Chapter 7

Simple T-extensions of Desarguesian Nets.

The aim of this chapter is to construct three methods for generating finite spreads π, and hence also translation planes. The distinguishing feature of these methods is that they each involve a partial spreadset $\mathcal{F}$ associated with a rational Desarguesian partial spread and another slope matrix 'T': the spread π is then 'generated', in some case-dependent sense, by $\{T\} \cup \mathcal{F}$.

The exact conditions for T to succeed depends on the individual case, but in each instance a wide range of planes can be constructed, in the sense that the dimensions over the kern can be almost arbitrary. Before describing the methods we need to take a closer look at spreadsets containing fields.

7.1 Spreadsets Containing Fields.

Let $\mathcal{S}$ be a finite spreadset, and suppose $\mathcal{F} \subset \mathcal{S}$, $|\mathcal{F}| > 1$. Hence, $\mathcal{F}$ is a field of linear maps iff it is additively and multiplicatively closed. We examine separately the meaning of additive and multiplicative closure of $\mathcal{F}$ using:

Hypothesis (*) *Let $\mathcal{S}$ is a spreadset associated with the additive group of a* finite *vector space V. Assume $\mathcal{S}$ is coordinatized by any one of the prequasifields $Q_e = (V, +, \circ)$, with $\circ$ specified by choosing the left identity $e \in V^*$. Let $\mathcal{F} \neq \{O\}$ be any non-empty subset of $\mathcal{S}$, and let $F \subset V$ be the set of all elements in V whose slope maps lie in $\mathcal{F}$ relative to the choice of e as the identity, thus:*

$$F := \{f \in V \mid f = (e)\phi \exists \phi \in \mathcal{F}\}.$$

and

$$\mathcal{F} = \{T_f \mid f \in F\},$$

where $T_x \in \mathcal{S}$ denotes the slope map of $x \in V$, relative to e as the left identity, i.e., $T_x : y \mapsto y \circ x$, $y \in V$.

First we consider the additive closure of $\mathcal{F}$.

Proposition 7.1.1 *Assume hypothesis (*), in particular, $F = (e)\mathcal{F} \subseteq V$. Then the following are equivalent.*

1. $\forall x \in V, f, g \in F : x \circ (f + g) = x \circ f + x \circ g.$

2. *$\mathcal{F}$ is an additive group.*

3. *$\mathcal{F}$ is additively closed.*

Proof: The condition

$$\begin{aligned} &\forall x \in V,\ f, g \in F : x \circ (f+g) = x \circ f + x \circ g \\ \iff &\forall x \in V,\ f, g \in F : x \circ T_{f+g} = xT_f + xT_g \\ &\qquad\qquad\qquad\iff T_{f+g} = T_f + T_g, \end{aligned}$$

and this cannot hold unless the slopeset of F is additively closed and, conversely, if the slopeset of F is additively closed then the element $M = T_f + T_g \in \mathcal{F}$ agrees with T_{f+g} at the non-zero element e. Hence $T_{f+g} = T_f + T_g$ is equivalent to $\mathcal{F}$ being additively closed. Finally, the additive closure of $\mathcal{F}$ is equivalent to it being an additive group by our finiteness hypothesis. ∎

Now we consider the multiplicative closure of $\mathcal{F}$.

Proposition 7.1.2 *Assume hypothesis (*), in particular, $F = (e)\mathcal{F} \subseteq V$. Then the following are equivalent.*

1. $\forall x \in V, f, g \in F : x \circ (f \circ g) = (x \circ f) \circ g.$

2. *$\mathcal{F}$ is a multipicative group.*

3. *$\mathcal{F}$ is multiplicatively closed.*

Proof: The the condition

$$\begin{aligned}
&\forall x \in V,\ f,g \in F : x \circ (f \circ g) = (x \circ f) \circ g \\
&\iff \forall x \in V,\ f,g \in F : xT_{f\circ g} = xT_fT_g \\
&\iff T_{f\circ g} = T_fT_g,
\end{aligned}$$

and this cannot hold unless the slopeset of F is multiplicatively closed and, conversely, if the slopeset of F is multiplicatively closed then $T_{f\circ g} = T_fT_g$ since they have the same value at the non-zero point e. Hence $T_{f\circ g} = T_fT_g$ is equivalent to $\mathcal{F}$ being multiplicatively closed. Finally, the multiplicative closure of $\mathcal{F}$ is equivalent to it being a multiplicative group since this hold for any finite multiplicative closed set of linear bijections. ∎

