Chapter 10

Large Planar Groups.

The aim of this chapter is to consider large planar groups acting on translation planes, or what amounts to the same thing, to consider quasifields that admit large automorphism groups, in one sense or another. The emphasis here is strongly on the finite case. We shall describe all the finite quasifields amitting maximal automorphism groups: those admitting automorphism groups that act transitively on their non-fixed points. We also treat comprehensively the structure of a Baer group and obtain a sharp upper bound for the size of a planar p-group of a finite translation plane of characteristic p.

10.1 Planar and Automorphism Groups.

In this section we make some general remarks concerning planar collineation groups of arbitrary [affine or projective] planes, and their identification with the automorphism groups of planar ternary rings coordinatizing the planes. Our interest is in the case where the planes are translation planes, but the arguments in the general case is exactly the same. The material covered here will be taken for granted in the sequel.

Let G be a planar group acting on a plane π, and let π_G be the fixed plane of G. Now G may be identified with an automorphism group $\hat{P}$ of any planar ternary ring Q obtained when π is coordinatized with the axis chosen in π_G. Thus π_G is coordinatized by a subplanar ternary ring R of Q, and the elements $g \in G$ are of form

$$g : (x, y) \mapsto \left(x^{\hat{g}}, x^{\hat{g}}\right),$$

for some $\hat{g} \in (AutQ)_R$. So the map $g \mapsto \hat{g}$ is a faithful permutation representation of G into $(AutQ)_R$, and this representation is permutation-isomorphic to the G-representation $G \to G^{\ell}$ obtained by restricting G to its action on any line ℓ that it leaves invariant. Conversely any subgroup $J \leq (AutQ)_R$ is of form $J = \hat{G}$ for some subgroup $G \leq Aut\pi$, obviously

$$G = \left\{g : (x, y) \mapsto \left(x^{\hat{g}}, x^{\hat{g}}\right) \mid \hat{g} \in J\right\},$$

and the fixed plane of G is just $\pi(J)$.

Hence any planar group G of a plane π, with fixed plane π_G, has a faithful representation ρ in $(AutQ)_R$, where Q is a planar ternary ring obtained when π is coordinatized by choosing axes in π_0, and R is the subternary ring coordinatizing π_G. The representation ρ may be chosen so that if H is a subgroup of G then $Fix(\rho(H)) = Q_H$ is the subternary of Q such that π_H is coordinatized by Q_H and $\rho(H) = (AutQ)_{Q_H}$.

Our interest is the case *when π is an affine translation plane and G is a planar group*, fixing the line at infinity. So π_G is a subaffine translation plane of π, and π may be coordinatized by a quasifield Q such that π_G is coordinatized by a subquasifield R, and the restriction representation of G on any component that it fixes is permutation isomorphic to the standard representation of G in $(AutQ)_R$, indicated above.

However, an additional tool is available in the case of translation planes: G and all its subgroups are linear over the kern field $F = R \cap K$, where $K = kern(Q)$. For example F may always be chosen to be the prime subfield in Q. Note that the choice of F may sometimes be more general than any type of kern field. The main examples arise when Q is a left or right vector space over a subfield F, relative to the quasifield operations. Such F can occur, for example, when Q is a semifield and F is some subfield not contained in the kern, or whenever $\pi(F)$ defines a rational Desarguesian partial spread of a translation plane $\pi(Q)$. In all these cases, not only G is F-linear, but the Baer condition provides a useful constraint:
If $(Q \geq A > B$ form a chain of quasifields that are also F-spaces then $2a \leq b$, where a and b are the dimensions of A and B treated as F-spaces.
However, all this easily generalizes to arbitrary finite planar ternary rings and finite planes. But translation planes admit further constraints when G is a Baer group and π_G is any Baer subplane. Roughly, we shall show in the next section that this means that when G gets 'large' π_G is forced to be Desarguesian. This leads to a sharp upper bound for arbitrary planar p-groups acting on arbitrary finite translation planes with the same characteristic.

