Chapter 11

Infinite Baer Nets.

In this chapter, we analyze the structure of a net embedded in a translation plane which contains at least one Baer subplane. Actually, it is not necessary that the translation plane be finite. In fact, we may analyze any vector space net containing a weaker version of subplane than Baer.

If a net contains a Baer subplane, it may contain exactly one. Or there may be exactly two Baer subplanes in the given net such that the subplanes share all of their parallel classes. In these lectures, we concentrate mainly on the case where there are at least three Baer subplanes sharing an affine point (the zero vector) and all of their infinite points (parallel classes).

11.1 Point-Baer And Line Baer Subplanes.

In any finite projective plane π of order n, a Baer subplane π_0 is just a subplane of order $\sqrt{n}$. Hence, to extend the notion of a Baer subplane usefully to the infinite case, it becomes necessary to replace the order-property of a Baer subplane by a characterization that can be used to define this concept in the infinite case. This lecture reviews some of the possible ways in which this has been attempted and also introduces a structure theorem of nets containing at least three Baer subplanes due to Johnson and Ostrom. This will be used in the next two lectures to extend the comprhensive characterization of such nets in the finite case, due to Foulser, to the infinite case.

A point-Baer subplane of a projective plane is a subplane such that every point of the plane is incident with a line of the subplane. Similarly, a line-Baer subplane is a subplane such that every line is incident with a point of

1. $\lambda \circ (f + g) = \lambda \circ f + \lambda \circ g$;
2. $(\lambda \circ f) \circ g = \lambda(\circ f \circ g)$.

However, since λ was chosen arbitrarily, and the above identities obviously apply even when λ is replaced by members of F, we conclude from the above (plus the quasifield distributive law):

Lemma 10.5.8 *F is a field and Q is a vector space over F [acting from the left] of dimension $N > 2$. Moreover G is a linear group of this vector space.*

Now view Q as the projective space $PG(N-1, q)$ and observe that the projective group G has two point orbits. Hence by an important result, G also has two hyperplane orbits, one of which must be all the hyperplanes through the 'point' F. The other hyperplane orbit must therefore include all the hyperplanes 'off' a point: this is the same number as the number of points off a hyperplane, viz., q^{N-1}. Thus we have shown

Lemma 10.5.9 *If $N > 2$ then G contains a p-group of order q^{N-1}, p being the characteristic of F.*

But now we have seen that this is impossible, unless $q = 2$ and $N = 4$, corresponding to the case when $F = GF(2)$. It can be shown however, that even in this case $AutQ$ contains *another* subgroup H that H fixes a Baer subfield K elementwise and acts transitively on $Q - K$, so in a technical sense we still have a generalized Hall plane. However, the first choice of F is also possible: corresponding to the Lorimer-Rahilly plane of order 16, and this is the only known finite plane which is tangentially transitive relative to a non-Baer subplane. Let us summarize our conclusions:

Theorem 10.5.10 *A finite translation plane π is tangentially transitive relative to a subplane π_0 iff π is a generalized Hall plane and π_0 is a Desarguesian Baer subplane (defining a derivable net) unless the order of the plane is 16 in which case π_0 may taken as a plane of order to when π is the Lorimer-Rahilly plane of order 16: and this is the only case where the non-Baer possibility can occur.*

Note that we have not verified here the claimed uniqueness of the Lorimer-Rahilly plane, although this has been established in the literature, see Walker [40]

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If a net contains a Baer subplane, it may contain exactly one. Or there may be exactly two Baer subplanes in the given net such that the subplanes share all of their parallel classes. In these lectures, we concentrate mainly on the case where there are at least three Baer subplanes sharing an affine point (the zero vector) and all of their infinite points (parallel classes).