Now consider any quasifield $Q = (V, +, \circ)$ such that a subset $F \subset V$ is a field relative to the quasifield operations and that for $x \in Q$ the following identities hold:

$$\begin{aligned}
x \circ (f + g) &= x \circ f + x \circ g \\
(x \circ f) \circ g &= x \circ (f \circ g)
\end{aligned}$$

It is clear from the axioms of a quasifield that $(V, +)$ is a vector space relative to the field F operating from the right via quasifield multiplication iff the above pair of conditions hold. Thus, when these conditions hold, we shall say *the quasifield Q is a* **right** *vector space over F*; it will be tacitly assumed that the vector space is defined relative to the quasifield operations. On comparing these conditions with propositions 7.1.1 and 7.1.2, we immediately deduce:

Proposition 7.1.3 *Let $\mathcal{S}$ be any finite spreadset, containing the identity map, associated with the additive group $(V, +)$ of some vector space; so $Q_e = (V, +, \circ)$ denotes the quasifield determined by $\mathcal{S}$ and $e \in V^*$. Assign to any $\{O\} \subset \mathcal{F} \subseteq \mathcal{S}$ the set of images F of e under $\mathcal{F}$, thus:*

$$F := \{f \in V \mid f = (e)\phi, \phi \in \mathcal{F}\}.$$

Then the following are equivalent:

1. *$\mathcal{F}$ is a field of linear maps.*
2. *$\mathcal{F}$ is closed under addition and composition.*
3. *For some non-zero e: F is a field and Q_e is a right vector space over F.*

4. For all non-zero e: F is a field and Q_e is a right vector space over F.

Suppose $Q_e = (V, +, \circ)$ is a finite quasifield, with identity e, such that Q_e is a right vector space over a subfield $F = (U, +, \circ)$, for some additive group $(U, +) \leq (V, +)$. Now $(V, +)$ may be assigned the structure of a field $K = (V, +, \bullet)$, such that:

$$\forall v \in V, f \in F : v \circ f = v \bullet f$$

The proof is an exercise in linear-algebra/field-extensions: if V is a k-dimensiona right vector space over a field $F = GF(q)$, then V can be given an F-linear identification with a right vector space K, where K is a k-dimensional field extension of the field F: for example, view F as the field of scalar $k \times k$ matrices in $Hom(k, q)$, and then choose as K a field of matrices of order $|F|^n$; this field exists in $Hom(k, q)$ by Galois theory.

Hence $y = x \circ f$ and $y = x \bullet f$ define the same subspace of $V \oplus V$, for all $f \in F$. Hence all these subspaces are components shared by the spreads $\pi(Q_e)$ and $\pi(K)$, and this clearly means that the rational partial spread associated with $\pi(F)$ is a subpartial spreads of both, $\pi(Q_e)$ and $\pi(K)$, and since the latter is Desarguesian, we conclude that $\pi(F)$ determines a *rational Desarguesian* partial spread.

We now consider the converse of this assertion. Hence, our goal is to demonstrate that if $\pi(Q_e)$, the spread associated with a finite quasifield $Q_e = (V, +, \circ)$, contains a rational Desarguesian partial spread δ whose components include X, Y and I, then Q_e contains a subfield F such that $(V, +)$ is a right vector space over F and the components of δ is the partial spread determined by $\pi(F)$, or equivalently, the $\pi(F)$ is a spread across δ).

Since δ is Desarguesian and rational, there is a Desarguesian spread $\Delta = \pi(K)$, where $K = (V, +, \bullet)$ is a field that may be chosen so that it contains a subfield F such that $\pi(F)$ is across δ, and contains (e, e). It is possible to insist further that e, the identity of $Q_e = (V, +, \circ)$, is also the identity of K, and hence of F: use the spreadset associated with K — it clearly contains the spreadset associated with δ — to define $\bullet$ in terms of e.