10.2 Baer Collineation Theory.

Let $G \leq (AutQ)_F$ be an automorphism group of a *finite* quasifield Q of order q^2 and characteristic p that fixes the Baer subquasifield F *elementwise.* We consider the structure of G, and its influence on the structure of F. Throughout the section, $B = (B_0, B_1)$ is any basis of Q relative to any *kern* field $K \subset F$ such that B_0 is a basis of F; so K can always be taken to be the prime subfield of Q. Now for each $f \in F$ its slope map T_f leaves F invariant and in fact T_f^F represents the slope map of $f \in F$, regarded as a member of the subquasifield F. Thus on any basis of type B, T_f has matrix form given by:

$$T_f = \begin{pmatrix} M_f & \mathbf{O} \\ A_f & B_f \end{pmatrix}, f \in F,$$

where M_f is the matrix of the slopemap T_f^F. Now, on the *same basis,* $g \in G$ has matrix form

$$g = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U_g & W_g \end{pmatrix}, g \in G$$

But since for $g \in G$ and $f \in F$ we have

$$(x \circ f)g = (x)g \circ (f)g = (x)g \circ f \Longrightarrow T_f g = gT_f,$$

which in matrix form may be written:

$$\forall f \in F, g \in G : T_f = \begin{pmatrix} M_f & \mathbf{O} \\ A_f & B_f \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U_g & W_g \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U_g & W_g \end{pmatrix}\begin{pmatrix} M_f & \mathbf{O} \\ A_f & B_f \end{pmatrix},$$

yielding

$$\forall f \in F, g \in G : \begin{pmatrix} M_f & \mathbf{O} \\ A_f + B_f U_g & B_f W_g \end{pmatrix} = \begin{pmatrix} M_f & \mathbf{O} \\ U_g M_f + W_g A_f & W_g B_f \end{pmatrix}. \tag{10.1}$$

Moreover, since $\{T_f \mid f \in F\}$ is a set of matrices any two distinct members of which differ by a non-singular matrix, the same applies to the B_f's and the number of these present is sufficient to form a spreadset (which clearly includes the identity), and so position $(2, 2)$ in the above matrix equation shows that W_g is in the kern of a spreadset B_f with identity, In particular:

Remark 10.2.1 *$\{W_g \mid g \in G\}$ form a multiplicative group in a field of matrices. Moreover, if $|\{W_g \mid g \in G\}| > \sqrt{|F|}$, then*

$$\{B_f \mid f \in F\}$$

is a field.

Next consider the possibility of a p-element $\rho \in G$, p being the characteristic of the quasifield. So ρ has only one eigenvalue in the algebraic closure of the prime field, *viz.* 1, since $\lambda^{p^i} = 1 \Rightarrow \lambda = 1$, so ρ must act trivially on the factor space Q/F, regarding Q and F as additive groups. Thus its matrix is of form:

$$\rho = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U_g & \mathbf{1} \end{pmatrix}$$

and by the eqn (10.1) we further have:

$$B_f U_g = U_g M_f, f \forall f \in F,$$

and since U_g intertwines two sets of irreducible matrices it must be in a field and hence non-singular. Thus we have shown:

Proposition 10.2.2 *$(AutQ)_F$ has a unique p-Sylow subgroup P, and this is elementary abelian of form:*

$$\left\{ \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U & \mathbf{1} \end{pmatrix} \mid f \in J \mid U \in J \right\} \tag{10.2}$$

where J is an additive group of matrices that is a subgroup of a field of matrices.

Moreover any $\rho \neq 1$, in the p-Sylow subgroup, can be expressed in the form where $U = \mathbf{1}$, provided the basis $B = (B_0, B_1)$ is modified to another basis $B' = (B'_0, B_1)$, without altering B_1 the basis of the *complement* F, but replacing the basis B_0 of F by a possibly different basis B'_0 of F. To see this, note that the matrix for ρ on the new basis is obtained by conjugating its given matrix by a matrix of type $Diag(C, \mathbf{1})$: thus we require non-singular C such that

$$Diag(C, \mathbf{1}) \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ U & \mathbf{1} \end{pmatrix} Diag(C^{-1}, \mathbf{1}) = \begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

and this works using $C = U^{-1}$.

Now return to the fundamental equation when B_2 is modified to ensure that the p-elements include the matrix

$$\begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

Feeding this into the fundamental equation shows that $B_f = M_f$ for all $f \in F$. Thus we have shown:

Proposition 10.2.3 *Suppose $(AutQ)_F$ includes a non-trivial p-element ρ. Then relative to a basis $B = (B_0, B_1)$, with B_0 chosen to be an arbitrary basis of F, and appropriate B_1, the following holds:*

1. *ρ has the form*
$$\begin{pmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} \end{pmatrix};$$

2. *$B_f = M_f$ for all $f \in F$;*

3. *The $\{U_g \mid g \in G\}$ forms an additive subgroup in the matrix field associated with the outer kern of $\{M_f \mid f \in F\}$.*
   In particular, if the p-Sylow subgroup in $(AutQ)_F$ has order $> \sqrt{|F|}$ then F is a field.