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A point-Baer subplane of a projective plane is a subplane such that every point of the plane is incident with a line of the subplane. Similarly, a line-Baer subplane is a subplane such that every line is incident with a point of the subplane. Every finite point-Baer subplane is line-Baer and conversely. However, in the infinite case, the concepts of point-Baer and line-Baer are independent (Barlotti [3]). So, a subplane is Baer if and only if it is both point-Baer and line-Baer. An affine point-Baer subplane is an affine plane which is point-Baer when the plane is considered projectively. A collineation σ of an affine plane which fixes a point-Baer subplane pointwise is said to be a point-Baer perspectivity if and only if the collineation fixes each subplane of a set C of point-Baer subplanes which form a cover of the points of the affine plane. The collineation σ is a point-Baer elation if and only if $Fix\sigma$ is in C. Otherwise, σ is a point-Baer homology. C is called the center of the collineation, the elements of C are called the central planes and $Fix\sigma$ is the axis.

If a collineation fixes a point-Baer subplane pointwise then, conceivably, it is not a point-Baer perspectivity. However, the structure of point-Baer collineations is essentially completely determined for translation planes. An axial-Baer perspectivity σ is a point-Baer perspectivity such that $Fix\sigma$ projectively nontrivially intersects each point-Baer subplane of the center.

The authors have recently provided a general structure theory for point-Baer and line-Baer perspectivities. In particular, the following result is fundamental.

Theorem 11.1.1 (Jha, Johnson [22].) *Let π be a translation plane and let σ be a collineation which fixes a point-Baer subplane pointwise.*

Then σ is either a point-Baer homology (and hence an axial-Baer homology) or σ is an axial-Baer elation and in this case all the planes of the center are proper Baer subplanes. In particular, in all cases, the axis $Fix\sigma$ is a proper Baer subplane of π and σ has a unique center.

Let N be a vector space net which admits at least three distinct point-Baer subplanes that share the same infinite points and mutually share an affine point. Assume that N has exactly these same infinite points.

In [10], Foulser completely determined the structure of N, when the planes are finite. In the section following this preliminary part, we show that this theory can also be determined in the more general situation when N is possibly infinite. When we can, we follow the general outlines of Foulser's argument. However, there are some situations which require different approaches so we will require a slightly different method paying particular attention to commutativity properties.

We have mentioned the notation of a direct product of affine planes. We shall require the following results of Johnson and Ostrom [28].

Theorem 11.1.2 *(Johnson-Ostrom [28, (4.20) and (5.1)].) Let M be an Abelian translation net. If M contains three distinct point-Baer subplanes incident with a point whose infinite points are the infinite points of M then M is a regular direct product net and each pair of the planes are isomorphic.*

Furthermore, M is then a vector space net over a field L and the point-Baer subplanes may be considered L-subspaces.

If one of the subplanes π_o has kernel K_o and M is isomorphic to $\pi_o \times \pi_o$ then M is a K_o-vector space net.

At least three of the point-Baer subplanes of the net which share an affine point and all of their parallel classes are K_o-subspaces but not all point-Baer subplanes are necessarily K- subspaces.

We point out that in (4.20) of [28], it is proved that L may be taken as the prime field of any of the affine planes.

In the following result, we specialize to the situation we are discussing.

Theorem 11.1.3 (Johnson-Ostrom (5.2) [28].) *Let M be a vector space net over a skewfield K where M is a regular direct product net of two isomorphic point-Baer subplanes with kernel K_o.*

Then M admits $\Gamma \cong GL(2, K_o)$ as a collineation group that fixes an affine point and fixes each parallel class.

Furthermore, Γ is generated by the groups which fix point-Baer subplanes pointwise.

If $M = \pi_o \times \pi_o$ and K_o is the kernel of π_o as a left K_o-subspace then the action of an invertible element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

on M is $(p_o, p_1) \to (ap_o + cp_1, bp_o + dp)$ for a, b, c, d in K_o and p_o, p_1 points of π_o.

Exercise 11.1.4 *Suppose M is a regular direct product net of two Desarguesian affine planes of order $q = p^r$. Using the above theorem, show there is a group isomorphic to $GL(2, q)$ acting on the net M.*

11.2 Regular Direct Products.

In this lecture, we consider a coordinate set for a regular direct product. However, before doing this we need to consider in some detail the meaning of the linear group $GL(2,K)$ when K is a non-commutative skewfield.

