for some constant c and hence that g is Einstein.

In [15] Y. Muto computed the second derivative of $A(g)$ at a critical point and showed that the index of $A(g)$ and the index of $-A(g)$ are both positive.

Y. Muto also considered the second derivative of $D(g)$ from the following point of view. Let $\mathcal{D}$ denote the diffeomorphism group of M; if $f \in \mathcal{D}$, then $D(f^*g) = D(g)$ and hence we have an induced mapping $\tilde{D} : \frac{\mathcal{M}}{\mathcal{D}} \longrightarrow \mathbf{R}$. We say that a metric g is a critical point of $\tilde{D}$ if its orbit under $\mathcal{D}$ is a critical point of $\tilde{D}$. As we have noted a Riemannian metric of constant curvature is a critical point of D; in [16] Y. Muto proved the following result.

Theorem 1.3 (Muto). *If M is diffeomorphic to a sphere and g is a metric of positive constant curvature, then the index of D and the index of $\tilde{D}$ are both zero and $\tilde{D}$ has a local minimum at g.*

2. Symplectic and contact manifolds

By a *symplectic manifold* we mean a C^∞ manifold M^{2n} together with a closed 2-form Ω such that $\Omega^n \neq 0$. By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* of the contact structure η. A contact structure is said to be *regular* if every point has a neighborhood such that any integral curve of ξ passing through the neighborhood passes through only once. The celebrated Boothby-Wang Theorem [10] states that a compact regular contact manifold is a principal circle bundle over a symplectic manifold of integral class. The Hopf fibration of an odd-dimensional sphere S^{2n+1} as a principal circle over complex projective space PC^n is a very well known example.

Let us now consider the Riemannian geometry of these manifolds. For a symplectic manifold M let k be any Riemannian metric and $X_1, \ldots, X_{2n}$ be a k-orthonormal basis. Consider the $2n \times 2n$ matrix $\Omega(X_i, X_j)$; it is non-singular and hence may be written as the product GF of a positive definite symmetric matrix G and an orthogonal matrix F. G then defines a new metric g and F defines an almost complex structure J; checking the overlaps of local charts, it is easy to see that g and J are globally defined on M. The key point is that $\Omega(X, Y) = g(X, JY)$ where g and J are created simultaneously by polarization. A metric g created in this way is called an *associated metric* and the set of these metrics will be denoted by $\mathcal{A}$. In particular $\mathcal{A}$ is the set of all almost Kähler metrics on M which have Ω as their fundamental 2-form. We note also that all associated metrics have the same volume element $dV = \frac{1}{2^n n!}\Omega^n$.

In the contact case we have a two step process for constructing associated metrics. Starting with any Riemannian metric k', define a metric k by

$$k(X, Y) = k'(-X + \eta(X)\xi, -Y + \eta(Y)\xi) + \eta(X)\eta(Y).$$

k is a Riemannian metric with respect to which η is the covariant form of ξ. Polarizing $d\eta$ on the contact subbundle $\{\eta = 0\}$ using k as in the symplectic case gives an associated metric

g and a tensor field ϕ of type (1,1) such that $\phi^2 = -I + \eta \otimes \xi$. As in the symplectic case $d\eta(X,Y) = g(X, \phi Y)$. We also refer to (η, g) or $(\phi, \xi, \eta.g)$ as a *contact metric structure*. For any associated metric $dV = \frac{1}{2^n n!}\eta \wedge (d\eta)^n$.

Given a contact metric structure $(\phi, \xi, \eta.g)$ we define a tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$. h is a symmetric operator which anti-commutes with ϕ; $h\xi = 0$ and h vanishes if and only if ξ is Killing. When ξ is Killing, the contact metric is said to be *K-contact*. We also have the following useful formulas involving h on a contact metric manifold.

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.1}$$

$$\frac{1}{2}\big(R(\xi, X)\xi - \phi R(\xi, \phi X)\xi\big) = h^2 X + \phi^2 X \tag{2.2}$$

$$(\nabla_\xi h)X = \phi X - h^2 \phi X - \phi R(X, \xi)\xi \tag{2.3}$$

$$Ric(\xi) = 2n - trh^2 \tag{2.4}$$

For a general reference to these ideas see [3].