Now consider the group homomorphism $\nu : g \mapsto W_g$; the kernel H of ν consists of all members in G that has $W_g = \mathbf{1}$, and this we have seen is simply the unique Sylow p-subgroup of G and so the image is a p-complement.So by Maschke's theorem a p-complement of F relative to H may be chosen and on that basis H has the form $Diag(M_f, B_f)$, with all the W_f's in the kern of the spreadset $\{B_f \mid f \in F\}$. In particular W_f's form a cyclic group so G is solvable and contains a Hall p' subgroup which is cyclic, and when this group has order $> \sqrt{|F|}$ then $\{B_f \mid f \in F\}$ is a field, and as we've seen above, this means that $\{M_f \mid f \in F\}$, and hence F also is a field provided a non-trivial p-element exists in G. We may summarize this as follows, in terms of the related translation plane.

Theorem 10.2.4 *Let π be a translation plane of order q^2, q a power of the prime p. Let G be a Baer group, so its fixed plane π_G has order q. Then G divides $q(q-1)$ and satisfies the following conditions:*

1. *G is solvable with a unique elementary abelian Sylow p-subgroup P, consisting of all the p-elements in G.*

2. *The kern of π_G has an additive subgroup isomorphic to P; so π_G is Desarguesian if $P > \sqrt{q}$.*

3. *The Hall p'-subgroups of G are cyclic and isomorphic to the multiplicative subgroups of the kern of π_G.*

Further properties are developed in the exercises below, based mainly on the discussion preceeding the theorem above. These exercises are of paramount importance in the study of translation planes!

Exercise 10.2.5 *Suppose G contains non-trivial p-elements and also a non-trivial p'-group of order $> \sqrt{|F|}$.*

1. *Relative to some basis the matrices T_f are of form:*

$$\{Diag(k, k^{\sigma}) \mid where k \in K\},$$

   where K is a field of matrices and σ is a field automorphism of K.

2. *Q is a vector space over F under quasifield automorphisms, F acting from the right.*

3. *The slopes of $\pi(F)$ in $\pi(Q)$ defines a derivable net.*

4. *If a Desarguesian Baer subplane ψ of a translation plane π of order q^2 is fixed elementwise by an element u such that $\gcd(u, p) = 1$, p is the characteristic, then the slopes of ψ define a derivable net in π.*

In the next lecture we shall obtain an upper bound for planar p-groups acting on translation planes. Our arguments crucially depend on a result that we establšihed in the present lecture: *large* Baer groups G have *Desarguesian* fixed plane π_G. Since no version of this result is known that applies to planes that are not translation planes (up to duality), the results of the following section are only known to hold for translation planes.

10.3 Planar p-Groups.

In this section Q is a finite quasifield with characteristic p, admitting an automorphism group P. Let $Fix(P) := F$; so F is a subquasifield of P, and $|Q| \geq |F|^2$, or P is trivial. Assume P is linear map of Q when this is viewed as a vector space over some field K, over which Q is known to be a vector space. So we may choose $K = GF(p)$, or, more generally, K may be taken to be any field contained in $F \cap Kern(Q)$, but it will prove useful to permit yet further possibilities for F: the most important case occurs when $\pi(F)$ contains a subplane that defines a rational Desarguesian partial spread in the spread associated with Q. We shall write f to denote the dimension of F over K: thus $|F| = q^f$.

In all cases, P leaves invariant a $GF(p)$-space $A \supset P$ such that $|A| = p|F|$: regard Q as a $GF(p)$ vector space and note that the number of rank-one extensions of a subspace of any subspace of a finite characteristic p vector space is $\equiv 1 \pmod p$. Now the restriction representation $\rho : P \to P^A$ acts semiregularly on the $|A - P| = p|F| - |F|$ points of $A - F$, and let $\partial_A P$ denote the kernel of ρ. Thus $|\partial_A P|| \geq q^f$. For the fixed-quasifield of $\partial_A P$, we use he notation:

$$\partial_A F := Fix(\partial_A P),$$

and observe that the Baer condition for subplanes, when applied to non-trivial P, implies that

$$|\partial_A F| \geq q^{2f} \geq |F|^2.$$

Thus we have established:

Remark 10.3.1 *For all rank-one $GF(p)$-extensions A of F in Q:*

1. $|\partial_A P|| \geq q^f$;

2. $|\partial_A F| \geq q^{2f} \geq |F|^2$.