Since δ is the rational partial spread determined by $\pi(F)$, and lies in both $\pi(K)$ and $\pi(Q_e)$, we have the subspace $y = x \circ f$, for $f \in F$, may be expressed as $y = x \bullet f'$ for some $f' \in F$, and vice versa. Choosing $x = e$ shows that in every case $f = f'$, since K and Q_e both have the same multiplicative identity e. Thus, we have the identity:

$$\forall x \in V, f \in F : x \circ f = x \bullet f.$$

Hence Q_e is a right vector space over F, because K has this property. So we have established that δ is the rational partial spread determined by $\pi(F)$, where F is a field in Q_e such that the latter is a vector space over F.

Hence we have shown that if a finite quasifield Q is a right vector space over a field F then $\pi(Q)$ the rational spread determined by $\pi(F)$ is a rational *Desarguesian* partial spread whose components include the standard components X, Y and I of $\pi(Q)$, and, conversely, a rational Desarguesian partial spread δ in $\pi(Q)$ that includes the standard components among its members must be determined by some $\pi(F)$, where F is a subfield of Q over which the latter is a right F-vector space. Thus the above theorem extends to include another equivalence: $\pi(F)$ determines a rational Desarguesian spread is equivalent to all the other parts of the theorem.

In the context of finite spreadsets $\mathcal{S} \supset \mathbf{1}$, associated with a vector space on $(V, +)$, the above has the following interpretation:
$\mathcal{F} \subset \mathcal{S}$ is a field of matrices iff the components associated with $\mathcal{F}$ in $\pi(Q_e)$ defines a rational Desarguesian partial spread that contains the three standard components X, Y and I of $\pi(Q_e)$.

Thus proposition 7.1.3 may be restated in more detail as follows:

Theorem 7.1.4 *Assume the hypothesis of proposition 7.1.3. Let $\mathcal{S}$ be any finite spreadset, containing the identity map, associated with the additive group $(V, +)$ of some vector space; so $Q_e = (V, +, \circ)$ denotes the quasifield determined by $\mathcal{S}$ and $e \in V^*$. Suppose $\mathcal{F} \subseteq \mathcal{S}$ and let*

$$F = \{f \in V : f = (e)\phi, \phi \in \mathcal{F}\}.$$

Then the following are equivalent:

1. *$\mathcal{F}$ is closed under addition and composition.*

2. *$\mathcal{F}$ is a field of linear maps.*

3. *$(F, +, \circ)$ is field and V is a right vector space over F, for some choice of $e \in V^*$.*

4. *$(F, +, \circ)$ is field and V is a right vector space over F, for all choice of $e \in V^*$.*

5. *The partial spread $\pi(F)$ in $\pi(\mathcal{S})$, that is $\pi(Q_e)$, determines a rational Desarguesian partial spread in $\pi(\mathcal{S})$ that includes its three standard components, X, Y and I.*

Note that in attempting to state the infinite analogue of the theorem above, care must be taken regarding two points: (1) multiplicative and additive closure will no longer force $\mathcal{F}$ to be a field, and (2) the field F may not be embeddable in a larger field of dimension k, where $k := dim_F V$.

7.2 T-extensions of Fields.

If $\mathcal{S}$ is a finite spreadset, in some $\overline{GL(V,+)}$, that contains a field $\mathcal{F}$, then the associated spread $\pi_{\mathcal{S}}$ contains the rational Desarguesian partial spread $\pi_{\mathcal{F}}$. In this section, we consider some ways of extending a field of matrices $\mathcal{F}$ to a spread $\mathcal{S}$ so that the latter is in some sense 'generated' by $\mathcal{F} \cup \{T\}$, where T is a suitably chosen in $GL(V,+) \setminus \{\mathcal{F}\}$. These procedures will yield classes of semifields, and also spreads of order q^3 admitting $GL(2,q)$.