We close this introduction to symplectic and contact manifolds with an example. Let M be an (n+1)-dimensional C^∞ manifold and $\bar{\pi} : TM \longrightarrow M$ its tangent bundle. If $(x^1, \ldots, x^{n+1})$ are local coordinates on M, set $q^i = x^i \circ \bar{\pi}$; then $(q^1, \ldots, q^{n+1})$ together with the fibre coordinates $(v^1, \ldots, v^{n+1})$ form local coordinates on TM. If X is a vector field on M, its *vertical lift* X^V on TM is the vector field defined by $X^V \omega = \omega(X) \circ \bar{\pi}$ where ω is a 1-form on M, which on the left side of this equation is regarded as a function on TM. For an affine connection D on M, the *horizontal lift* X^H of X is defined by $X^H \omega = D_X \omega$. The *connection map* $K : TTM \longrightarrow TM$ is defined by

$$KX^H = 0, \;\; K(X_t^V) = X_{\bar{\pi}(t)}, \;\; t \in TM.$$

TM admits an almost complex structure J defined by

$$JX^H = X^V, \;\; JX^V = -X^H.$$

Dombrowski [12] showed that J is integrable if and only if D has vanishing curvature and torsion.

If now G is a Riemannian metric on M and D its Levi-Civita connection, we define a Riemannian metric $\bar{g}$ on TM called the *Sasaki metric*, by

$$\bar{g}(X,Y) = G(\bar{\pi}_* X, \bar{\pi}_* Y) + G(KX, KY)$$

where X and Y are vector fields on TM. Since $\bar{\pi}_* \circ J = -K$ and $K \circ J = \bar{\pi}_*$, $\bar{g}$ is Hermitian for the almost complex structure J.

On TM define a 1-form β by $\beta(X)_t = G(t, \bar{\pi}_* X)$, $t \in TM$ or equivalently by the local expression $\beta = \sum G_{ij} v^i dq^j$. Then $d\beta$ is a symplectic structure on TM and in particular $2d\beta$ is the fundamental 2-form of the almost Hermitian structure $(J, \bar{g})$. Thus TM has an almost Kähler structure which is Kählerian if and only if (M, G) is flat (Dombrowski [12], Tachibana and Okumura [21]).

Let $\mathbf{R}$ denote the curvature tensor of G, $\bar\nabla$ the Levi-Civita connection of $\bar g$ and $\bar R$ the curvature tensor of $\bar g$. Complete formulas for $\bar\nabla$ and $\bar R$ can be found in [14]; here we give just two of the four formulas describing the connection.

$$(\bar\nabla_{X^V} Y^H)_t = -\frac{1}{2}(\mathbf{R}(X,t)Y)^H \tag{2.5}$$

$$(\bar\nabla_{X^H} Y^H)_t = (D_X Y)_t^H - \frac{1}{2}(\mathbf{R}(X,Y)t)^V \tag{2.6}$$

The tangent sphere bundle $\pi : T_1M \longrightarrow M$ is the hypersurface of TM defined by $\sum G_{ij}v^iv^j = 1$. The vector field $N = v^i\frac{\partial}{\partial v^i}$ is a unit normal, as well as the position vector for a point t. The Weingarten map A of T_1M with respect to the normal N is given by $AU = -U$ for any vertical vector U and $AX = 0$ for any horizontal vector X (see e.g. [3,p.132]). Thus many computations on T_1M involving horizontal vector fields can be done directly on TM.