Note that $\partial_A P$ and $\partial_A F$ might vary with the choice of A, we shall only require the inequalities to hold; accordingly we simplify our notation by writing:

Notation 10.3.2 *If P is a non-trivial p-group in $AutQ$ with fixed subquasifield F then choose some P-invariant $GF(p)$-space $A \supset F$, where $|A|/|P| = p$ and define:*

1. $\partial P := \partial_A P$.

2. $\partial P := \partial_A P$.

3. $\partial^{k+1} P = \partial\partial^k P$ *and* $\partial^{k+1} F = \partial\partial^k F$ *whenever* $\partial^k P$ *is non-trivial.*

By repeatedly applying remark 10.3.1:

$$\begin{array}{rcl} |\partial P| & \geq & |P|/q^f \\ \text{and } |\partial F| & = & q^{2f+d_1} \exists d_1 \geq 0 \\ \text{so} & & \\ |\partial^2 P| & \geq & |P|/q^f q^{2f+d_1} \end{array}$$

$$\begin{aligned}
\text{and } |\partial^2 F| &= q^{2^2 f+2d_1+d_2} \exists d_2 \geq 0 \\
&\text{so} \\
|\partial^3 P| &\geq |P|/q^f q^{2f+d_1} q^{2^2 f+2d_1+d_2} \\
\text{and } |\partial^3 F| &= q^{2^3 f+2^2 d_1+2d_2+d_3} \exists d_3 \geq 0 \\
&\text{so} \\
|\partial^4 P| &\geq |P|/q^f q^{2f+d_1} q^{2^2 f+2d_1+d_2} q^{2^3 f+2^2 d_1+2d_2+d_3} \\
\text{and } |\partial^4 F| &= q^{2^4 f+2^3 d_1+2^2 d_2+2d_3+d_4} \exists d_4 \geq 0,
\end{aligned}$$

and in general:

$$\begin{aligned}
|\partial^{k+1} P| &\geq |P|/q^f q^{2f+d_1} q^{2^2 f+2d_1+d_2} q^{2^3 f+2^2 d_1+2d_2+d_3} \ldots q^{2^k f+2^{k-1} d_1+2^{k-2} d_2+\ldots} \\
\text{and } |\partial^{k+1} F| &= q^{2^{k+1} f+2^k d_1+2^{k-1} d_2+\ldots+d_{k+1}} \exists d_{k+1} \geq 0,
\end{aligned}$$

provided $\partial^k P$ is non-trivial. We rewrite these as:

$$\begin{aligned}
|\partial^{k+1} P| &\geq \frac{|P|}{q^{f+(2f+d_1)+(2^2 f+2d_1+d_2)+(2^3 f+2^2 d_1+2d_2+d_3)\ldots+(2^k f+2^{(k-1)} d_1+2^{(k-2)} d_2+\ldots}} \\
\text{and } |\partial^{k+1} F| &= q^{2^{k+1} f+2^k d_1+2^{k-1} d_2+\ldots+d_{k+1}} \exists d_{k+1} \geq 0,
\end{aligned}$$

and so

$$\begin{aligned}
|\partial^{k+1} P| &\geq |P|/q^{f(2^{k+1}-1)+d_1(2^k-1)+d_2(2^{k-2}-1)+d_3(2^{k-3}-1)\ldots d_k} \\
\text{and } |\partial^{k+1} F| &= q^{2^{k+1} f+2^k d_1+2^{k-1} d_2+\ldots+d_{k+1}} \exists d_{k+1} \geq 0.
\end{aligned}$$

Now choose k so that ∂^{k+1} is the trivial (after which ∂ is no longer defined. Then we have

$$|P| = q^{f(2^{k+1}-1)+d_1(2^k-1)+d_2(2^{k-2}-1)+d_3(2^{k-3}-1)\ldots d_k},$$

and

$$|Q| = q^{2^{k+1} f+2^k d_1+2^{k-1} d_2+\ldots+d_{k+1}}.$$

So

$$|P| q^{f+d_1+d_2+d_3\ldots+d_{k+1}} = q^{2^{k+1} f+2^k d_1+2^{k-1} d_2+\ldots+d_{k+1}} = |Q|,$$

so we get our main result:

Proposition 10.3.3

$$|P| = \frac{|Q|}{q^{f+d_1+d_2+d_3\ldots+d_{k+1}}}.$$

Corollary 10.3.4 *Let quasifield Q with $Kern(Q) \supset K \cong GF(q)$, so $|Q| = q^n$ for some positive integer n. Then the Sylow p-subgroups in $(AutQ)_K$ have order $\leq q^{n-1}$.*

