Chapter 5

Coordinatization.

The theme of this chapter is coordinatization of structures that are associated with translation planes. In particular, we emphasize how spreads are coordinatized by spreadsets and (pre)quasifields, and also on how spreadsets may themselves be coordinatized by (pre)quasifields.

5.1 Spreads and Quasifields.

Recall that, by definition 1.1.17, a spread $\pi = (V, \mathcal{S})$ is a collection of additive subspaces $\mathcal{S}$, of an additive group V, such that every $x \in V$ lies in some *component* $\sigma \in \mathcal{S}$, and

$$\alpha, \beta \in \mathcal{S} \Longrightarrow V = \alpha \oplus \beta \vee \alpha = \beta.$$

We now assign to each prequasifield Q an associated spread $\pi(Q)$, said to be *coordinatized* by Q. We summarize some related notation which will be very extensively used: the notation is essentially that of elementary coordinate geometry in the context of quasifields; it is kept sufficiently flexible to consider the classification of quasifields among zero-linked structures, defintion 4.2.1; variants of the notation are useful in studying partial spreads and nets.

Notation 5.1.1 *Let $Q = (W, +, \circ)$, where $(W, +)$ is the additive group of a vector space and $\circ$ is a binary operation on W. Then on the vector space $W \oplus W$ we define the following subsets.*

1. *The X-axis and the Y-axis are respectively $X := W \oplus \mathbf{O}$ and $Y := \mathbf{O} \oplus W$.*

2. *The unit line is the set* $\{(x,x) \mid x \in W\}$, *and denoted by* Z *or* $vecI$.

3. *The* non-vertical lines *are the sets of type*

$$\forall m, b \in W : \text{`}y = x \circ m + b\text{'} := \{(x, x \circ m + b \mid x \in W\}.$$

4. *The vertical lines are all sets of type:*

$$\forall b \in W : \text{`}x = b\text{'} := \{(b, y) \mid y \in W\}.$$

The quotation marks above are often dropped. Also note that, in the context of a translation algebra $(Q, +, \circ)$, the collection of all lines whether or not they are vertical, coincides with $\Pi(Q)$, the incidence structure associated with $(Q, +, \circ)$. As $\Pi(Q)$ is determined by the lines through zero, we shall introduce a special notation for this structure: we write $\pi(Q)$ for the lines $\Pi(Q)$ through the origin:

Definition 5.1.2 *If* $Q = (W, +, \circ)$ *is any zero-linked structure, see definition 4.2.1 structure,then* $\pi(Q) := (V, \mathcal{S})$, *where* $V = W \oplus W$ *and*

$$\mathcal{S} = Y = \{\mathbf{O} \oplus W\} \cup \{\text{`}y = x \circ m\text{'} \mid m \in Q\};$$

the members of $\mathcal{S}$ *are the components of* $\pi(Q)$; *thus the components are the lines of* $\Pi(Q)$ *through the origin.*

A fundamental but elementary result is that $\pi(Q)$ is a spread iff the given zero-linked structure Q is at least a pre-quasifield.

Remark 5.1.3 *Let* $Q = (W, +, \circ)$ *be a zero-linked structure, definition 4.2.1, and the sfield* K *its kern: thus* K *is the centralizer of the slopeset of* Q *in the ring* $Hom(W, +)$. *Then* $\pi(Q)$, *is a spread iff* Q *is a pre-quasifield;* $\pi(Q)$ *is said to be* coordinatized by *the prequasifield* Q.

Proof: $\Leftarrow$ is straightforward. To establish the converse we assume that Q is a zero-linked structure and that π_Q is a spread on V; we must deduce that Q is a prequasifield. Consider $\Pi(Q)$, c.f., definition 4.2.1, the incidence structure associated with $Q = (W, +, \circ)$; so the pointset of $\Pi(Q)$ is $V := W \oplus W$ and the lines of $\Pi(Q)$ are all the subspaces and cosets of $(V, +)$ that are of form '$x = c$' or '$y = x \circ m + c$', for $m, c \in W$. Hence, the lines of $\Pi(Q)$ through the 'origin' $\mathbf{O} = \mathbf{0} \oplus \mathbf{0}$ consists of the components of the spread π_Q.

So every subspace of type '$x = c$' or '$y = x \circ m + c$' is a translate of a component of π_Q. Hence $\Pi(Q)$ is the translation plane associated with the spread π_Q, and $W \oplus W$ may be identified with its translation group in the obvious way. Thus Q is a translation algebra, as defined in 4.2.1, that coordinatizes a translation plane, and hence must be a pre-quasifield by proposition 4.2.3. ∎

5.2 Quasifields and Spreadsets.

We introduced in an earlier chapter, see definitions 1.3.4 and 1.3.11, the notion of a [partial] spreadset, and we described how they give rise to [partial] spreads. In this lecture, we similarly explore the connection between spreadsets and quasifields.

Although in some theoretical sense, spreadsets, quasifelds and spreads all turn out to be 'equivalent', the correspondence is not one-one: for example, many non-isomorphic quasifields are associated with the same spread and most spreadsets are associated with several non-isomorphic quasifields that they 'coordinatize'. Thus, spreadset and [pre]quasfields provide essentially distinct approaches to the study of spreads and translation planes.

To keep this lecture self-contained, we review the definition of a partial spreadset in the following exercise: it provides a characterization of the concept as given in our earlier definition 1.3.11. For the convenience of the reader, the rest of this lecture tacitly treats this exercise as defining a [partial] spreadset.