When K_o is a skewfield which is not a field, there are some important differences between the commutative and noncommutative case with the consideration of the group $GL(2,K_o)$. Actually, the use of the notation is a bit problematic as the elements are not necessarily K-linear mappings in the traditional sense.

Consider a Desarguesian affine plane (x,y) considered as a 2-dimensional left vector space over a skewfield $(K,+,\cdot)$. Since we may also consider the affine plane as a 2-dimensional right space over K, we take components to have the form $y = x\alpha$ for α in K and $x = 0$ and note that $y = x\alpha$ and $x = 0$ are 1-dimensional left K-subspaces. We may consider the mappings called the kernel mappings

$$T_\beta : (x,y) \to (\beta x, \beta y).$$

It follows easily that $\{T_\beta \mid \beta \in K\}$, forms a field isomorphic to $K \equiv (K,+,\cdot)$ and fixes each component of the Desarguesian plane.

Now consider the mappings $(x,y) \to (xa + yb, xc + yd)$ such that the corresponding determinant $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defined as $ac^{-1}d - b \neq 0$ if $c \neq 0$ and $ad \neq 0$ otherwise. Then it follows easily that each mapping is a $\{T_\beta\}$-linear mapping. Hence, we may justify the designation $GL(2,K)$.

Traditionally, the kernel of a translation plane is the set of endomorphisms which leave each component of the plane invariant. Hence, $\{T_\beta \mid \beta \in K\} = K^\circ$ is the kernel of the Desarguesian plane π. Furthermore, the full collineation group of π which fixes the zero vector (the translation complement) is $\Gamma L(2,K)$ or is $\Gamma L(2,K^\circ)$. Since the use of K or $K^\circ \cong K$ is merely in the distinction between K and the associated kernel mappings, we also refer to K as the kernel of the plane. So, considering the translation complement of π as $\Gamma L(2,K)$ then $K^{\circ^*} = K^\circ - \{T_o\}$ is a group of semilinear mappings and as a collineation T_β is in $\Gamma L(2,K)$ but is in $GL(2,K)$ if and only if β is in $Z(K_o)$. That is to say that the elements of $GL(2,K)$ are the elements of $\Gamma L(2,K)$ which commute with K° and T_β commutes with K° if and only if β is in $Z(K)$. The notation can be particularly tricky if one considers βx as a linear mapping over the prime field P of K. For example, βx is normally written $x\beta$, when considering β as a P-linear endomorphism. Then considering an element u in K as a P-linear endomorphism, it follows then that $\beta u = u\beta$ when considering the elements as linear endomorphisms whereas it is not necessarily the case that $\beta u = u\beta$ when the operation juxtaposition is considered as skewfield multiplication.

Note that elements of $GL(2, K)$ act on the elements (x, y) on the right whereas T_β acts on the elements (x, y) on the left.

In the statement of the above theorem, and we have π_o with kernel K_o, we have $GL(2, K_o)$ acting on the left side on a subnet instead of the right. If it occurs that a subgroup R of this group acts as a collineation of a translation plane Σ with kernel K containing π_o , then R is a subgroup of $\Gamma L(\Sigma, K)$. We now consider this situation.

We shall consider an affine translation plane with kernel K as follows: Let X be a left K-subspace and form $V = X \oplus X$. We denote points by (x, y) for x, y in X.

When we have an affine translation plane Σ with kernel K, we similarly consider the lines through $(0,0)$ (components) in the form $x = 0, y = xM$ where M is a K-linear transformation. The kernel K then gives rise to a set of kernel mappings

$$\{T_\beta : (x, y) \to (\beta x, \beta y) \mid \beta \in K\} = K^\circ.$$

In the finite dimensional case, we may take M as a matrix with entries in K_o say as $[a_{ij}]$ and define $xM = (x_1, x_2, ...x_n)M$ as $(\Sigma x_i a_{i1}, ..., \Sigma x_i a_{in})$. It follows that M becomes a left K-linear mapping with scalar multiplication defined by $\beta x = (\beta x_1, \beta x_2, ..., \beta x_n)$ and furthermore, $\{(x, xM)\}$ is a left K-linear subspace. In this case K° is a skewfield isomorphic to K and as a collineation group of Σ, $K^{\circ *}$ is a semilinear K-group. Similar to the Desarguesian case, one may consider the left scalar multiplication as a linear endomorphism over the prime field P of K. When we do this, we shall use the notation $K_\circ$. Hence, the $M's$ now commute with the elements of $K_\circ$.