Consider the extremal case $|P| = q^{n-1}$: so $f = 1$ and all the d_i's vanish. This means we have a strict Baer chain of quasifields

$$GF(q) = F = Q_0 \subset Q_1 \subset Q_2 \ldots \subset Q$$

such that $(AutQ_{i+1})_{Q_i}$ is divisible by $|Q_i|$, and so all the Q_i's with the possible exception of the last one, viz. Q, are fields. But fields Q_{i+1} cannot admit $|Q_i|$ automorphisms fixing the Baer subfield $|Q_i|$ unless $|Q_i| = 2$. Thus either $|Q_1| = q$, as happens in, say, the Hall planes, or $Q \supset Q_1 \supset F$ where $F = GF(2)$, $Q_1 = GF(4)$, and Q has order 4^2. Thus we have shown

Corollary 10.3.5 *If a quasifield of order q^n admits an automorphism p-group P of order q^{n-1} that fixes a kern plane of order q elementwise then either Q is two-dimensional over its kern or $|Q| = 16$.*

Specialising to $q = p$ we obtain an absolute bound for the Sylow p-subgroup of the automorphism group of a quasifield:

Corollary 10.3.6 *A quasifield of order n cannot admit an automorphism group of order n.*

Thus a translation plane of order n does not admit planar groups of order n. Actually the above corollary may be refined to the following:

Corollary 10.3.7 *A quasifield of order p^n cannot admit an automorphism p-group of order $\geq p^{n-1}$, unless $n = 2$ or $p^n = 16$.*

As already indicated both cases do occur.

10.4 Klein Groups On Odd-Order Spreads.

Every finite p-group S, p a prime, contains maximum order elementary abelian p-subgroup A, and the rank of S is defined to be r if $|A| = p^r$; thus the rank of S is the rank of the maximum $GF(p)$-subspaces that it contains. For an arbitrary finite group G, its p-rank is defined to be the rank of its Sylow p-subgroups.

In the context of translation planes the importance of p-rank stems from the fact that in certain cases there is a tendency for the p-rank of a group G acting on a spread π of order u^n to force n to be very large, *provided* $\gcd(u,p) = 1$. For Chevalley-type groups, representation theory leads to such results but are too advanced to introduce at this stage.

However, for $p = 2$, Ostrom has proved a remarkable theorem, using only very elementary ideas, that lead to similar conclusions: and these conclusions apply to *all* groups with large 2-ranks — not just to the Lie-Chevalley type of groups. Here we prove Ostrom's theorem.

We are concerned with the action of elementary abelian 2-groups A on spreads $\pi = (V, \Gamma)$ of odd order p^r, $p > 2$ an odd prime. Ostrom's theorem implies that $|A|$ divides r, thus generalising the standard result on Baer involutions. Hence the two rank of any finite group G implies information concerning the lower bound for the size of the *odd order* spreads π on which it may act.

Theorem 10.4.1 (Ostrom's Baer Trick.) *Let A be an elementary abelian 2-group in $Aut(V, \Gamma)$, where $\pi = (V, \Gamma)$ is a spread of odd order q^n, whose kern contains the field $F = GF(q)$. Suppose all the involutions in A are Baer collineations, linear over the kern field F. Then $|A|$ divides n.*

Proof: We may write $|A| = 2^R$. For $R = 1$ the result holds because n is even if π admits a Baer involution. We use induction on the exponent R to complete the proof.
Let α and β be any two distinct involutions in A, and consider the Klein group

$$K = \{\alpha, \beta, \alpha\beta, \mathbf{1}\}.$$

Since A is abelian π_α is K-invariant. Now β cannot act trivially on π_α because this would force π_α to be elementwise fixed by a Klein group, and this cannot occur in spreads of odd order.
To establish that β induces a *Baer* involution on π_α, we need to rule out the possibility that $\beta|\pi_\alpha$ is an involutory central collineation.
First consider the case the possibility that β induces on π_α a *kern* involution $\hat{\beta} = \beta|\pi_\alpha$; now clearly $\hat{\alpha} = \alpha|\pi_\beta$ is also a kern involution. Thus $\hat{\beta}$ and $\hat{\alpha}$ are both $-\mathbf{1}$, on the spaces π_β and π_α respectively. But since $V = \pi_\alpha \oplus \pi_\beta$, because the two subspaces are disjoint and of rank $n/2$, we clearly have

$$\alpha\beta = \hat{\alpha} \oplus \hat{beta} = -\mathbf{1} \oplus -\mathbf{1} = -\mathbf{1}.$$

