibre, we may write

$$Y = \sum_i (\alpha_{2i-1} X_{2i-1} + \beta_{2i} X_{2i}) + \gamma\xi$$

where the coefficients are periodic functions.

Now suppose that the critical point g is not a K-contact metric. Since ϕ and h anti-commute, we may assume that all the $\lambda_{2i-1}, i = 1,\dots,n$ are non-negative. Also from equation (4.2) it is easy to see that if some of the λ_{2i-1} vanish, the zero eigenspace of h is parallel along ξ and hence we may choose the corresponding X_{2i-1} and X_{2i} parallel along a fibre. Again since M is regular we may choos a vector field Y on $\mathcal{U} \times S^1$ such that at least some $\alpha_{2i-1} \not\equiv 0$ for some $\lambda_{2i-1} \neq 0$ and Y is horizontal, i.e. $\eta(Y) = 0$, and projectable, i.e. $[\xi, Y] = 0$. Writing $Y = \sum_i (\alpha_{2i-1} X_{2i-1} + \beta_{2i} X_{2i})$ along a fibre we have using (2.1)

$$0 = [\xi, Y] = \nabla_\xi Y - \nabla_Y \xi$$

$$= \sum_i \big((\xi\alpha_{2i-1})X_{2i-1} + \alpha_{2i-1}\nabla_\xi X_{2i-1} + (\xi\beta_{2i})X_{2i} + \beta_{2i}\nabla_\xi X_{2i}$$

$$+\alpha_{2i-1}X_{2i} + \lambda_{2i-1}\alpha_{2i-1}X_{2i} - \beta_{2i}X_{2i-1} + \lambda_{2i-1}\beta_{2i}X_{2i-1}\big).$$

Taking components we have

$$0 = \xi\alpha_{2j-1} + \sum_i \alpha_{2i-1} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\beta_{2j},$$

$$0 = \xi\beta_{2j} + \sum_i \beta_{2i} g(\nabla_\xi X_{2i}, X_{2j}) + \lambda_{2j-1}\alpha_{2j-1}.$$

Multiplying the first of these by β_{2j}, the second by α_{2j-1} and summing on j we have

$$\xi(\sum_j \alpha_{2j-1}\beta_{2j}) = -\sum_j \lambda_{2j-1}(\alpha_{2j-1}^2 + \beta_{2j}^2) \leq 0.$$

Thus $\sum_j \alpha_{2j-1}\beta_{2j}$ is a non-increasing, non-constant function along the integral curve, contradicting its periodicity.

In preparation for a sketch of the proof of Theorem 4.2 we state the following lemma of Cartan.

Lemma 4.3. *Let (M, G) be a Riemannian manifold, D the Levi-Civita connection of G and $\mathbf{R}$ its curvature tensor. Then (M, G) if locally symmetric if and only if*

$$(D_X\mathbf{R})(Y, X, Y, X) = 0$$

for all orthonormal pairs $\{X, Y\}$.

Proof of Theorem 4.2. As we have seen the tangent sphere bundle, T_1M, inherits a contact structure from the symplectic structure on TM and a natural associated metric from the Sasaki metric on TM. After computing $(R(U,\xi)\xi)_t$, $t \in T_1M$ for a vertical tangent vector U, we consider the critical point condition (4.1) and compare horizontal and vertical parts. This yields for any orthonomal pair $\{X, t\}$ on the base manifold (M, G)

$$(D_t\mathbf{R})(X, t)t = 0 \tag{4.3}$$

and

$$\mathbf{R}(\mathbf{R}(X, t)t, t)t = X, \tag{4.4}$$

see [7] for more details. From (4.3) and Lemma 4.3 we see that (M, G) is locally symmetric.