Exercise 5.2.1 *Let τ be a set of homomorphisms of the additive group $(W, +)$ of a vector space. Then τ is a partial spreadset on W iff*

$$\alpha, \beta \in \tau \Rightarrow \alpha - \beta \in \overline{GL(W, +)}.$$

A partial spreadset τ is a spreadset iff $\mathbf{O} \in \tau$ and τ is a transitive set of maps on W, which means:

$$\forall x, y \in W : \exists t \in \tau \ni y = x^t.$$

If W is a vector space over a [skew]field K then τ is a [K-linear] spreadset if the members of τ are K-linear.

Much of the following remark amounts to restating the meaning of a spreadset, in terms of its characterization in the exercise above, and also reviews the connection between spreads and spreadsets as discussed in section 1.3. However, the main point of the remark is to establish the theoretical equivalence between spreadsets and [pre]quasifields.

Remarks 5.2.2 *Let $(W,+)$ be the additive group of a vector space and suppose τ is a set of additive maps of W such that $\mathbf{O} \in \tau$ and $\tau^* \subset GL(V,+)$. Then:*

1. *τ is a spreadset iff τ^* is regular on W^*, that is:*
$$\forall x, y \in W^* : \exists! t \in \tau \ni y = x^t.$$
2. *If $|W|$ is finite then τ is a partial spread iff $\tau^* \subset GL(W,+)$ is such that $A - B$ is also non-singular, whenever A and B are distinct members of τ.*
3. *If $|W|$ is finite then τ is a spread iff $\mathbf{O} \in \tau$ and τ contains $|W|$ elements any two of which differ by a non-singular map or zero.*
4. *Let $Q = (W, +, \circ)$ be a [pre]quasifield. Then the set of its slope maps, see definition 4.3.2, τ_Q form a spreadset, called the spreadset* associated *or* coordinatized *by Q.*

Proof: We only consider case (4), as this is the least trivial case. The slope maps $T_m : x \mapsto x \circ m$, $m \in Q^*$, are bijections because Q^* is a quasigroup, and the distributive law for Q means that every such $T_m \in GL(W,+)$. Next we must show that the additive map $T_a - T_b$, for $a, b \in W$ is bijective, assuming $a \neq b$. If $x(T_a - T_b) = \mathbf{0}$ then $x \circ a = x \circ b$, contradicting the quasigroup property for multiplication. Thus $T_a - T_b$ is injective. To show this map is surjective, consider $w \in W^*$. Now $w = x(T_a - T_b)$ for some $x \in W$ iff

$$w = x \circ a - x \circ b \exists x \in W,$$

and this holds by proposition 4.2.4. It only remains to check that if x and y are non-zero then $y = x^t$, for a unique $t \in \tau$. This equivalent to checking that $y = x \circ t$ has a unique solution for t, and this again follows from the quasigroup property. ■

We now associate with any spreadset, in the sense of definition 5.2.1, several related algebraic systems that turn out to be at least prequasifields: this will lead to the correspondence between spreadsets and [pre]quasifields mentioned earlier.

Definition 5.2.3 (Sytems Coordinatizing Spreadsets.) *Let τ be a spreac set on $(W, +)$, the additive group of a vector space. To each $e \in W^*$ assign the system $Q_e := (W, +, \circ)$, where $\circ$ is defined by:*

$$x \circ y = x^{t(e \mapsto y)},$$

where $t(e \mapsto y)$ denotes the unique element of τ that maps e to y. The system Q_e is said to coordinatize τ at e.

It is immediately obvious that $e \circ y = y$, so Q_e has e as a left identity. Moreover, when $\mathbf{1} \in \tau$ then e actually becomes a two-sided identity. Now consider whether Q_e is a [pre]quasifield. The non-singularity of the non-zero members in τ shows that $x \circ a = c$ has a unique solution for x when $a \neq 0$. The additive property of linear maps provides the right distributive law. Also, the condition

$$x \circ a - x \circ b = x(\tau_a - \tau_b), \tag{5.1}$$

shows the LHS, as a function of x, is bijective on W because, by definition, any two distinct members of a spreadset differ by a non-singular W-bijection. Finally an equation of type $a \circ x = b$ has a unique solution for x because of the 'regularity hypothesis'. Thus we conclude:

Remark 5.2.4 (The Quasifields Coordinatizing A Spreadset.) *Let τ be a spreadset on some some $(W, +)$, the additive group of a vector space. Then for each $e \in W^*$ the system Q_e coordinatizing τ, as in definition 5.2.3, is a [pre]quasifield, which we call the [pre]quasifield* coordinatizing *τ at e. The [pre]quasifield has e as a left identity, and hence Q_e is a quasifield (with identity e) iff τ includes the identity map.*

Corollary 5.2.5 (The centralizer of a spreadset is the kern.) *The centralizer of τ in $Hom(W, +)$ is a [skew]field K, and K is the external kern of all the [pre]quasifields Q_e, $e \in W^*$, coordinatizing τ. In particular, if W is a vector space over a [skew]field F and if τ is a spread set of F-linear maps then F is in the external kern of the [pre]quasifield Q_e.*

We shall gradually get less pedantic with kern terminology: for instance, we shall usually not specify whether the kern considered is 'internal' or 'external'.