Now the group K contains a *kern* involution of π, contrary to our hypothesis that the non-trivial elements in A are all Baer collineations.
It remains to rule out the case when β induces an *affine* homology on π_α, with axis, say, $C \in \Gamma$. Now $C_0 = C \cap \pi_\alpha$ is the fixed subspace on C common to π_α and π_β. As α and β are both F-linear involutions of the vector space C with the same fixed space C_0 (neither fixed space can be larger because we are dealing with Baer involutions) they must coincide on C,that is,

$$\alpha|C = \mathbf{1}_{C_0} \oplus -\mathbf{1}_D = \beta|C,$$

where D is any complement of C_0 in C. But now $\alpha\beta$ is a homology with axis C, contradicting again our hypothesis that A contains only Baer involutions. Thus we see that A induces on π_α a group of *Baer* involutions A_1 of order 2^{R-1}. Now by our inductive hypothesis 2^{R-1} divides the dimension $R/2$ of π_α, and the desired result follows by induction. ∎

Corollary 10.4.2 *Let π be a spread of odd order q^n containing $GF(q)$ in its kern. If π admits an automorphism group G with two-rank r then 2^{r-1} divides n.*

Proof: Let A be an elementary abelian group of G of rank n. So A is semilinear on V, the vector space associated with π, over the kern field $K = GF(q)$. Now the K-linear part of A has order $\geq |A|/2$, and Ostrom's Baer trick can be applied to it. ∎

Corollary 10.4.3 *Let π be a spread of odd order q^n containing $GF(q)$ in its kern. If π admits an elementary abelian 2-group of order 2^r and the involutions in A form a single conjugacy class in Autπ then 2^r divides n, provided $|A| > 2$.*

Proof: If A contains even one Baer involution then the conjugacy hypothesis allows us to apply the Ostrom Baer trick. So assume all the involutions in A are homologies, and consider a Klein subgroup $H \leq K$. Now Ostrom has observed that there are (in any projective plane) only two possibilities for such H: (1) all its elements share the same axis and center; or (2) each of the three non-trivial elements of H have as center and axis the opposite sides of a triangle: each of the three anti-flags of the triangle corresponding to one of the three non-trivial elements of H.

Possibility (1) cannot occur since then on the common coaxis W we the Klein group H acting semirgularly and faithfully: this is easily seen to be impossible: e.g. H becomes an elementary abelian non-cyclic Frobenius complement on W (in a Frobenius group whose kernel consists of all the maps $x \mapsto x+w$, $w \in W$, of W).
Possibilty (2) cannot occur, in the context of our conjugacy hypothesis, for then the homology whose axis is the ideal line, would be conjugate to a homology with an affine line as axis. ∎

10.5 Tangentially Transitive Planes.

Let π be any projective [resp. affine] plane, and π_0 be a proper subprojective [resp. subaffine] plane. Then a line is a *tangent* [line] to π_0 if it meets it at exactly one point. Similarly, a point is a *tangent* [point] if it meets exactly one line of π_0.

Now suppose G is a planar group with fixed plane π_G. Then it is clear that G permutes the tangents to π_G through any element of π_G, that is, G leaves invariant the set of non-fixed elements $\Theta(\epsilon)$ though each of its fixed elements $\epsilon \in \pi_G$. It is easy to see that all the restriction maps $\rho_\epsilon : G \longrightarrow G^{\Theta(\epsilon)}$, for $\epsilon \in \pi_G$, are faithful representations of G that are permutation isomorphic, and hence G is transitive on all the tangents through some fixed element of π_G iff it is transitive on the tangents through each element of π_G. When this happens we say G is tangentially transitive.

Definition 10.5.1 *Let G be a planar collineation group of a plane π with fixed plane π_G. Then G is said to be* tangentially transitive *relative to π, and π_G is called a* tangentially transitive *subplane iff G acts transitively on the tangents through some (and hence each element of π_G). π is called tangentially transitive (tt) iff it is tt relative to some proper subplane.*

The definition may easily be characterised in algebraic terms, by noting the equivalence between planar groups and automorphisms of coordinatizing ternary rings, c.f. section 10.1.