Let g' denote the metric on T_1M induced from $\bar g$ on TM. Define ϕ', ξ' and η' on T_1M by

$$\xi' = -JN,\ JX = \phi'X + \eta'(X)N.$$

η' is the contact form on T_1M induced from the 1-form β on TM as one can easily check. However $g'(X,\phi'Y) = 2d\eta'(X,Y)$, so strictly speaking (ϕ',ξ',η',g') is not a contact metric structure. Of course the difficulty is easily rectified and we shall take

$$\eta = \frac{1}{2}\eta',\ \xi = 2\xi',\ \phi = \phi',\ g = \frac{1}{4}g' \tag{2.7}$$

as the standard contact metric structure on T_1M. In local coordinates

$$\xi = 2v^i(\frac{\partial}{\partial x^i})^H;$$

on TM the vector field $v^i(\frac{\partial}{\partial x^i})^H$ is the so-called geodesic flow.

We can now compute $\nabla\xi$ in two ways, by equation (2.1) and by using (2.5) and (2.6). Comparing these we can determine the tensor field h for the standard contact metric structure on T_1M. For a vertical vector U at $t \in T_1M$ we have

$$hU_t = U_t - (\mathbf{R}_{KU,t}t)^V.$$

For a horizontal vector X orthogonal to ξ we have

$$hX_t = -X_t + (\mathbf{R}_{\pi_*X,t}t)^H.$$

For example, if the base manifold (M,G) is of constant curvature $+1$, the structure on T_1M is K-contact (Tashiro [23]). If the base manifold is flat then the non-zero eigenvalues of h are ±1, each with multiplicity n, and T_1M is locally $E^{n+1}\times S^n(4)$, 4 being the constant curvature of the sphere owing to the homothetic change in metric (2.7). If the base manifold is of constant curvature -1, the non-zero eigenvalues of h are ± 2, each with multiplicity n.

3. Integrals of scalar curvatures on symplectic and contact manifolds

We now want to consider a number of integral functionals defined on the set of metrics associated to a symplectic or contact structure. To begin we need to see how the set $\mathcal{A}$ of associated metrics sits in the set $\mathcal{M}$ of all Riemannian metrics with the same total volume; for a more detailed treatment see [5].

Let M be a symplectic manifold and $g_t = g + tD + O(t^2)$ be a path of metrics in $\mathcal{A}$. We will use the same letter D to denote D as a tensor field of type (1,1) and of type (0,2), $D^i{}_j = g^{ik} D_{kj}$. Now

$$g(X, JY) = \Omega(X, Y) = g_t(X, J_tY) = g(X, J_tY) + tg(X, DJ_t) + O(t^2)$$

from which

$$J = J_t + tDJ_t + O(t^2).$$

Applying J_t on the right and J of the left we have

$$J_t = J + tJD + O(t^2).$$

Squaring this yields $JDJ - D = 0$ and hence $JD + DJ = 0$. Conversely if D is a symmetric tensor field which anti-commutes with J, then $g_t = ge^{tD}$ is a path of associated metrics. We summarize this and the corresponding result in the contact case as follows (cf.[5],[6]).

Lemma 3.1. *Let M be a symplectic or contact manifold and $g \in \mathcal{A}$. A symmetric tensor field D is tangent to a path in $\mathcal{A}$ at g if and only if*

$$DJ + JD = 0 \tag{3.1}$$

in the symplectic case and

$$D\xi = 0, \ D\phi + \phi D = 0 \tag{3.2}$$

in the contact case.

Similar to the role played by Lemma 1.1 in critical point problems on $\mathcal{M}$, we have the following lemma for critical point problems on $\mathcal{A}$.

Lemma 3.2. *Let T be a second order symmetric tensor field on M. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ for all symmetric tensor fields D satifying (3.1) in the symplectic case and (3.2) in the contact case if and only if $TJ = JT$ in the symplectic case and $\phi T - T\phi = \eta \otimes \phi T\xi - (\eta \circ T\phi) \otimes \xi$ in the contact case (i.e. ϕ and T commute when restricted to the contact subbundle).*

Proof. We give the proof in the symplectic case; the proof in the contact case being similar. Let $X_1, \ldots X_{2n}$ be a local J-basis defined on a neighborhood $\mathcal{U}$ (i.e. $X_1, \ldots X_{2n}$ is