In the finite case, spreadsets have a particularly simple characterization:

Remark 5.2.6 *Let $\tau \subset \overline{GL(n,q)}$ such that $\mathbf{O} \in \tau$. Then τ is a spread iff $|\tau| = q^n$ and any two members of τ differ by an element of $GL(n,q)$. More $GF(q)$, associated with the scalar maps, is in the kern of the quasifields associated with π_τ.*

Proof: This is just a restatement of remark 5.2.2(4), bearing in mind that, by the corollary above, the centralizer of a spreadset corresponds to the kern of all the quasifields associated it. ∎

We now verify that every spreadset τ determines a spread π_τ and this coincides with all spread as $\pi(Q_e)$, as Q_e ranges over the quasifields coordinatizing τ. We first fix our notation in the context of *partial* spreadsets $\tau \subset Hom(W,+)$.

Definition 5.2.7 *Let $(W,+)$ be an additive group of a vector space and $\tau \subset Hom(V,+)$ such that:*

$$A, B \in \tau \Longrightarrow A - B \in \overline{GL(W,+)}.$$

The τ is a PARTIAL SPREADSET *and the associated partial spread is the collection of additive subspaces of $V = W \oplus W$ given by:*

$$\pi_\tau := \{[y = xT] \mid T \in \tau\} \cup \{Y\},$$

and we define

$$\pi_{\underline{\tau}} := \{[y = xT] \mid T \in \tau\}.$$

The more elaborate notation is chosen for the simpler structure because in most contexts the Y-axis needs to be included.

Theorem 5.2.8 *Let τ be a spreadset on a vector space W. Then the collection of subspaces defined on $W \oplus W$ by:*

$$\pi_\tau = \{[y = xT] \mid T \in \tau\} \cup \{\mathbf{0} \oplus W\}$$

is a spread, called the spread associated with τ. Moreover, for each $e \in W^*$ the spread $\pi(Q_e) = \pi_\tau$, where Q_e is the [pre]quasifield, coordinatizing τ at e.

Proof: Consider any $y = xT$ that lies in π_τ. Putting $\theta = eT$ we have

$$x \circ \theta := x^{t(e \mapsto \theta)} = xT,$$

so $y = x \circ \theta$ is the same subspace of $W \oplus W$ as $y = xT$. Conversely any $y = x \circ \theta$ may, by definition, be expressed as $y = xT$ where $T \in \tau$ maps e to θ. Thus π_τ is the same set of subspaces of $W \oplus W$ as in $\pi(Q_e)$. However, the latter is a spread because, by remark 5.2.4, Q_e is always a quasifield. Hence π_τ is also a spread and the desired result follows. ∎

Theorem 5.2.9 *Let $\pi(Q)$ be a spread coordinatized by a [pre]quasifield $Q = (W, +, \circ)$, and suppose K is the [external] kern of K. Then the standard action of K on $\pi(Q)$ coincides with the action of kern of $\pi(Q)$, that is, the standard action of K^* on $W \oplus W$ is the same the action as that of the full group of kern homologies of $\pi(Q)$.*

Proof: The non-vertical components of $\pi(Q)$ are of form $y = x \circ m$, or equivalently, yxM, where M is in the spreadset determined by Q. Now the kern of Q are the members $k \in End(W, +)$ that centralize all such M, so the standard action of k on $W \oplus W$ yields:

$$(x, xT) \mapsto (xk, xTk) = (xk, xkT) \in [y = xT],$$

and hence every $y = xT$ is left invarant by k. Hence K may be identified with a subfield of the [skew]field of kern endomorphisms of the spread $\pi(Q)$. Now consider the converse.

Let τ be the slopeset of Q. So the non-vertical components of the spread $\pi(Q)$ are all of form $y = xT$, $T \in \tau$. Moreover, we may regard Q as being Q_e for some e. Consider any homology leaving every member of $\pi(Q)$ invariant. Since this fixes Y and X it must be of from $\alpha \oplus \beta \in GL(W, +) \oplus GL(W, +)$ and satisfy the condition:

$$\forall x \in W : (x, xT) \mapsto (x\alpha, xT\beta) \in [y = xT],$$

so $\alpha^{-1}T\beta = T$ so:

$$\forall T \in \tau : \alpha T = T\beta.$$

Now apply Schur's lemma above. ∎

We now consider the problem of deciding when a spreadset is a quasifield and when it is a pre-quasifield without an identity.

Corollary 5.2.10 *Let τ be a spreadset. Then the following are equivalent:*

1. *τ contains the identity map.*
2. *Some prequasifield Q_e coordinatizing τ is a quasifield.*
3. *All prequasifield Q_e coordinatizing τ are quasifields.*
4. *The spread π_τ includes the unit line $y = x$.*

Thus it becomes desirable to 'reduce' a spreadset τ to an equivalent spread containing the identity; we regard two spreadsets as being *equivalent* if the corresponding spreads are isomorphic. So, when are two spreadsets equivalent? A simple sufficiency condition is the following:

Remark 5.2.11 *If τ is a spreadset on W then so is $A^{-1}\tau B$, whenever $A, B \in GL(W, +)$ and the map $\theta : (x, y) \mapsto (xA, yB)$, of $W \oplus W$, is an isomorphism from the spread π_τ onto the spread $\pi_{A^{-1}\tau B}$. Moreover θ leaves invariant the common components $X = W \oplus \mathbf{0}$ and $Y = \mathbf{0} \oplus W$.*

Thus we may simplify a spreadset to an equivalent one such that the unit line belongs to it, and hence the coordinatizing prequasifields are all quasifields:

Corollary 5.2.12 *Let τ be a K-linear spreadset on a K-space W, K any field: so the components of the spread π_τ are K-subspaces of the ambient space $W \oplus W$, and the subspaces $X = W \oplus \mathbf{0}$ and $Y = \mathbf{0} \oplus W$ are among the components of π_τ. Then the spreadset τ is equivalent to a K-linear spreadset θ such that its associated spread π_θ has the same ambient space $W \oplus W$ as π_τ, and the components of π_θ are K-subspaces of $W \oplus W$ that include not only X and Y, but also the unit line $I = \{(w, w) \mid w \in W\}$.*

Thus, all spreads that are coordinatized by spreadsets, i.e. are of form π_τ for some spreadset τ, may be [re]-coordinatized by a spreadset σ such that σ includes the identity.