Remark 10.5.2 *Let T be a ternary ring and suppose $G \leq AutT$ is transitive on $T - Fix(G)$; so $S = F(G)$ is the subternary ring of T consisting of the* *fixed elements of* G. *Then* $\pi(T)$, *the plane coordinatized by* T, *is tangentially transitive relative to* $\pi(S)$, *with respect to the group:*

$$\hat{G} := \{(x,y) \mapsto (x^g, y^g) \mid g \in G\}.$$

Conversely, suppose π *is a plane admitting a tangentially transitive group* G *coordinatized by a ternary ring* T *when the axes are chosen in the fixed plane* π_G. *Then* π_G *is coordinatized by a subternary ring* S *and* $(AutT)_S$ *contains a subgroup* $\tilde{G}$ *such that* $\tilde{G}$ *is transitive on* $T - S$, *with* $Fix(\tilde{G}) = S$.

We saw in an earlier lecture that the Hall quasifields Q are two dimensional over their kern K, by part of their definition, and that $(AutQ)_K$ is transitive on $Q - K$, theorem 5.4.3. Hence the algebraic characterization of tt above yields

Remark 10.5.3 *A Hall plane* π *is tangentially transitive relative to some Baer subplanes* π_0 *coordinatized by the kern.*

A direct explanation of why Hall planes are tangentially transitive may be given in terms of derivation. A Hall plane H is derived from a Desarguesian plane $\Delta = \pi(F)$, the field F being a Baer extension of a field K, and Δ is derived relative to the slopes of $\pi(K)$. Part of the inherited group includes a group of central collineations with Y-axis leaving $\pi(K)$ invariant, viz:

$$G : \{(x,y) \mapsto (xa+b, y) \mid a \in K^*, b \in K\}.$$

Notice G is transitive on $\{\lambda a + b \mid a \in K^*, b \in K\}$, the set of slopes shared by the Desarguesian plane and the derived Hall plane. Thus on the derived side Y becomes a Baer subplane and G acts tangentially transitively relative to Y.

This can be generalized, by using a semifield D, two dimensional over its middle nucleus N_m, instead of a field. Now, by repeating the above argument, $\pi(D)$ when derived yields a translation plane tt relative to the Baer subplane, corresponding to the Y-axis of $\pi(D)$. Thus we have established:

Remark 10.5.4 *Let* D *be a semifield plane with middle nucelus* M, *which we assume to be a commutative field. Then* π' *the plane obtained by deriving relative to the slopeset of* $\pi(M)$ *is tangentially transitive relative to a Baer subplane. The plane* π' *is called a* GENERALISED HALL PLANE.

The procedure above can be repeated in more general contexts. Take any affine plane π of order n^2 admitting a group of central collineations G of order $n^2 - n$ that fixes an affine line Y elementwise and leaves invariant a derivable net Δ that includes Y and is left invariant by G. Then on the derived side G becomes a Baer group of order $n^2 - n$ and hence must act transitively on all the tangent points on any fixed line of π_G, its fixed Baer subplane.

This procedure permits the construction of tangentially transitive planes in several Lenz-Barlotti classes, apart from translation planes. The fact that duals of two dimensional translation planes are derivable and admit large groups of central collineations makes them promising candidates from this procedure. It is an exercise to verify that this procedure actually does work. Similarly verify that the derived Ostrom-Rosatti planes are tangentially transitive relative to some Desarguesian planes.

Notice, however, that in the constructions we have sketched so far, because they are based on derivation, the planes π are tangentially transitive relative to subplanes that are both Desarguesian and Baer. This invites the obvious questions:

If π is tt relative to π_0 then does π_0 have to be (1) Desarguesian (2) Baer.

In the finite case there is only one known case where π_0 can be *chosen* to be non-Baer — although a Baer choice is also possible in this case — in the remarkable Lorimer-Rahilly translation plane of order 16, see p 66. In all known cases, finite or infinite, π_0 is Desarguesian.

In this section we consider tangentially transitive *finite translation planes.* We show that in this case all tt planes are generalized Hall planes (including the Lorimer-Rahilly plane), and this essentially answers the two questions raised above in the affirmative. This leaves open the question of describing explicitly the generalized Hall planes, or rather, the finite semifield planes that are two-dimensional over their middle nucleus. We hope to provide a satisfactory answer to this question too. Note that the Hughes-Kleinfeld planes are coordinatized by semifields that are two-dimensional over their middle nucleus.

The rest of the section is devoted to showing that if a finite translation plane π is tangentially transitive relative to a subplane π_0 then it is a generalized Hall plane.

We begin by stating a special case of remark 10.5.2, relevant to the translation plane case.