The first method is based on having available a quasifield $Q = (V, +, \circ)$, of suqare order, that contains a subfield F, such that Q is a two-dimensional vector space over F. Since such situations arise iff the spread $\pi(Q)$ is derivable relative to the slopes of $\pi(F)$, we shall refer to the corresponding spreads as being obtained by T-derivation. This method yields a range of semifields that are two dimensional over at least one of their seminuclei, and, in a somewhat vaccuous sense yields them 'all': every such semifield 'yields itself' by the procedure to be described. However, the method is also effective in genuinely constructing long chains of two-dimensional semifields when used sensibly.

The next method is concerned with 'cyclic T-extensions' of a field $\mathcal{F}$ that also yields semifields of non-square order, but this time the field $\mathcal{F}$ lies in at least two semi-nuclei: N_m and N_r, but these can be changed by dualising and/or transposing. Thus neither of the two constructions indicated so far entirely replace the other.

The final construction we discuss is a modification of the above indicated method in the three dimensional case. This yields semifields spreads (*not* semifield spreads) of order q^3 that admit $GL(2,q)$, acting as it does on the Desarguesian spread of order q^3. The dimension of the spread over its kern can be made arbitrarily large, demonstrating that non-solvable groups can act on spreads of arbitrarily large dimensions: so far this phenomenon is known in suprisingly few cases.

We now describe each of the above indicated constructions.

7.2.1 T-Derivations.

We describe here a method of constructing semifields of order q^2 that have $GF(q)$ over their middle nucleus. By transposing and/or dualising the resultant semifield plane, the $GF(q)$ can be can be taken to be any of the three seminuclei. Hence, we focus on the middle or right nucleus case (as the treatment is almost identical) and we shall generally ignore the right nucleus (which involves dualising the left nucleus).

Basically, the method begins with a quasifield $Q = (V, +, \circ)$, of arbitrary order q^2, that contains a subfield $(F, +, \circ) \cong GF(q)$ such that $(V, +)$ is a right vector space over F. Such quasifields, as we saw earlier, are essentially those obtainable from spreadsets $\mathcal{S}$ on $(V, +)$ that contain a subfield $\mathcal{F}$, or equivalently, from spreads of order q^2 that contain rational Desarguesian partial spreads of degree $q + 1$.

The key idea is that for any choice of $T \in \mathcal{S} \setminus \mathcal{F}$, regardless of the Q yielding $\mathcal{S}$, the additive group $\mathcal{F}+\mathcal{F}T$ is an additive spreadset. We shall refer to spreads constructed in this manner, as arising by applying a T-*extensions* to $\mathcal{S}$:

Proposition 7.2.1 (T-Derivations.) *Let $\mathcal{S}$ be a spreadset (or even a partial spreadset!) on a finite additive group $(V, +)$ such that $\mathcal{S} \supset \mathcal{F}$, where $\mathcal{F}$ is a field $\cong GF(q)$,and V has order q^2. Then for any $T \in \mathcal{S} - \mathcal{F}$, the additive set of matrices*

$$\Theta := \tau(T, \mathcal{F}) = \{a + Tb \mid a, b \in \mathcal{F}\}$$

is a spreadset, and hence so is the transpose:

$$\Theta^T = \{a + bT^T \mid a, b \in \mathcal{F}^T\}.$$

In particular, $\Theta\mathcal{F} = \Theta$ and $\mathcal{F}^T\Theta^T = \Theta^T$.

Proof: If $x\alpha + xT\beta = \mathbf{O}$, for $\beta \neq \mathbf{O}$, then $x\alpha\beta^{-1} - = xT$ so $F - T$ is singular for some $F \in \mathcal{F}$, contradicting the hypothesis that $\{T\} \cup \mathcal{F}$ is a subset of the (partial!) spreadset $\mathcal{S}$. Thus $\Theta = \Theta\mathcal{F}$. The rest follows easily. ∎

Note that by allowing $\mathcal{S}$ to be a partial spread, the method can be extended even to cartesian groups $Q = (V, +, \circ)$ of order q^2 that are right vector spaces over a subfield $F = GF(q)$, provided that some $t \in Q - F$ defines an additive map $x \mapsto x \circ t$ on $(V, +)$.

