Chapter 14

The Translation Planes of order q^2 that admit $SL(2, q)$.

In this final chapter, we consider the set of translation planes of order q^2 that admit $SL(2, q)$ in the translation complement and mention a classification. The theory developed from Walker's thesis who classified all such translation planes of odd order that have $GF(q)$ in their kern, and Sch—äefer dealt with the even order case. Foulser and Johnson showed that no further cases occur when the kern hypothesis is dropped.

The resulting classification, of translation planes of order q^2 admitting $SL(2, q)$, constitutes one of the most powerful tools in finite translation plane theory. As a demonstration, we show how the classification allows us to completely determine the translation planes that admit large Baer groups that generate a nonsolvable group.

We first consider the examples that arise in the classification.

14.0.3 Desarguesian Planes.

A Desarguesian plane of order q^2 may be coordinated by a field $F \simeq GF(q^2)$ and admits $\Gamma L(2, q^2)$ in the translation complement where the p-elements are elations where $p^r = q$. In particular, there is a regulus net R which is left invariant by a subgroup isomorphic to $GL(2, q)$.

14.0.4 Hall Planes.

If the net R is derived, the group $GL(2,q)$ is inherited as a collineation group of the derived plane. Hence, the Hall planes admit $GL(2,q)$ where the p-elements are Baer p-collineations.

14.0.5 Hering and Ott-Schaeffer Planes.

The reader is referred to Lüneburg [31] for details.

Definition 14.0.22 *Let $\mathcal{Q}$ be any set of $q+1$ points in $PG(3,q)$ such that no four of the points are coplanar. Then $\mathcal{Q}$ is called a $(q+1)$-arc.*

The $(q+1)$-arcs are all determined as follows:

Theorem 14.0.23 *Let $\mathcal{Q}$ be a $(q+1)$-arc then $\mathcal{Q}$ may be represented as follows:*

(1) (Segre [38]) If q is odd then the representation is $\{(s^3, s^2t, st^2, t^3); s,t$ in $GF(q)$, $(s,t) \neq (0,0)\}$. Even if q is even, if an arc has this representation, we call this a 'twisted cubic' $\mathcal{Q}^3$.

(2) (Casse and Glynn [8]) If q is even then the representation is $\mathcal{Q}^\alpha = \{s^{\alpha+1}, s^\alpha t, st^\alpha, t^{\alpha+1}); s,t$ in $GF(q), (s,t) \neq (0,0)\}$ where α is an automorphism of $GF(q)$ which is a generator.

Theorem 14.0.24 *Let V_4 denote a 4-dimensional vector space over $K \simeq GF(q)$. Consider the following matrix group:*

$$S^\beta = \left\langle \begin{bmatrix} a^{\beta+1} & ba^\beta & ab^\beta & b^{\beta+1} \\ ca^\beta & da^\beta & cd^\beta & db^\beta \\ ac^\beta & bc^\beta & ad^\beta & bd^\beta \\ cc^\beta & dc^\beta & cd^\beta & d^{\beta+1} \end{bmatrix} ; a,b,c,d \in K \text{ and } ad - bc \neq 0 \right\rangle .$$

(1) If q is not 3^r or 2 and $\beta = 2$ then $S^{\beta=2}$ is isomorphic to $GL(2,q)$ and acts triply transitive on the points of the twisted cubic $\mathcal{Q}^3$. Furthermore, S^2 acts irreducibly on V_4.

(2) If $q = 2^r$ and β is an automorphism α of K then $S^{\beta=\alpha}$ is isomorphic to $GF(2,q)$ and acts triply transitive on the points of the $(q+1)-$ arc, $\mathcal{Q}^\alpha$. Furthermore, S^α acts irreducibly on V_4.

Theorem 14.0.25 *Let Σ be $PG(3,q)$ and consider the plane $x_4 = 0$ where the points are given homogeneously by (x_1, x_2, x_3, x_4) for x_i in $GF(q)$, $i = 1,2,3,4$.*

(1) Then $x_1x_3 = x_2^\beta$ for $\beta \in \{2, \alpha\}$ defines an oval cone C_β with vertex $(0,0,0,1)$ and oval $\mathcal{O}_\beta = \{(1,t,t^\beta,0),(0,0,1,0); t \in GF(q)\}$ in $x_4 = 0$.