Remark 10.5.5 *Let π be an affine translation plane and π_0 an affine sub-*

plane. Then π is tangentially transitive relative to π_0 iff it can be coordinatized by a quasifield Q such that π_0 is coordinatized by a subquasifield F such that $(AutQ)_F$ is transitive on $Q - F$.

We note that the case $|Q| = |F|^2$ has already been covered.

Lemma 10.5.6 *If $|Q| = |F|^2$ and $(AutQ)_F$ is transitive on $Q - F$ then F is a field and Q is a a vector space over F in the sense that for all $f, g \in F$ and $x, y \in Q$:*

1. $(x + y) \circ f = x \circ f + y \circ f$;
2. $x \circ (f + g) = x \circ f + x \circ g$;
3. $(x \circ f) \circ g = x \circ (f \circ g)$.

Proof: Recall exercise 2.(2). ∎
Now the condition that Q is a rank-two right vector space over F means that the slopes of $\pi(F)$ in $\pi(Q)$ define a rational (Baer) Desarguesian partial spread in $\pi(Q)$, and such partial spreads are [generic] derivable partial spreads. The derived spread admits a group of central collineations of order $n^2 - n$ where $|Q| = n^2$: the group is just the inherited group corresponding to the Baer group acting on $\pi(Q)$:

$$\{\hat{g} : (x, y) \mapsto (x^g, y^g) \mid g \in G\}.$$

Now it is an exercise to check that a spread of order n^2 admitting a Baer group of order $n(n-1)$ is a semifield spread with $GF(n)$ in N_m.

Thus we have shown:

Corollary 10.5.7 *If $|Q| = |F|^2$ then the plane $\pi(Q)$ is obtained by deriving a a plane coordinatized by a semifield relative to the slopeset of its middle nucleus. This by definition means that $\pi(Q)$ is a generalized Hall plane.*

Thus from now on we may assume that $|Q| > |F|^2$. Choose any $\lambda \in Q - F$. Then since G is transitive on $Q - F$ we see that $N_G(G_\lambda)$ induces a regular group on $Fix(G_\lambda) \cap Q - F$. However, $Fix(G_\lambda)$ is a quasifield Q_λ containing F, so we now have a quasifield $Q_\lambda \supset F$ such that $(AutQ_\lambda)_F \supset N_\lambda$ such that N_λ is regular on $(Q_\lambda)_F$. However N_λ must contain a Baer involution so the regularity is contradicted unless Q_λ is a Baer extension of F, in which case lemma 10.5.6 so F is a field and additionally the following identities apply, for $f, g \in F$:

1. $\lambda \circ (f + g) = \lambda \circ f + \lambda \circ g$;

2. $(\lambda \circ f) \circ g = \lambda(\circ f \circ g)$.

However, since λ was chosen arbitrarily, and the above identities obviously apply even when λ is replaced by members of F, we conclude from the above (plus the quasifield distributive law):

Lemma 10.5.8 *F is a field and Q is a vector space over F [acting from the left] of dimension $N > 2$. Moreover G is a linear group of this vector space.*

Now view Q as the projective space $PG(N-1, q)$ and observe that the projective group G has two point orbits. Hence by an important result, G also has two hyperplane orbits, one of which must be all the hyperplanes through the 'point' F. The other hyperplane orbit must therefore include all the hyperplanes 'off' a point: this is the same number as the number of points off a hyperplane, viz., q^{N-1}. Thus we have shown

Lemma 10.5.9 *If $N > 2$ then G contains a p-group of order q^{N-1}, p being the characteristic of F.*

But now we have seen that this is impossible, unless $q = 2$ and $N = 4$, corresponding to the case when $F = GF(2)$. It can be shown however, that even in this case $AutQ$ contains *another* subgroup H that H fixes a Baer subfield K elementwise and acts transitively on $Q - K$, so in a technical sense we still have a generalized Hall plane. However, the first choice of F is also possible: corresponding to the Lorimer-Rahilly plane of order 16, and this is the only known finite plane which is tangentially transitive relative to a non-Baer subplane. Let us summarize our conclusions:

Theorem 10.5.10 *A finite translation plane π is tangentially transitive relative to a subplane π_0 iff π is a generalized Hall plane and π_0 is a Desarguesian Baer subplane (defining a derivable net) unless the order of the plane is 16 in which case π_0 may taken as a plane of order to when π is the Lorimer-Rahilly plane of order 16: and this is the only case where the non-Baer possibility can occur.*

Note that we have not verified here the claimed uniqueness of the Lorimer-Rahilly plane, although this has been established in the literature, see Walker [40]