Recall, theorem5.3.3, that for additive spreadsets $\mathcal{S}$ the middle nucleus corresponds to the largest subset $\mathcal{F}$ such that $\mathcal{FS} = \mathcal{S}$, and the right nucleus corresponds to the transpose situation, viz., the largest $\mathcal{F} \subset \mathcal{F}$ such that $\mathcal{S} = \mathcal{S}\mathcal{F}$, theorem 5.3.4. Hence, for convenience and for its future role, we shall usually only comment on the middle nucleus situation. We note that any semifield spreadset of order q^2 is obtained by applying a T-extension to itself.

Remark 7.2.2 *If $\mathcal{T}$ is a spreadset of order q^2 containing a field $\mathcal{F} \cong GF(q)$ such that $\mathcal{F}\mathcal{T} \subset \mathcal{T}$ then*

$$\mathcal{T} = \mathcal{F} + \mathcal{F}T,$$

whenever $T \in \mathcal{T} \setminus \mathcal{F}$; in particular, $\mathcal{T}$ coincides with $\tau(T, \mathcal{F})^T$, using the notation of proposition 7.2.1.

Thus all semifields that are two-dimensional over their middle (or left) nucleus are T-extensions — of themselves! However, the process of T-extensions can be effectively used to yield a variety of examples of semifields that are two dimensional over the middle nucleus, and indeed, by transposing and dualising, over any semifield. To generate such examples, using T-extensions, one can arbitrarily repeat arbitrary long chains of steps, each step involving one of dualising-transposing-T-deriving-recordinatising and collecting the required spreadsets at each stage, for example by adopting using a loop such as the following:

Generating Two Dimensional Semifields.

a Choose spread with derivable partial spread δ.

b Coordinatise by a quasifield Q so that δ is coordinatized by a field F.

c Now either form Q' containing field F' such that Q' coordinatizes the transpose spread and Q' is a right vector space over F' a field isomorphic to F, or simply choose $Q' = Q$ and $F' = F$.

d Obtain two-dimensional semifield associated with any $t \in Q' - F'$, with middle nucleus F'.

e Dualise and/ or transpose the semifield and/or derive relative to F'-slopes.

f Return to step [a] or stop.

Certainly many non-isomorphic spreads arise thus, and as indicated above, 'all' finite semifields that are two dimensional over a seminucleus are of this form, albiet in a somewhat vaccuous sense; although T-extensions provide a useful method for generating examples of two-dimensional semifields it is not meaningful to ask if their are 'other' semifields of order q^2, with $GF(q)$ in seminucelus.

7.2.2 Cyclic Semifields.

Let W be a finite n-dimensional vector space, $n > 1$ over a field F and suppose $T \in \Gamma L(W,F) \setminus GL(W,F)$ is a strictly semi-linear bijection of W, regarded as an F-space; also let K be a subfield of F such that $T \in GL(W,K)$, for example K might be chosen to be the prime subfield of F.

We are interested here in the case when T is F-irreducible, that is, when T does not leave invariant any non-trivial proper F-subspace of W. Examples of such T are easily constructed, for instance on choosing $S \in GL(W,F)$ to correspond to a Singer cycle of $PG(n-1,F)$, $\sigma \in Gal(F)^*$, we might define $T = S\hat{\sigma}$; it is also not hard to see that S^k, for many values of k, work as well as S itself.

We now observe that the F-subspace of $Hom(W,F)$, generated by the powers of T, form an additive spreadset and thus yields a semifield; the strict F-semiinearity of T ensures that these semifields will not be a field. We shall call these semifields *cyclic.*

Proposition 7.2.3 *Suppose W is a finite n-dimensional vector space, $n > 1$, over a field F and that $T \in GL(W,K)$, where K is a proper subfield of F. If $T \in \Gamma L(W,F) \setminus GL(W,F)$ is F-irreducible, then viewing T and $f \in F$ as elements of $GL(W,K)$, the set:*

$$\Delta(T,F) := \{\mathbf{1}a_0 + Ta_1 + \ldots + T^{n-1}a_{n-1} \mid a_0, a_1, \ldots a_{n-1} \in F\}$$

is an additive spreadset over the field K. Such spreadsets will be called **cyclic semifield** *spreadsets.*