We have seen that every prequasifield $(Q, +.\circ)$ may be 'converted' to a quasifield $(Q, +, *)$ by choosing $e \in Q^*$ and defining $*$:

$$(x \circ e) * (e \circ y) = x \circ y,$$

and now $e \circ e$ becomes the identity. We now demonstrate that the associated spreads are isomorphic and hence both sytems have the same [outer] kern.

Let $S_x : x \mapsto x \circ a$, $a \in Q$, and $T_a : x \mapsto x * a$, $a \in Q$ denote respectively the slopemap of a in the prequasifield $(Q, +, \circ)$ and the quasifield $(Q, +, *)$ respectively. Thus the identity above yields $S_e T_{(e \circ y)} = S_y$, for all $y \in Q$ and so the slopeset τ_Q of the quasifield $(Q, +, *)$ is given by $\tau_Q = {S_e}^{-1}\sigma_Q$, where σ_Q is the slopeset of the prequasifield $(Q, +, \circ)$. We shall state this result in terms of:

Definition 5.2.13 *Let $(Q, +, \circ)$ be a prequasifield. Define $Q_e := (Q, +, *)$ by*

$$\forall x, y \in Q : (x \circ e) * (e \circ y) = x \circ y.$$

Then Q_e is the quasifield that normalizes *the prequasifield $(Q, +, \circ)$ at e.*

Thus we have established:

Proposition 5.2.14 *Let Q be a prequasifield normalized by a quasifield R at $e \in Q^*$. Let τ_Q and τ_R be respectively the slopeset of two systems. Then $\tau_R = E^{-1}\tau_Q$, where E is the slopemap of e regarded as member of Q. In particular, the spreads defined by a prequasifield is isomorphic to the spreads obtained by any of its normalized quasifields, and the external kernel of the two systems are the same.*

It is worth stressing that normalising a prequasifield to a quasifield is equivalent to introducing a multiplicative identity in its spreadset τ by replacing τ by $T^{-1}\tau$, where T is any non-zero element in τ.

5.3 Substructures of Quasifields.

In this lecture, we introduce certain additive and multiplicative substructures associated with quasifields and prequasifields and consider their connection with the associated spreadsets.

Note that we have already considered the most important case, viz., the kern: all the quasifields coordinatizing a translation plane, and, a fortiori, those associated with a given spreadset, have isomorphic kerns since they may be identified with the group of homologies with the ideal line as axis.

The aim here is to consider several other substructures of prequasifields that extend the notion of the kern in various ways, and thus have some geometric significance. Our main concern here is the extent to which these structure are invariant, as the quasifields from which they arise range over all the quasifields associated with a fixed spreadset.

There are basically two types of substructures that we consider here: the extreme case of each type being nearfields (associative quasifields) and semifields (distributive quasifields).

In nearly every case, our goal is to show that each type of substructure is an invariant for all the prequasifields coordinatizing a fixed spreadset $\mathcal{S}$. This reflects the fact, as we shall see in the next chapter, that the substructures we consider are nearly always associated with certain maximal groups of central collineations of the spread coordinatized by $\mathcal{S}$.

We deal first with the multiplicative substructures associated with a (pre)quasifield Q, and then turn to an additive analogue. In the multiplicative cases, the structures we refer to are just the seminuclei of the multiplicative quasigroup structure of Q^*, and we have already met these in the context of loops (rather than just quasigroups).

Although our definitions are formulated to hold for the general case, to maintain clarity, all the results in this section are established only for the finite case. We begin by repeating the definition of the nuclei of a loop in the context of prequasifields.

Definition 5.3.1 *Let $\mathcal{Q} = (Q, +, \circ)$ be a finite prequasifield. Then the middle, left and right nucleus are respectively defined as follows:*

1.

$$N_m = \{f \in Q \mid (x \circ f) \circ y = x \circ (f \circ y) \forall x, y \in Q\}$$

2.

$$N_r = \{f \in Q \mid (x \circ y) \circ f = x \circ (y \circ f) \forall x, y \in Q\}$$

3.

$$N_\ell = \{f \in Q \mid f \circ (x \circ y) = (f \circ x) \circ y \forall x, y \in Q\}$$

Each of the above are called semi-nuclei of of Q_e, and their intersection N is the nucleus of Q_e.

We consider here the nuclei of the [pre]quasifields Q_e associated with a fixed spreadset $\mathcal{S}$. Since the choice of Q_e depends on the choice of the left identity e, it is reasonable to ask to what extent the nuclei depend on the choice of e, for a fixed spreadset $\mathcal{S}$. Our aim is to show that, in the finite case, the right and middle nuclei are essentially independent of the choice of e.

As far as the left nucleus N_ℓ is concerned, there is no general coherent theory, probably because this is the only type of nucleus that turns out not to have a geometric interpretation in the general case. However, the kern of a quasifield is contained in its left nucleus and this certainly has a geometric meaning, and is arguably fully understood. Hence we shall not consider further the left nucleus in this section, apart from noting that in the case of finite quasifelds its non-zero elements, as well as those of the other seminuclei, form a multiplicative group.