(2) The $(q+1)$-arc $\mathcal{Q}^\beta = \{(1,t,t^\beta,t^{\beta+1}),(0,0,0,1); t \in GF(q)\}$ is contained in C_β and the q lines $L_t = \left\langle(0,0,0,1),(1,t,t^\beta,t^{\beta+1})\right\rangle$ intersect $\mathcal{O}_\beta$ in $(1,t,t^\beta,0)$. Hence, there is a unique line $L_\infty = \langle(0,0,0,1),(0,0,1,0)\rangle$ of the oval cone which does not contain a point of $\mathcal{Q}_\beta$.

We shall call L_∞ the 'tangent' line to $(0,0,0,1)$. More generally, any image of L_∞ under an element of the group S^β is called the tangent line at the corresponding image point.

(3) Consider the plane $x_1 = 0$ which intersects $\mathcal{Q}^\beta$ in exactly the point $(0,0,0,1)$. We shall call $x_1 = 0$ the 'osculating' plane at $(0,0,0,1)$. Each image of $x_1 = 0$ under an element of S^β is also called an osculating plane and the corresponding image point.

Theorem 14.0.26 *If $\mathcal{Q}^\beta$ is a twisted cubic then the set of $q+1$-tangents form a partial spread $\mathcal{T}$.*

Theorem 14.0.27 *Assume q is even and $\beta = \alpha$ for some automorphism of $GF(q)$. Let S_2 denote a Sylow 2-subgroup of S^α.*

(1) Then S_2 fixes a unique point P of $\mathcal{Q}^\alpha$ and fixes the tangent plane $T(P)$.

(2) Choose any point Q of $\mathcal{Q}^\alpha - \{P\}$ and form the lines XQ and then the intersection points $I = T(P) \cap XQ$ and then the lines PI of $T(P)$ incident with P. Let $N_i(P)$ denote the two remaining lines of $T(P)$ incident with P for $i = 1,2$.

Then $\mathcal{R}_i = N_i(P)S^\alpha$ is a regulus and $\mathcal{R}_j$ is the opposite regulus to $\mathcal{R}_i$ for $i \neq j$.

To construct the Hering and Ott-Schaeffer planes we require that $q \equiv -1 \pmod 3$.

Theorem 14.0.28 *When $q \equiv -1 \pmod 3$ any element ρ of order 3 in S^β fixes a 2-dimensional subspace M pointwise.*

(1) There is a unique Maschke complement L for ρ such that $V_4 = L \oplus M$.

(2) If $\beta = 2$ and q is odd then $\mathcal{T} \cup LS^2 \cup MS^2$ is the unique S-invariant spread of V_4.

The corresponding translation plane is called the 'Hering plane'of order q^2.

(3) If $\beta = \alpha$ *and* q *is even then* $\mathcal{R}_i \cup LS^\alpha \cup MS^\alpha$ *is a* S*-invariant spread of* V_4 *for* $i = 1$ *or* 2 *and for any automorphism* α *of* $GF(q)$.

The corresponding translation planes are called the 'Ott-Schaeffer planes'.

Remark 14.0.29 *(1) The Hering and Ott-Schaeffer planes admit affine homologies of order* 3 *with* $q(q-1)$ *distinct axes.*

(2) Schaeffer determine the planes when α *is the Frobenius automorphism and Ott generalized this to arbitrary automorphisms. (See Hering [17], Schaeffer [37] and Ott [33].)*

(3) Each Ott-Schaeffer plane is derivable. If α *is an automorphism for a given Ott-Schaeffer plane then* α^{-1} *is the automorhpism for its corresponding derived plane. (See e.g. Johnson [27]. If* $q = 2^r$ *it turns out that the number of mutually non-isomorphic planes is* $\varphi(r)$ *as the automophisms used in the construction are generators of the cyclic group of order* r.

14.0.6 The Three Walker Planes of order 25.

Let

$$\tau_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ 3s^2 & s & 1 & 0 \\ s^3 & 3s^2 & s & 1 \end{bmatrix}; s \in GF(5)$$

and

$$\rho = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then $\langle \tau_s, \rho \rangle = S \simeq SL(2,5)$.

Furthermore, let

$$H = \left\langle \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1} \end{bmatrix}; t \in GF(5) - \{0\}. \right\rangle$$

Then, there are exactly three mutually nonisomorphic spreads π_2, π_4, π_6 of order 25 that admit S such that H fixes exactly 6 components of each plane and ρ fixes either 2, 4, or 6 of these components respectively. These planes are determined by Walker in [41].

14.0.7 The Translation Planes with Spreads in $PG(3,q)$ admitting $SL(2,q)$.

The translation planes or order q^2 with kernels containing $GF(q)$ and admitting $SL(2,q)$ as a collineation group are completely determined by Walker and Schaeffer.