Proof: If some $\mathbf{1}a_0 + Ta_1 + \ldots T^i a_i + \ldots + T^k a_k$, for $0 \le i \le k \le n-1$, where $a_k \neq 0$, is singular then there is an $x \in W^*$ such that:

$$\begin{aligned} \mathbf{0} &= (x)\mathbf{1}a_0 + Ta_1 + \ldots + T^k a_k \\ \text{so } (x)T^k &= (x)\left(\mathbf{1}a_0 + Ta_1 + \ldots + T^{k-1}a_{k-1}\right)\frac{1}{a_k} \end{aligned}$$

and hence the F-subspace of W generated by $\{x, xT, xT^2 \ldots, xT^{k-1}\}$ is T-invariant contradicting the F-irreducibilty of T. Thus all elements of type $\mathbf{1}a_0 + Ta_1 + \ldots + T^{n-1}a_{n-1}$, other than when $a_0 = a_1 = \ldots = a_{n-1}$, are non-singular, and hence $\Delta(T, F)$ is an additive group of linear non-singular K-linear maps that has the correct size to be a spreadset. The result follows. ■

Remark 7.2.4 *The kern of $\Delta(T, F)$ is isomorphic to the centralizer of $\{T\} \cup \mathcal{F}$ in $Hom(W, +)$.*

Proof: The kern is the centralizer of the slope set of $\Delta(T, F)$ and this lies in the subalgebra, over the prime field, of $Hom(W, +)$ generated by $\{T\} \cup \mathcal{F}$. ■

The Sandler semifields and the finite Hughes-Kleinfeld semifields are cyclic semifields, and as pointed out by Kallaher [29], almost all cyclic semifields are of these types. Thus cyclic semifields may be regarded as providing a uniform characterization of the finite Hughes-Kleinfeld and Sandler semifields, in slightly generalized form.

7.2.3 T-Cyclic $GL(2, q)$-spreads

We now define spreads that are *never* semifield-spreads, but still based on a field $\mathcal{F}$ of K-linear maps of an n-dimensional K-vector space W, K any finite field.

The construction is best described directly, as a spread on $V = W \oplus W$, rather than via a spreadset, so it becomes convenient to work with matrices, relative to a chosen K-basis of W, and we make the identifications $W = K^n$, $V = K^n \oplus K^n$. Now the field of linear maps associated with the scalar action of F on W, viz., $\hat{f} : x \mapsto xf$, becomes identified with a field $\mathcal{F}$ of $n \times n$ matrices over K, acting on K^n, and $T \in GL(n, K)$ is still required to be strictly $\mathcal{F}$-semilinear on K^n, or equivalently:

$$T \in N_{(GL(n,K))}(\mathcal{F}) - C_{(GL(n,K))}(\mathcal{F}),$$

and we shall insist that T does not leave invariant any non-trivial $\mathcal{F}$-subspace of rank ≤ 2, rather than insisting that T acts irreducibly, as in the previous case.

We shall demonstrate that the orbit τ of the subspace $y = xT$ of V, under the standard action of $\mathcal{G} = GL(2, \mathcal{F})$ on V, forms a partial spread that extends to a larger $\mathcal{G}$-invariant partial spread $\pi(T, \mathcal{F}) := \pi_{\mathcal{F}} \cup \tau$, where $\pi_{\mathcal{F}}$ is the [rational Desarguesian] partial spread associated with $\mathcal{F}$. On specialising to the case $dim_F W = 3$, the partial spread $\pi(T, \mathcal{F})$ becomes a non-Desarguesian spread of order q^3 admitting $GL(2, q)$, where $\mathcal{F} \cong GF(q)$.

Proposition 7.2.5 *Let $W = K^n$ be the standard n-dimensional vector space over a finite field $K = GF(q)$, for $n > 3$. Suppose $\mathcal{F} \subset GL(n, K)$ is a field, containing the scalar field K, and*

$$T \in N_{(GL(n,K))}(\mathcal{F}) - C_{(GL(n,K))}(\mathcal{F}),$$

so there is a non-trivial field automorphism $\sigma \in Gal(\mathcal{F}/K)^*$ such that

$$\forall X \in \mathcal{F} : X^\sigma = T^{-1} X T.$$

Let $\pi_{\mathcal{F}}$ be the rational Desarguesian partial spread determined on $V := W \oplus W$ by the spreadset $\mathcal{F}$, and let τ be the orbit of the K-subspace $y = xT$, of V, under the group:

$$\mathcal{G} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{F}, ad - bc \neq 0 \right\} \cong GL(2, \mathcal{F}),$$

in its standard action on V.
Put:

$$\pi(T, \mathcal{F}) := \tau \cup \pi_{\mathcal{F}}.$$

Suppose T does not leave invariant any non-zero $\mathcal{F}$-subspace of W that has rank ≤ 2. Then the following hold.