Remark 5.3.2 *Let Q be a finite quasifield with multiplicative identity e. Then $N_m^*(Q)$, $N_r^*(Q)$ and $N_\ell^*(Q)$ are multiplicative groups, with identity element e.*

Proof: Trivial. ∎
We now show the invariance of the middle nucleus of all the quasifields coordinatizing a given finite spreadset.

Theorem 5.3.3 *Let τ be a finite spreadset. Let $\alpha \subset \tau$ be the largest non-zero subset of τ^* satisfying the condition $\alpha\tau^* \subseteq \tau$; note that this is equivalent to $\alpha\tau^* = \tau$ and α is a group [under map composition] iff the identity is in τ. Let Q_e be the (pre)quasifield coordinatizing τ relative to some chosen left identity $e \in Q^*$. Then the (semi)group*

$$\alpha \cong \left\{\hat{f} : x \mapsto x \circ f \mid f \in N_m^*(Q_e)\right\} \cong N_m^*(Q_e),$$

where $N_m^*(Q_e)$ is viewed as a multiplicative (semi)group.

Proof: The element $f \in Q^*$ lies in $N_m^*(Q_e)$ iff for $x, y \in Q$:

$$\begin{aligned} (x \circ f) \circ y &= x \circ (f \circ y) \\ \iff (xT_f)T_y &= xT_{f\circ y} \\ \iff T_fT_y &= T_{f\circ y} \end{aligned}$$

and this is equivalent to $T_f \in \alpha$, and also shows that that $f \mapsto T_f$ defines a semigroup isomorphism from $N_m^*(Q_e)$ onto α of the required type. The result follows. ∎
Now we consider the analogue of the above with the middle nucleus replaced by the right nucleus.

Theorem 5.3.4 *Let τ be a finite spreadset. Let $\alpha \subset \tau$ be the largest non-zero subset of τ^* satisfying the condition $\tau^*\alpha \subseteq \tau$ note that this is equivalent to $\tau^*\alpha = \tau$ and α is a group [under map composition] iff the identity is in*

τ. Let Q_e be the (pre)quasifield coordinatizing τ relative to some chosen left identity $e \in Q^*$. Then the (semi)group

$$\alpha \cong \left\{\hat{f} : x \mapsto x \circ f \mid f \in N_r^*(Q_e)\right\} \cong N_r^*(Q_e),$$

where $N_r^*(Q_e)$ is viewed as a multiplicative (semi)group..

Proof: The element $f \in Q^*$ lies in $N_r^*(Q_e)$ iff for $x, y \in Q$:

$$\begin{aligned} (x \circ y) \circ f &= x \circ (y \circ f) \\ \iff (xT_y)T_f &= xT_{y \circ f} \\ \iff T_yT_f &= T_{y \circ f} \end{aligned}$$

and this is equivalent to $T_f \in \alpha$, and also shows that that $f \mapsto T_f$ defines a semigroup isomorphism from $N_r^*(Q_e)$ onto α of the required type. The result follows. ∎

We now specialize to nearfields.

Definition 5.3.5 *A quasifield with associative product is called a nearfield.*

A classical theorem of Zassenhaus gives a complete classification of all finite nearfields: apart from fields they are either the Dickson nearfields, introduced ahead, or they are among a finite list of sporadic nearfields called *irregular nearfields*. The results above imply that

Corollary 5.3.6 *Let S be a finite spreadset containing the identity. Then the following are equivalent:*

1. *S^* is a group of non-singular linear maps.*
2. *Some quasifield Q_e coordinatizing S is a nearfield.*
3. *All quasifields Q_e coordinatizing S are nearfields.*

Moreover, if S^* is a group, then all the nearfields coordinatizing S^* have isomorphic multiplicative groups.

In fact, inspecting the isomorphism from α to its nuclei, developed above shows:

Corollary 5.3.7 *All the nearfields coordinatizing a given spreadset are isomorphic as spreadsets.*

So far we have considered multiplicatively closed subsets α of spreadset $\mathcal{S}$. We now turn to the additive version of this theory. To emphasize the analogy with the multiplicative case we introduce a non-standard definition.

Definition 5.3.8 *Let Q be any prequasifield. Then its distributor is the additive semigroup:*

$$\delta(Q) = \{c \in Q \mid x \circ (c + y) = (x \circ c) + (x \circ y) \forall x, y \in Q\}$$

So, at least in the finite case, $\delta(Q)$ is an additive subgroup of Q.

Theorem 5.3.9 *Let τ be a finite spreadset over a finite field K, and $\alpha \subset \tau$ be the largest non-zero subset of τ^* satisfying the condition $\tau^* + \alpha \subseteq \tau$, or equivalently, the condition $\tau^* + \alpha = \tau$; thus α is an additive group of linear maps over K. Let Q_e be the (pre)quasifield coordinatizing τ relative to some chosen left identity $e \in Q^*$. Then there is an additive group isomorphism:*

$$\alpha \cong \left\{\hat{f} : x \mapsto x \circ f \mid f \in \delta(Q_e)\right\} \cong \delta(Q_e)$$

Proof: The element $c \in Q_e$ lies in $\delta(Q_e)$ iff for $x, y \in Q_e$:

$$\begin{aligned} x \circ c + x \circ y &= x \circ (c + y) \\ \iff xT_c + xT_y &= x(T_{c+y} \\ \iff T_c + T_y &= T_{c+y} \end{aligned}$$

and this is equivalent to $T_c \in \alpha$, and also shows that that $c \mapsto T_c$ defines an additive group isomorphism from $\delta(Q_e)$ onto α of the required type. The result follows. ∎

A distributive (pre)quasifield Q is called a pre(semifield). We state this definition in terms of $\delta(Q)$:

Definition 5.3.10 *A (pre)quasifield $(Q, +, \circ)$ is a (pre)semifield if $\delta(Q) = Q$. A semifield is said to be* proper *if its multiplictaion is not associative.*

Theorem 5.3.9 above immediately yields the following characterization of the spreadsets whose associated (pre)quasifields are semifields.