To be clear, we now have three different uses of the term kernel of a translation plane. We always consider the translation plane as $X \oplus X$ where X is a left vector space over a skewfield K, the kernel mappings are denoted by K° and the component kernel mappings thought of as prime field endomorphism are denoted by $K_\circ$. All three skewfields are isomorphic and each is called the kernel of the translation plane where context usually determines which skewfield we are actually employing.

We now consider a coordinate set for a regular direct product net.

We point out that in the proof of (4.20) of [28], it is noted that two point-Baer subplanes that share the same infinite points and an affine point sum to the entire vector space and furthermore their intersections on any line concurrent with the common affine sum to the line.

We may identify any point-Baer subplane as π_o within the direct product so that the points of the net have the general form (p_1, p_2) for p_1 and p_2 in π_o and the lines have the form $L_1 \times L_2$ for L_1 and L_2 parallel lines of π_o. It follows that the net M is $\pi_o \times \pi_o$ with the identity mapping defined on the set of parallel classes.

Considering the translation plane π_o with kernel K_o , we specify two lines incident with the zero vector as $x_o = 0$ and $y_o = 0$. We further decompose π_o in terms of these two subspaces and write the elements of π_o as (x_o, y_o) where x_o, y_o are in a common K_o-subspace W_o . We may take $y_o = x_o$ as the equation of a line of π_o incident with the zero vector so that the remaining lines are of the general form $y_o = x_o M$ where M is a K_o-linear transformation of W_o for M in a set Π_o.

The points of the net now have the general form (x_o, y_o, x_1, y_1) where x_o, y_o, x_1, y_1 are in W_o. The lines of the net are as follows: $(y_o = x_o M + c_o) \times (y_o = x_o M) + c_1)$ for all M in Π_o containing I and O and $(x_o = c_o) \times (x_o = c_1)$.

Note change bases by the mapping $\chi : (x_o, y_o, x_1, y_1) \longmapsto (x_o, x_1, y_o, y_1)$

Finally, we write $(x_o, x_1) = x$ and $(y_o, y_1) = y$ when (x_o, y_o, x_1, y_1) is a original point of the net or (x_o, x_1, y_o, y_1) is a point after the basis change.

Note that, before the basis change χ, the lines of the net are sets of points

$$\{(x_o, x_o M + c_o, x_1, x_1 M + c_1) \text{ for all } x_o, x_1 \text{ in } W_o\} \text{ , for fixed } c_o \text{ and } c_1 \text{ in } W_o$$

and

$$\{(c_o, y_1, c_1, y_2) \text{ for all } y_1, y_2 \text{ in } W_o\}, \text{ for fixed } c_o \text{ and } c_1 \text{ in } W_o.$$

Hence, after the basis change, the lines of the net have the basic form

$$x = (c_o, c_1) \quad \text{and } y = x \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} + (c_o, c_1).$$

Before the basis change χ, the point-Baer subplanes incident with the zero vector which are in a $GL(2, K_o)$ orbit of π_o have the following form:

$\rho_\infty = \{(0,0,x_1,y_1)$ for all x_1, y_1 in $W_o\}$ and $\rho_\alpha = \{(x_o, x_1, \alpha x_o, \alpha x_1)$ for all x_o, x_1 in $W_o\}$ for each α in K_o. **We shall call these subplanes ρ_∞, or ρ_α the base subplanes.**

We now observe that the group $GL(2, K_o)$ acting on the right is represented by mappings of the form $(x_o, x_1, y_1, y_2) \longmapsto (ax_1 + bx_2, cx_1 + dx_2, ay_1 + by_2, cy_1 + dy_2)$.