1. *τ is a partial spread containing $q(q^2 - 1)$ components and the global stabilizer of $y = xT$ in $\mathcal{G}$ is the diagonal group*

$$\{Diag[A, A^\sigma] \mid A \in \mathcal{F}^*\}.$$

2. *The rational Desarguesian partial spread $\pi_{\mathcal{F}}$ is a $\mathcal{G}$-orbit, and $\mathcal{G}$ acts triply transtively on its components.*

3. *The $\mathcal{G}$-orbits, τ and $\pi_{\mathcal{F}}$, do not share any components and $\pi(T, \mathcal{F})$ is also a partial spread.*

4. *$\pi(T,\mathcal{F})$ is a spread iff $dim_{\mathcal{F}}W = 3$. In this case, the spread admits $\mathcal{G} = GL(2,\mathcal{F})$ so that this group partitions the components of $\pi(T,\mathcal{F})$ into two orbits, viz., τ and $\pi_{\mathcal{F}}$, and $\mathcal{G}$ acts triply transitively on the orbit $\pi_{\mathcal{F}}$ and transitively on the orbit τ.*
   The kern of $\pi(T,\mathcal{F})$ is isomorphic to the centralizer of $\{T\} \cup \mathcal{F}$ in $Hom(W,+)$; hence $K = GF(q)$ is always in the kern, and $\mathcal{F}$ is not: so the spread is non-Desarguesian.

Proof: The image of (x, xT), $x \in K^n$, $x \neq \mathbf{0}$, under an element of $\mathcal{G}$:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is $(xA + xTC,\ xB + xTD)$, and this meets the component $y = xT$ iff the conditions $u = xA+xTC$ and $uT = xB+xTD$ hold simultaneously for some $u \in W^*$, and this is equivalent to

$$(xA + xTC)T = xB + xTD,$$

and since T normalizes $\mathcal{F}$, and induces σ on it the above is equivalent to:

$$xT^2C^\sigma = xB + xT(D - A^\sigma),$$

and this means that the $\mathcal{F}$-subspace generated by $\{x, xT\}$ is T-invariant, contradicting the hypothesis that T cannot leave invariant non-trivial $\mathcal{F}$-subpace of dimension ≤ 2, unless $B = C = \mathbf{O}$ and $D = A^\sigma$. Now the image of (x, xT) is (xA, xAT), for all x.
Thus, the orbit τ of the component $y = xT$ under $\mathcal{G}$ contains, in addition to $y = xT$, only subspaces that are disjoint from $y = xT$ and, additionally, the global stabilizer of $y = xT$ is given by

$$\mathcal{G}_{\{y=xT\}} = \{Diag(A, A^\sigma) \mid A \in \mathcal{F}^*\},$$

so

$$|\tau| = |GL(2,q)|/(q-1) = q(q^2-1).$$

Thus we have established that the $\mathcal{G}$-orbit of $y = xT$, viz., τ, is a collection of $q(q^2-1)$ subspaces that have the same size as $y = xT$ and all members of $\tau \setminus \{y = xT\}$ are disjoint from $y = xT$. It follows that if R and S are *any* two distinct members of τ, then they are disjoint because if $R \cap S \neq \mathbf{O}$ then we may choose $g \in \mathcal{G}$ such that $(R)g = (y = xT)$ and now $y = xT$ meets the element $(S)g \in \tau$.