Theorem 5.3.11 *Let τ be a finite spreadset. Then the following are equivalent.*

1. *Some quasifield Q coordinatizing τ is a (pre)semifield.*

2. *τ is additively closed iff every (pre)quasifield Q coordinatizing τ is a (pre)semifield.*

5.4 Hall Systems

Let K be any field. Choose an indeterminate t and consider the rank two left K vector space defined on $Q_t = K + Kt$, where $x + yt \in Q$ is identified with $(x, y) \in K^2$: so addition and scalar multiplication on Q_t are done componentwise:

$$\begin{aligned} \forall x, x', y, y' \in K : (x + yt) + (x' + y't) &= (x + x') + (y + y')t \\ \text{and } \forall k, x, y \in K : k(x + yt) &= kx + (ky)t. \end{aligned}$$

Any quasifield that has rank two over its kernel K may thus be regarded as being of form $(Q_t, +, \circ)$, where addition is standard and the multiplication $\circ$ is an extension of left multiplication by the scalars in $K \subset Q$ with the general elements of Q. Morover, for each $a \in Q_t$ the map

$$\begin{aligned} R_a \;:\; & Q \to Q \\ & x \mapsto x \circ a \end{aligned}$$

is required to be a K-linear bijection of Q_t, and the quasifield $(Q_t, +, \circ)$ is completely specified when all the *slope-maps* R_a, for $a \in Q$ are specified. To specify the R_a's it is now sufficient to write the 2×2 matrix over K for the linear maps R_a relative to the basis $(1, t)$ of Q_t; so R_0 is assigned the zero matrix, and the quasifield identity is assigned the identity matrix.

We now seek to classify all the quasifield $(Q_t, +, \circ)$ associated with the K-vector space Q_t, such that the following conditions hold:

Condition 5.4.1 (Hall Conditions.)

1. *$(Aut(Q_t, +, \circ))_K$ is transitive on $Q - K$; and*

2. *K is central In Q.*

This classification here is the first step towards classification of all the *finite* quasifields that admit maximally transitive automorphism groups, i.e. acting transitively on the non-kern elements.

Since K centralizes Q it centralizes the standard basis $(1, t)$, so the matrices [always relative to the standard basis] of its elements are just the scalars:

$$\forall k \in K : R_k := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

That the above definition is expressed using different notation in definition 6.3.1 ahead.

Partial spreads *all* whose components lie across some subspread are called rational partial spreads.

Definition 5.7.2 *Let (V, Γ) be a partial spread and let A be a non-zero additive subspace of V such that A is a subspread of (V, Γ) and additionally:*

$$\gamma \in \Gamma \Longrightarrow \gamma \cap V \neq \mathbf{O};$$

thus A is a subspread (or a 'subplane') across Γ. *The partial spread (V, Γ) is called a* **rational** *partial spread if Γ has at least one subspread across it. If, additionally, (V, Δ) is a Desarguesian spread such that $\Delta \supset \Gamma$ then (V, Γ) is called a rational Desarguesian partial spread.*

Note that essentially the same definition, but in different terminolgy is covered by definitions 6.3.3 and 6.3.1 ahead.

If Q is a quasifield and R is a subquasifield then the spread $\pi(Q)$, coordinatized by Q, has a subspread that may be identified with $\pi(R)$ and, by definition, the partial spread Γ determined by $\pi(R)$ is rational, with $\pi(R)$ across it. The converse is also true: any rational spread $\Gamma \subset \mathcal{S}$, contained in a spread $(V, \mathcal{S})$, may be 'coordinatized' by a subquasifield R of a quasifield Q coordinatizing $(V, \mathcal{S})$. We now verify this elementary, but fundamental, property of rational partial spreads; it reflects the fact that *subplanes $\mathcal{A}_0$, of any affine plane $\mathcal{A}$, are coordinatized in the classical sense by some subternary ring T_0 of a ternary ring coordinatizing $\mathcal{A}$.*

Remark 5.7.3 *Let $\mathcal{S}$ be a spreadset defined on a vector space T. Suppose that $\pi_{\mathcal{S}} := (T \oplus T, \Sigma)$, the spread coordinatized by $\mathcal{S}$, contains a rational partial spread $\Gamma \subseteq \Sigma$ such that Γ contains the standard components $X = T \oplus \mathbf{O}$, $Y = \mathbf{O} \oplus T$ and $I = \{(t,t) \mid t \in T\}$. Let $U \leq T \oplus T$ be any subspread of $\pi_{\mathcal{S}}$ that lies across Γ. Then*

$$R := \{r \in T \mid r \oplus r \in I \cap U\},$$

is a subspace of T such that $U = R \oplus R$, and for each $e \in R^*$, the quasifield $Q_e = (T, +, \circ)$, coordinatizing the spreadset $\mathcal{S}$, contains the system $G_e = (R, +, \circ)$ as a subquasifield and the standard isomorphism from $\pi(Q_e)$ onto $\pi_{\mathcal{S}}$:

$$\Psi : \pi(Q_e) \to \pi(T \oplus T, \mathcal{S}),$$

and hence, since R_k is the scalar map $k\mathbf{1}_2$, Q_t is not a quasifield unless

$$a \in Q_t - K \Longrightarrow R_a \text{ has no eigenvalues in K.}$$

But the eigenvalues of R_t are just the roots of $f(x)$ so we have established:

