1. The classical integral functionals

The study of the integral of the scalar curvature, $A(g) = \int_M R\, dV_g$, as a functional on the set $\mathcal{M}$ of all Riemannian metrics of the same total volume on a compact orientable manifold M is now classical. Moreover other functions of the curvature have been taken as integrands, most notably $B(g) = \int_M R^2\, dV_g$, $C(g) = \int_M |Ric|^2\, dV_g$, and $D(g) = \int_M |Riem|^2\, dV_g$, where Ric denotes the Ricci tensor and Riem denotes the full Riemannian curvature tensor; the critical point conditions for these have been computed by Berger [2]. A Riemannian metric g is a critical point of $A(g)$ if and only if g is an Einsten metric. Einstein metrics are critical for $B(g)$ and $C(g)$ and metrics of constant curvature and Kähler metrics of constant holomorphic curvature are critical for $D(g)$ but not necessarily conversely.

Our study in these lectures is primarily motivated by two kinds of questions. 1. Given an integral functional restricted to a smaller set of metrics, what is the critical point condition; one would expect a weaker one. The smaller sets of metrics we have in mind are the sets of metrics associated to a symplectic or contact structure. 2. Given these sets of metrics, are there other natural integrands depending on the structure as well as the curvature?

To set the stage for our study let us first prove that a Riemannian metric is critical for $A(g)$ if and only if it is Einstein. Let M be a compact orientable manifold and $\mathcal{M}$ the set of all Riemannian metrics normalized by the condition of having the same total volume, usually taken to be 1, but we don't insist on the particular value in a given problem. We begin with the following lemma.

Lemma 1.1. *Let T be a second order symmetric tensor field on M. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ for all symmetric tensor fields D satifying $\int_M D^i_i\, dV_g = 0$ if and only if $T = cg$ for some constant c.*

Proof. Let X, Y be an orthonormal pair of vector fields on a neighborhood $\mathcal{U}$ on M and f a C^∞ function with compact support in $\mathcal{U}$. Regarding X and Y as part of a local orthonormal basis, define a tensor field D on M by $D(X,X) = f$ and $D(Y,Y) = -f$, with all other components equal to zero and $D \equiv 0$ outside $\mathcal{U}$. Then $\int_M (T(X,X) - T(Y,Y)) f\, dV_g = 0$ for any C^∞ function with compact support and hence $T(X,X) = T(Y,Y)$ for every orthonormal pair X, Y. Therefore $T = cg$ for some function c and it remains to show that c is a constant. To see this let X be any vector field and $D = \mathcal{L}_X g$, where $\mathcal{L}$ denotes Lie differentiation (i.e. D is tangent to the orbit of g under the diffeomorphism group). Then since the integral of a divergence vanishes,

$$0 = \int_M T^{ij}(\nabla_i X_j + \nabla_j X_i)\, dV_g = -2 \int_M (\nabla_i T^{ij}) X_j\, dV_g,$$

but X is arbitrary so that $\nabla_i T^{ij} = 0$ from which we see that c must be a constant. The converse is immediate.

Now the approach to these critical point problems is to differentiate the functional in question along a path of metrics. So let $g(t)$ be a path of metrics in $\mathcal{M}$ and

$$D_{ij} = \left.\frac{\partial g_{ij}}{\partial t}\right|_{t=0}$$

its tangent vector at $g = g(0)$. We define two other tensor fields by

$$D_{ji}{}^h = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji})$$

$$D_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h$$

where ∇ denotes the Riemannian connection of $g(0)$ and we note that

$$D_{kji}{}^h = \left.\frac{\partial R_{kji}{}^h}{\partial t}\right|_{t=0}$$

where $R_{kji}{}^h$ denotes the curvature tensor of $g(t)$.