11.3 Baer Nets: Structure Theory.

As we indicated earlier, Foulser has completely determined in the finite case the structure of vector space nets that admit at least three Baer subplanes that share the same slopeset. In this lecture, we extend Foulser's analysis to the infinite case.

We assume that we have a translation plane Σ with kernel K and there are at least three point-Baer subplanes as above with kernel K_o which are left invariant under the mappings K^{o*} or equivalently are K-subspaces. Then there is a regular direct product net N isomorphic to $\pi_o \times \pi_o$ embedded in Σ. The translation complement of Σ is a subgroup of $\Gamma L(\Sigma, K)$ with the elements acting on the left. Furthermore, there is group of the direct product net N which is isomorphic to $GL(2, K_o)$ and naturally embedded in $GL(4, K_o)$ with the elements acting on the right. It is easy to see that if a collineation g of Σ fixes a K-subspace π_o pointwise then g is in $GL(\Sigma, K)$ and hence commutes with the mappings T_β. Now any kernel homology group K^{o*} induces a faithful kernel group on any invariant point-Baer subplane so K may be considered a subskewfield of K_o.

We shall use the notation (∞) to denote the parallel class containing the line $x = 0$ and (0) to denote the parallel class containing the line $y = 0$. We shall use both the original direct product point notation and the notation after the basis change χ more-or-less simultaneously. After our main structure theorem, we shall use the representation after the basis change exclusively.

Lemma 11.3.1 *Let Σ be any point-Baer subplane incident with the zero vector and sharing all parallel classes with the net. Then $(0, x_o, 0, x_1)$ is in $\Sigma \cap (x = 0)$ if and only if $(x_o, 0, x_1, 0)$ is in $\Sigma \cap (y = 0)$.*

Proof: Let the infinite points of $x = 0, y = x \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ be denoted by

(∞) and (M) respectively.

Let $(x_o, 0, x_1, 0)$ be a point of $\Sigma \cap (y = 0)$. Form the line $(\infty)(x_o, 0, x_1, 0)$ $\equiv (x = (x_o, x_1))$ and intersect the line $y = x$ to obtain $(x_o,\ x_o, x_1, x_1)$. Since all such lines are lines of Σ , the intersection is a point of Σ. Now form the line of Σ , $(0)(x_o, x_o, x_1, x_1)$ and intersect $x = 0$ to obtain $(0, x_o, 0, x_1)$ in $\Sigma \cap (x = 0)$.

Exercise 11.3.2 *Is there any difference between the proof of the above lemma in the infinite case and in the finite case?*

Lemma 11.3.3 *Now assume the subplane Σ is not a base subplane.*

For $(0, x_o, 0, x_1)$ in $\Sigma \cap (x = 0)$ define a mapping λ on W_o which maps x_o to x_1.

Then λ is a $1-1$ and onto additive transformation of W_o.

Furthermore, $\Sigma = \{(x_o, y_o, \lambda x_o, \lambda y_o)$ for all x_o, y_o in $W_o\}$.

Proof: It is easy to check that no two distinct point-Baer subplanes incident with a common affine point and sharing all of their parallel classes can share two distinct affine points. Hence, $x_o = 0$ if and only if $x_1 = 0$ when $(0, x_o, 0, x_1)$ is a point of Σ and Σ is not the base subplane ρ_∞ or ρ_o.

It follows that the subplane Σ is a translation affine subplane and hence a subspace of the underlying vector space taken over at least over the prime field.

Hence, it follows that λ is $1-1$ since the intersections with any of the base subplanes contain exactly the zero vector and it is also now clear that λ is additive. It remains only to show that λ is an onto mapping.