Lemma 5.4.2 *Q_t cannot be a quasifield satisfying the Hall conditions 5.4.1 unless the common quadratic $f(x) = -x^2 + \alpha x + \beta$ is irreducible over K.*

Hence we shall assume that $f(x)$ is irreducible from here on: so $\beta \neq 0$. But since the determinant of R_{a+bt} is just β it follows that every non-zero R_{a+bt} is non-singular and the quasigroup condition on multiplication $(x, y) \circ (a, b) = (c, d)$ is met. To meet the remaining condition for quasigroup multiplication $(a, b) \circ (x, y) = (c, d)$, where (x, y) is the 'unkown', we first note that if $(c, d) = k(a, b)$ then $(x, y) = (k, 0)$ is a solution. Thus our main task, to show that quasigroup multiplication works, requires us to show that a solution for (x, y) exists in the following matrix equation:

$$(a, b)\begin{pmatrix} x & y \\ \frac{1}{y}f(x) & \alpha - x \end{pmatrix} = (c, d), \quad ad - bc \neq 0, \tag{5.4}$$

and, tacitly assuming $ad - bc \neq 0$, the equation may be written

$$\begin{aligned} ax + \frac{b}{y}f(x) &= c \\ ay + b(\alpha - x) &= d \end{aligned}$$

which obviously has a solution if $b = 0$. So assuming from now on that $b \neq 0$, we obtain from the above:

$$\begin{aligned} axy + b(\beta + \alpha x - x^2) &= cy \\ axy + b(\alpha x - x^2) &= dx \end{aligned}$$

yielding on recalling equation (5.5):

$$\begin{gathered} cy - dx = \beta \\ ay - bx = d - b\alpha \end{gathered}$$

and now our assumption $ad - bc \neq 0$ shows that this equation has a unique solution for (x, y), and this back-traces to establish a unique solution for the equation (5.4). Thus the multiplication specified is a loop, and as a consequence $(Q_t, +, \circ)$ is a quasifield: the reader is invited to check the minor details that have not been explicitly discussed.

To verify that the group $G = (Aut(Q_t, +, \circ))_K$ does act on the quasifield $(Q_t, +, \circ)$ we note that G, as a matrix group relative to $(1, t)$, is clearly the group:

$$\left\{ \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \mid u \in K, v \in K^* \right\},$$

and it can be directly verifed that this group preserves the multiplication. Thus we have established:

Theorem 5.4.3 *Suppose K is a field and $f(x) = -x^2 + \alpha x + \beta$ is an irreducible quadratic over K. Let $Q = K \oplus \mathbf{O}$ and define $Q_f := (Q, +, \circ)$, where $+$ is the standard addition on $K \oplus K$, by*

$$\forall a \in K : (a, b) \circ (x, 0) = (ax, bx),$$

and

$$\forall x \in K, y \in K^* : (a, b) \circ (x, y) = \begin{pmatrix} x & y \\ \frac{1}{y} f(x) & \alpha - x \end{pmatrix}.$$

Then Q_f is a quasified iff $f(x)$ is irreducible in K, and when this is the case $\overline{K} = K \oplus K$ is in the kern of Q_f and centralizes the quasifield multiplicatively.

Let $G = Aut(Q_f)_{\overline{K}}$ be the elementwise stabliser of the kern field $\overline{K}$ in the automorphism group of the quasifield. Then G is regular on the set of all non-kern elements $K \oplus K - K \oplus 0$ of Q_f. Such Q_f are called Hall systems.

Conversely if a quasifield Q is rank two over its kern K such that K centralizes Q and $(AutQ)_K$ has $Q - K$ as an orbit then Q is a Hall sytem.

Exercise 5.4.4

1. *Show that $GF(4)$ may be regarded as a Hall system and all other Hall systems are of dimension exactly two over K.*

2. *Show that $GF(4)$ is the only Hall sytem which is also a field.*

3. *Show that no Hall system can have an algebraic-closed field as its kern.*

5.5 Coordinatizing Spreads by Spreadsets.

Let $\pi = (V, \Gamma)$ be a spread over a skewfield K; so all its components are isomorphic as vector spaces to a common vector space W. We seek to identify π with τ_W, the spread on $W \oplus W$ coordinatized by some spreadset τ; we shall regard a K-linear isomorphism ψ from π to τ_W as being a coordinatization of π by τ.

Our goal here is to show that every spread is coordinatized by at least one spreadset τ, and that τ can be chosen so that it contains the identity. It is also possible to ensure that τ and the coordinatizing isomorphism ψ may be chosen so that any ordered triple of distinct components (X_V, Y_V, Z_V) are mapped under ψ to the ordered triple $(y = 0, x = 0, y = x)$, in $W \oplus W$. However, it is desirable to consider the more general situation, where X_V and Y_V are mapped respectively to $W \oplus \mathbf{0}$ and $\mathbf{0} \oplus W$, but where no component is necessarily required to be mapped to the unit line $x = y$; for example, it is often useful to have ψ send a Baer subplane of (V, Γ) onto the unit line of $W \oplus W$.