Theorem 14.0.30 *Let π be a translation plane of order q^2 with spread in $PG(3,q)$ that admits $SL(2,q)$ as a collineation group.*

Then π is one of the following types of planes:

(1) Desarguesian,
(2) Hall,
(3) Hering and q is odd
(4) Ott-Schaeffer and q is even
(5) one of three planes of order 25 of Walker.

14.0.8 Arbitrary Dimension.

There are exactly three semifields planes of order 16 one each with kernel $GF(2), GF(4)$ and $GF(16)$ each of which is derivable. We have considered the planes derived from the semifields planes with kernel $GF(4)$ that admit $PSL(2,7)$ as a collineation group. The semifield plane with kernel $GF(2)$ derives the Dempwolff plane of order 16 which admits $SL(2,4)$ as a collineation group. Furthermore, the kernel of the Dempwolff plane is $GF(2)$ (see e.g. Johnson [26]).

Using methods of combinatorial group theory and linear algebra, Foulser and I were able to prove that the only translation plane of order q^2 that admits $SL(2,q)$ as a collineation group and whose spread is not in $PG(2,q)$ is, in fact, the Dempwolff planes.

Theorem 14.0.31 *(Foulser-Johnson [13]). Let π be a translation plane of order q^2 that admits a collineation group isomorphic to $SL(2,q)$ in its translation complement.*

Then either the plane has its spread in $PG(3,q)$ or is the Dempwolff plane of order 16.

Actually, the way that the proof was given, it was not necessarily to assume that $SL(2,q)$ acts faithfully on the translation plane. That is, it is possible that $PSL(2,q)$ acts on the plane. In fact, this essentially never occurs.

Corollary 14.0.32 *Let π be a translation plane of order q^2 that admits a collineation group isomorphic to $PSL(2,q)$ then π is Desarguesian.*

14.0.9 Applications.

Let π be a translation plane of odd order p^r that admits at least two Baer p-groups B_1 and B_2 in the translation complement with distinct Baer axes. Assume that $\mid B_i \mid > \sqrt{p^r} \geq 3$. Then, by Foulser's work (which works in the characteristic 3 case in this situation), it follows that the Baer axes lie in the same net of degree $p^r + 1$. The Baer groups generate a group G isomorphic to $SL(2,p^s)$ for $p^s > p^{r/2}$. From here, it follows that the group G must be $SL(2,q)$. Applying the previous theorem, we have:

Theorem 14.0.33 *(Jha and Johnson [23]) Let π be a translation plane of odd order p^r that admits at least two Baer p-groups of order $> \sqrt{p^r} \geq 3$. Then π is the Hall plane of order p^r.*

Recall, that Foulser's result is not necessarily valid in translation planes of even order but there is considerable incompatibility between elation and Baer 2-groups.

Dempwolff analyzed the groups generated by two Baer 2-groups with distinct axes and orders $\sqrt{2^r}$ if the translation plane is of order 2^{2r}.

Theorem 14.0.34 *(Dempwolff [9]) Let π be a translation plane of even order q^2 and let G be a collineation group in the translation complement which contains at least two Baer 2-groups of orders $> \sqrt{q}$ with distinct axes. Let N denote the subgroup of G generated by affine elations.*

Then one of the following situations occur:

(1) $q^2 = 16, G \simeq SL(3,2)$ and π is either the Lorimer-Rahilly or Johnson-Walker plane, or

(2) $G/N \simeq SL(2,2^z)$ where $2^z > \sqrt{q}$ and $N \subseteq Z(G)$.

Using the incompatibility results previous mentioned, we know that any elation group centralizing a Baer 2-group can have order ≤ 2. If, in fact, the order is 1 then we argue that, in fact, we obtain $SL(2,q)$ so that the results of Foulser and myself apply. If the order of is 2 then some group representation theory shows that $G \simeq SL(2,2^z) \oplus N$ and we argue that $SL(2,2^z)$ contains a Baer group of order $> \sqrt{q}$ which again shows that $SL(2,q)$ is a collineation group. We note that the Dempwolff plane of order 16 does not occur here since there are no large Baer 2-groups in this plane.

Hence, we may show:

Theorem 14.0.35 *(Jha and Johnson [24]) Let π be a translation plane of even order q^2 that admits at least two Baer groups with distinct axes and orders $> \sqrt{q}$ in the translation complement.*

Then, either π is Lorimer-Rahilly or Johnson-Walker of order 16 or π is a Hall plane.