From the above remarks, any two distinct point-Baer subplanes sharing a common affine point and their infinite points sum to the vector space and their intersections with a line incident with the common point sum to the line. Hence, given any element x_1^* of W_o consider the vector $(0, 0, 0, x_1)$ there exists vectors $(0, x_o^*, 0, 0\ )$ in $\rho_o \cap (x = 0)$ and $(0, x_o, 0, x_1)$ in $\Sigma \cap (x = 0)$ such that

$$(0, 0, 0, x_1^*) = (0, x_o^*, 0, 0) + (0, x_o, 0, x_1).$$

It follows that $x_1 = x_1^*$ so there exists a vector $(0, x_o, 0, x_1^*$ in Σ. Hence, the mapping $\lambda : x_o \rightarrow x_1$ is onto.

If $(0, x_o, 0, \lambda x_o, 0)$ is in $\Sigma \cap (x = 0)$ then $(x_o, 0, \lambda x_o, 0)$ is in $\Sigma \cap (y = 0)$ so that $(x_o, y_o, \lambda x_o, \lambda y_o)$ is in Σ for all x_o, y_o in W_o as Σ is the direct sum of any two components. Let $(x_o^*, y_o^*, x_1^*, y_1^*)$ be any point of Σ then it follows that

Σ also contains $(0, 0, \lambda x_o^* - x_1^*, \lambda y_o^* - y_1^*)$ and since $\Sigma \cap \rho_\infty = (0, 0, 0, 0)$ this forms $x_1^* = \lambda x_o^*$ and $y_1^* = \lambda y_o^*$. This completes the proof of the lemma.■

Exercise 11.3.4 *If the plane is finite, how would the above proof be able to be simplified?*

To see that it is not possible that Σ is not a base subplane, we show that, in fact, λ is in K_o.

Lemma 11.3.5 *For* $y = x \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ *a line of the net and* $(x_o, 0, x_1, 0)$ *in* Σ *then* (x_o, x_oM, x_1, x_1M) *is also in* Σ.

Proof: We have seen this previously in the preliminary section. We form $(x_o, 0, x_1, 0)(\infty) \equiv (x = (x_o, x_1))$ and intersect $y = x \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ to obtain the point (x_o, x_oM, x_1, x_1M). Since all of the points and lines are points and lines of Σ, it follows that the intersection point is also in Σ. ■

The previous lemma shows that if $(x_o, 0, x_1, 0)$ is in Σ then so is (x_o, x_oM, x_1, x_1M) which, in turn, implies that $(0, x_oM, 0, x_1M)$ is in Σ. However, also we have that $x_1 = \lambda x_o$ and we know that $(0, x_oM, 0, \lambda(x_oM))$ is in Σ. Subtracting, since Σ is additive, we have that $(0, 0, 0, (\lambda x_o)M - \lambda(x_oM))$ is in Σ for all x_o. Since $\Sigma \cap \rho_\infty = (0, 0, 0, 0)$, it follows that $(\lambda x_o)M) = \lambda(x_oM)$.

Let L_o be any skewfield such that $\{M$ for M in $\Pi_o\}$ is a set of L_o-linear transformations. Then it follows that L_o must be contained in the kernel K_o of $\pi_o = \rho_\infty$. Hence, λ is in $L_o \subseteq K_o$.

Hence, we have proved the following result:

Theorem 11.3.6 *Let M be any Abelian net which contains three point-Baer subplanes that share the same affine point and share all of their parallel classes.*

Then there is a skewfield K_o such that M is a K_o- vector space net and there is a K_o-space W_o such that the points of M may be identified with $W_o \oplus W_o \oplus W_o \oplus W_o$. The set of all point-Baer subplanes of M that share the zero vector is isomorphic to the set $\{\{(0, 0, y_o, y_1)$ *for all* (y_o, y_1) *in* $W_o \oplus W_o\} \cup_{\alpha \epsilon K_o} \{(x_o, y_o, \alpha x_o, \alpha y_o)$ *for all* (x_o, y_o) *in* $W_o \oplus W_o\}\}$.