Now working on (M, G), for each unit tangent vector $t \in T_mM$, let $[t]^\perp$ denote the subspace of T_mM orthogonal to t and define a symmetric linear transformation $L_t : [t]^\perp \longrightarrow [t]^\perp$ by $L_tX = \mathbf{R}(X, t)t$. Then from (4.4) we have that $(L_t)^2 = I$ and hence that the eigenvalues of L_t are ± 1. Now M is irreducible, for if M had a locally Riemannian product structure, choosing t to one factor and X tangent to the other we would have $\mathbf{R}(X, t)t = 0$, contradicting the fact that the only eigenvalues of L_t are ± 1. However the sectional curvature of an irreducible locally symmetric space does not change sign. Thus if for some t, L_t had both +1 and -1 occuring as eigenvalues, there would be sectional curvatures equal to +1 and -1. Consequently only one eigenvalue can occur and hence (M, G) is a space of constant curvature +1 or -1.

Conversely if (M, G) has constant curvature c, let U be a vertical vector tangent to T_1M and X a horizontal vector orthogonal to ξ. Then at a point t, $hU_t = (1-c)U_t$, $hX_t = (c-1)X_t$, $(R(\xi, U)\xi)_t = -c^2U_t$ and $(R(\xi, X)\xi)_t = (3c^2 - 4c)X_t$. Substituting these into the critical point condition (4.1) we see that it is satisfied when $c = \pm 1$.

Remarks: I. Recently Mr. S. R. Deng has begun the study of the second variation for the functional $L(g)$.

Proposition 4.4 (Deng). *Let $g \in \mathcal{A}$ be a critical point of $L(g)$, then at g, $\frac{d^2L}{dt^2}$ is nonpositive.*

II. Clearly in dimension 3 by Perrone's form of the critical point condition for A and the form (4.2) for L, we see that if g is a critical point for both of them, g is a K-contact metric. Now the *Webster scalar curvature* W on a 3-dimensional contact metric manifold is defined by

$$W = \frac{1}{8}(R + \frac{1}{4}|\tau|^2 + 2);$$

by virtue of (2.4) and $\frac{1}{4}|\tau|^2 = |h|^2$, W becomes

$$W = \frac{1}{8}(R - Ric(\xi) + 4).$$

Chern and Hamilton [11] studied the functional $E_W(g) = \int_M W\, dV_g$ for 3-dimensional contact manifolds as a functional on $\mathcal{A}$ regarded as the set of CR-structures on M and proved the following Theorem.

Theorem 4.5 (Chern-Hamilton). *Let M be a compact 3-dimensional contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_W(g) = \int_M W\, dV_g$ if and only if g is K-contact.*

An alternate proof was given by D. Perrone [19]. In view of the work we have done so far we can prove this theorem as follows.

Proof of Theorem 4.5. Clearly it is enough to consider $\int_M R - Ric(\xi)\, dV_g$ and having computed the derivatives of each term separately we see that

$$\frac{d}{dt}\int_M R - Ric(\xi)\, dV_g\bigg|_{t=0} = \int_M (-R^{ki} + h^i{}_m h^{mk} + R^k{}_{rs}{}^i \xi^r \xi^s - 2h^{ik}) D_{ik}\, dV_g.$$

Thus the critical point condition is

$$(Q\phi - \phi Q) - (l\phi - \phi l) - 4\phi h = -\eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi.$$

So far we have not used the fact that we are in dimension three and hence this is the critical point condition for the intergral of the generalized Tanaka-Webster scalar curvature as defined by Tanno [22]. Now in dimension 3 we can combine this condition with (3.6) to get $h = 0$.

5. The Abbena-Thurston manifold as a critical point

In 1976 W. Thurston [24] gave an example of a compact symplectic manifold with no Kähler structure. We will begin by discussing this manifold briefly and then turn to a natural Riemannian metric on this manifold introduced by E. Abbena [1]. For details of the topological obstructions to a Kähler structure we refer to [24] or [1] and simply remark here that the first Betti number of this manifold is 3 whereas the odd-dimensional Betti numbers of a compact Kähler are even.

Let G be the closed connected subgroup of $GL(4, \mathbf{C})$ defined by

$$\left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i a} \end{pmatrix} \middle|\, a_{12}, a_{13}, a_{23}, a \in \mathbf{R} \right\},$$