Theorem 1.2. *Let M be a compact orientable C^∞ manifold and $\mathcal{M}$ the set of all Riemannian metrics on M with unit volume. Then $g \in \mathcal{M}$ is a critical point of $A(g) = \int_M R\, dV_g$ if and only if g is Einstein.*

Proof. The proof is to compute $\frac{dA}{dt}$ at $t = 0$ for a path $g(t)$ in $\mathcal{M}$. First note that from $g_{ij}g^{jk} = \delta_i^k$,

$$\left.\frac{\partial g^{ij}}{\partial t}\right|_{t=0} = -D^{ij}.$$

Differentiation of the volume element gives

$$\frac{d}{dt} dV_g = \frac{d}{dt}\sqrt{det(g(t))}dx^1 \wedge \cdots \wedge dx^n = \frac{1}{2det(g(t))}\left(\frac{d}{dt}det(g(t))\right)dV_g$$

$$= \frac{1}{2}g^{ij}\left(\frac{d}{dt}g_{ij}\right)dV_g = \frac{1}{2}D_i^i dV_g.$$

Now

$$\left.\frac{dA}{dt}\right|_{t=0} = \left.\frac{d}{dt}\int_M R_{kji}{}^k g^{ji}\, dV_g\right|_{t=0}$$

$$= \int_M (D_{kji}{}^k g^{ji} - R_{ji}D^{ji} + \frac{1}{2}Rg^{ji}D_{ji})\, dV_g$$

$$= \int_M (-R^{ji} + \frac{1}{2}Rg^{ji})D_{ji}\, dV_g$$

since the integral of a divergence vanishes. On the other hand differentiation of $\int_M dV_g = 1$ gives $\int_M D_i^i\, dV_g = 0$. Thus setting $\frac{dA}{dt}\big|_{t=0} = 0$ and applying the lemma, we have

$$R_{ji} - \frac{1}{2}Rg_{ji} = cg_{ji}$$

for some constant c and hence that g is Einstein.

In [15] Y. Muto computed the second derivative of $A(g)$ at a critical point and showed that the index of $A(g)$ and the index of $-A(g)$ are both positive.

Y. Muto also considered the second derivative of $D(g)$ from the following point of view. Let $\mathcal{D}$ denote the diffeomorphism group of M; if $f \in \mathcal{D}$, then $D(f^*g) = D(g)$ and hence we have an induced mapping $\tilde{D} : \frac{\mathcal{M}}{\mathcal{D}} \longrightarrow \mathbf{R}$. We say that a metric g is a critical point of $\tilde{D}$ if its orbit under $\mathcal{D}$ is a critical point of $\tilde{D}$. As we have noted a Riemannian metric of constant curvature is a critical point of D; in [16] Y. Muto proved the following result.

Theorem 1.3 (Muto). *If M is diffeomorphic to a sphere and g is a metric of positive constant curvature, then the index of D and the index of $\tilde{D}$ are both zero and $\tilde{D}$ has a local minimum at g.*

2. Symplectic and contact manifolds

By a *symplectic manifold* we mean a C^∞ manifold M^{2n} together with a closed 2-form Ω such that $\Omega^n \neq 0$. By a *contact manifold* we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$ called the *characteristic vector field* of the contact structure η. A contact structure is said to be *regular* if every point has a neighborhood such that any integral curve of ξ passing through the neighborhood passes through only once. The celebrated Boothby-Wang Theorem [10] states that a compact regular contact manifold is a principal circle bundle over a symplectic manifold of integral class. The Hopf fibration of an odd-dimensional sphere S^{2n+1} as a principal circle over complex projective space PC^n is a very well known example.

Let us now consider the Riemannian geometry of these manifolds. For a symplectic manifold M let k be any Riemannian metric and $X_1, \dots, X_{2n}$ be a k-orthonormal basis. Consider the $2n \times 2n$ matrix $\Omega(X_i, X_j)$; it is non-singular and hence may be written as the product GF of a positive definite symmetric matrix G and an orthogonal matrix F. G then defines a new metric g and F defines an almost complex structure J; checking the overlaps of local charts, it is easy to see that g and J are globally defined on M. The key point is that $\Omega(X, Y) = g(X, JY)$ where g and J are created simultaneously by polarization. A metric g created in this way is called an *associated metric* and the set of these metrics will be denoted by $\mathcal{A}$. In particular $\mathcal{A}$ is the set of all almost Kähler metrics on M which have Ω as their fundamental 2-form. We note also that all associated metrics have the same volume element $dV = \frac{1}{2^n n!}\Omega^n$.

In the contact case we have a two step process for constructing associated metrics. Starting with any Riemannian metric k', define a metric k by

$$k(X, Y) = k'(-X + \eta(X)\xi, -Y + \eta(Y)\xi) + \eta(X)\eta(Y).$$

k is a Riemannian metric with respect to which η is the covariant form of ξ. Polarizing $d\eta$ on the contact subbundle $\{\eta = 0\}$ using k as in the symplectic case gives an associated metric