Furthermore, there is a collineation group Γ of the net isomorphic to $GL(2, K_o)$ which fixes $(0,0,0,0)$ and all parallel classes and acts triply transitively on the set of all point-Baer subplanes incident with $(0,0,0,0)$. Moreover, if B denotes the set of all point-Baer subplanes of M and $\Gamma_{[\pi_o]}$ is the pointwise stabilizer of a subplane π_o of B then

$$\Gamma = \left\langle \Gamma_{[\pi_o]} \mid \pi_o \in B \right\rangle .$$

Exercise 11.3.7 *Restate this theorem in the finite case assuming that M is a net of degree q^2 and degree $q+1$ that contains three Baer subplanes. Let the kernel of any one of the subplanes be $GF(h)$. How many Baer subplanes are in the net?*

Corollary 11.3.8 *Let M be Abelian net which contains three point-Baer subplanes that share the same affine point A and all of their parallel classes.*

If one of the point-Baer subplanes has kernel $K_o$then the set of all point-Baer subplanes of M incident with A is isomorphic to $PGL(1, K_o)$.

Proof: We consider the above representation after the basis change χ. The group

$$\left\langle Diag \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \text{ such that } \lambda \epsilon K_o \right\rangle \cdot \left\langle Diag \begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix} \text{ such that } \beta \epsilon K_o - \{0\} \right\rangle$$

fixes $\pi_o = \rho_\infty$pointwise and acts doubly transitively on the point-Baer subplanes. Note that $DiagA = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$.

Exercise 11.3.9 *Restate the corollary in the finite case assuming that one of the subplanes has kernel $GF(h)$.*

Below, we completely determine the collineation group of a net of type in the statement of the above theorem. We first verify the following result.

Theorem 11.3.10 *Let R be any Abelian net which contains three point-Baer subplanes that share the same affine point A and all of their parallel classes. Let π_o be any point-Baer subplane incident with A. Then π_o is an affine translation plane with kernel K_o. Let G_{π_o} denote the full linear translation complement of π_o.*

Then there is a collineation group of R isomorphic to G_{π_o} which leaves π_o invariant.

Proof: We have noted that R is a regular direct-product net. The result then follows from a previous exercise.■

Theorem 11.3.11 *Let R be any Abelian net which contains three point-Baer subplanes that share the same affine point A and all of their parallel classes. Let π_o be any point-Baer subplane incident with A.*

Then π_o is an affine translation plane with kernel K_o. Let G^{π_o} denote the full linear translation complement of π_o obtained as a collineation group of R which leaves π_o invariant.

Then the full collineation group of R which fixes π_0 is isomorphic to the product of G^{π_o} by $GL(2, K_o)$. The two groups intersect in the group kernel of π_o naturally extended to a collineation group of R.

Exercise 11.3.12 *Assume that R is a finite net of order q^2 and degree $q+1$ and that the kernel of a Baer subplane is $GF(q)$. Show the net defines a regulus in $PG(3,q)$. Consider the group of the regulus net acting in $PG(3,q)$. Show there is a subgroup isomorphic to $PGL(2,q) \times PGL(2,q)$.*

Proof of the theorem: The group $GL(2, K_o)$ acts 3-transitively on the point-Baer subplanes of the net R and fixes R componentwise. Hence, we may assume that a collineation fixes the zero vector and permutes the point-Baer subplanes $\pi_\infty = \{(0,p)$ such that $p \in \pi_o\}, \pi_\lambda = \{(p, \lambda p)$ such that $p \in \pi_o$ and λ in the kernel of $\pi_o\}$ (when $\lambda = 0$ the subplane π_o is identified with $\pi_o \times 0$.

So, if a collineation g of R which fixes the zero vector then we may assume that g leaves π_∞, π_o, and π_1 invariant. Hence, g is in G_{π_o} as it acts faithfully on π_o.

Since $GL(2, K_o)$ fixes R componentwise, assume g fixes R componentwise. Then g induces the kernel mappings on π_o and on π_1 and is fixed-point-free as it also leaves π_∞ invariant. Thus, the faithful stabilizer of π_o in $GL(2, K_o)$ which fixes π_∞, π_o, and π_1 is $\left\langle \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \text{ such that } \beta \epsilon K_o \right\rangle$ in this representation. It then follows that the collineation group of R is the product as maintained.■