Instructive Diversion. *This is a case of a simple but useful principle: (a) if a rank r subspace A of a vector space V of rank $2r$ has an orbit $\mathcal{A}$ under a subgroup $G \le GL(V,+)$ such that $A - A^g$ is non-singular or zero for all g then $\mathcal{A}$ is a partial spread that is G-invariant; (b) if the subspace A is disjoint from all the members of a G-invariant partial spread $\mathcal{B}$ then $\mathcal{A} \cup \mathcal{B}$ is also a partial spread.*

Next, to apply the second part of the above principle, consider the possibility that $y = xT$ meets $\pi_{\mathcal{F}}$, the rational Desarguesian spread coordinatized by $\mathcal{F}$. If $T - A$ is singular for $A \in \mathcal{F}^*$, then $xT = xA$, for some $A \in \mathcal{F}^*$, $x \in W^*$. Thus $y = xT$ and $y = xA$ are disjoint subspaces of V, for $A \in \mathcal{F}^*$: otherwise T leaves invariant the rank-space $x\mathcal{F}$, contrary to hypothesis. Moreover, $y = xT$ is certainly disjoint from $\mathbf{O} \oplus W$. Hence $y = xT$ is disjoint from the rational Desarguesian partial spread coordinatized by the spreadset $\mathcal{F}$. But this partial spread, viz.,

$$\pi_{\mathcal{F}} := \{y = xA \mid A \in \mathcal{F}\} \cup \{Y\}.$$

is also invariant under $\mathcal{G}$ because

$$(\mathbf{0}, u)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (uc, ud)$$

shows that Y is left invariant when $c = \mathbf{0}$, and otherwise, when $cu \neq \mathbf{0}$, Y maps to $y = x(uc)^{-1}ud$, which is a component of type $y = xf$, $f \in \mathcal{F}$. Similarly, we can determine that $y = xf$, $f \in \mathcal{F}$, maps under $\mathcal{G}$ into the rational Desarguesian partial spread $\pi_{\mathcal{F}}$:

$$(y = xf)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} (y = x(a + fc)^{-1}(b + fd)) & \text{if } a + fc \neq \mathbf{O}; \\ (x = \mathbf{0}) & \text{otherwise.} \end{cases}$$

In particular, Y is not $\mathcal{G}$ invariant, and the global stabilizer $\mathcal{G}_{\{Y\}}$ of Y is doubly transitive on all the other components of $\pi_{\mathcal{F}}$: for example, note that $\mathcal{G}_{\{}Y\}$ does not leave X invariant and the global stabilizer of X in $\mathcal{G}_{\{Y\}}$ is transitive on the components in $\pi_{\mathcal{F}} \setminus \{X, Y\}$. Hence $\mathcal{G}$ leaves $\pi_{\mathcal{F}}$ invariant and acts 3-transitively on its components.

Thus, recalling that the members of $\pi_{\mathcal{F}}$ are disjoint from $y = xT$, we see that the orbit $(y = xT)\mathcal{G}$ is a partial spread such that its members all have trivial intersection with the members of $\pi_{\mathcal{F}}$.

Now specialize to the case $\mathcal{F} = GF(q)$ and $dim_{\mathcal{F}}W = 3$. Now the partial

spreads $\pi_{\mathcal{F}}$ and τ together contribute $q+1+q(q^2-1)=q^3+1$ components of the partial spread $\pi(T,\mathcal{F})$, and this is the size needed to make it into a spread. Since the $\mathcal{G}$-orbit τ now has the size of $\pi(T,\mathcal{F})\setminus\pi_{\mathcal{F}}$, we conclude that $\mathcal{G}$ is transitive on the components of the spread outside $\pi_{\mathcal{F}}$.
This spread is coordinatized by a spreadset $\mathcal{S}\supset\mathcal{F}\cup\{T\}$, that includes the identity and yet $\mathcal{S}$ is not a field because T does not centralize $\mathcal{F}$. The slope-set of $\pi(T,\mathcal{F})$ is clearly in $Hom(W,+)$ so its kern is as claimed. ∎
By varying T, for a fixed choice of $\mathcal{F}$, it is possible to ensure that the dimension of the spread $\pi(T,\mathcal{F})$, over its kern, can be made arbitrarily large; in partciular this means that non-Desarguesian translation planes of order q^3 that admit $SL(2,q)$ can be chosen to have arbitrarily large dimension. We leave this verification as an exercise for the reader.