Theorem 5.5.1 (Coordinatizing Spreads By Spreadsets.) *Let $\pi = (V, \Gamma)$ be a spread over a skewfield K, such that all the components in Γ are isomorphic as K-vector spaces to a K vector space W. Let $\alpha : X \to W$, $\beta : Y \to W$ be arbitrary vector space isomorphisms from two distinct $X, Y \in \Gamma$ onto W. Then*

1. *There is a unique linear bijection:*

$$\alpha \oplus \beta : V \to W \oplus W,$$

whose restrictions to X and Y are respectively α and β.

2. *Each $T \in \Gamma - \{X, Y\}$ is associated with a unique pair of linear bijections*

$$(X_T : T \to X, Y_T : T \to Y),$$

such that:

$$T = \{(t)X_T + (t)Y_T : t \in T\}.$$

3. *The set of linear maps on W specified by:*

$$\tau = \left\{\alpha^{-1} X_T{}^{-1} Y_T \beta \mid T \in \Gamma\right\} \cup \{\mathbf{0}_W\},$$

is a spreadset on W, and $\alpha \oplus \beta : V \longrightarrow W \oplus W$ is a K-linear isomorphism from the spread (V, Γ) onto the spread π_τ coordinatized by τ, see definition 5.2.7.

4. *τ contains the identity, or equivalently the unique line lies in W_τ if and only if $\alpha = \beta$.*

Proof: We give a sketch; it is left to the reader to make the proof more precise. The components of V, regarded as $X \oplus Y$, are of form $((t)X_T, (t)Y_T$, and may be rewritten $(x, (x){X_T}^{-1}Y_T)$, or $(x, M_T(x))$, where $M_T := {X_T}^{-1}Y_T$ is essentially the slope of T. Now τ is essentially the set of images of the M_T's, together with the zero-map of W, induced on W when V is identified iwth $W \oplus W$ using $\alpha \oplus \beta$. ∎

Thus all spreads are coordinatized by some spreadset. Hence we may assume that any spread is of type W_τ and three selected components are $x = 0$, $y = 0$ and $z = 0$ respectively.

5.6 Inventory of Quasifields Coordinatizing a Fixed Spread.

From now on, *a prequasifield will always be assumed to have at least left identity.* We are here concerned with the description of all the non-isomorphic prequasifields Q such that the associated spread is is isomorphic to a given spread π. If ϕ is an isomorphism from π to $\pi(Q)$ then ϕ will be called a *coordinatization* of π by Q. Thus we are concerned with the description of all the non-isomorphic prequasifields that coordinatize π.

We now describe a concrete procedure that yields a Q coordinatizing the given π uniquely once certain geometric choices are made, and also leads to a unique isomorphism Ψ from π onto $\pi(Q)$, in terms of certain 'geometric' options: the choice of the x-axis, the y-axis, etc. We shall see that the isomorphism types of all (pre)quasifelds Q such that $\pi(Q) \cong \pi$ may be obtained as an image of some Ψ determined by fixing the geometric options.

5.6.1 Coordinatization Algorithm.

There are two basic situations to consider: determine all the isomorphism types for the prequasifields coordinatizing a spread, and also all the quasifields, with a two-sided multiplicative identity, coordinatizing the spread. We first describe all the prequasifield coordinatizing a given spread, and then specialize to describe all the quasifields coordinatizing it.

Let $\pi = (V, \Gamma)$ be a spread on a K-space V, K a skew-field, such that the components are all K-subspaces of V.

1. Choose distinct components $X_V, Y_V \in \Gamma$; these are called the x and y-axis of the coordinatization scheme.

2. Choose a *unit point* $u \in V - (X_V \cup Y_V)$, and hence: $u = u_x \oplus u_y$; so u_x and u_y are the projections of u on X and y.

3. Let W be a K-space isomorphic to the members of Γ, and choose an *identity* $e \in W - \{0\}$.

4. Select linear bijections $\alpha : X_V \to W$, and $\beta : Y_V \to W$ such that $\alpha(u_x) = \beta(u_y) = e$.

5. The linear bijection $\alpha \oplus \beta : V \to W \oplus W$ defines a spread on $W \oplus W$ whose component-set is given by:

$$\Delta = \{\alpha \oplus \beta(\gamma) \mid \gamma \in \Gamma\} \cup \{0 \oplus W\}.$$

   Thus $\alpha \oplus \beta$ is a K-linear isomorphism from V onto $W \oplus W$ that is also an isomorphism from the spread (V, Γ) onto the W-labelled spread $(W \oplus W, \Delta)$, and this isomorphism sends u to (e, e).

6. Let Q_e be the standard prequasifield coordinatizing $(W \oplus W, \Delta)$, and let σ be the associated K-linear isomorphism from $(W \oplus W, \Delta)$ onto $\pi(Q_e)$.

The K-linear bijection

$$\sigma(\alpha \oplus \beta) : V \to W \oplus W$$

is a K-linear spread isomorphism from (V, Γ) onto $\pi(Q_e)$ such that u is mapped onto e. The prequasifield Q_e is said to coordinatize (V, Γ) relative to the axes X_V, Y_V, the unit point u and identifiers α and β; the kern of Q_e contains K.

Let v denote the component in Γ that passes through u. Choose any K-linear bijection $\Xi : v \to W$ such that $\Xi(u) = e$, and define

$$\forall z \in v \alpha(\pi_X(z)) = \Xi(z) = \beta(\pi_Y(z)),$$

where π_X and π_Y denote the projection of V onto V_X and V_Y respectively. Such α and β are completely determined Ξ and satisfy all the requirements of α and β as defined earlier. In this case the resulting prequasifield is a quasifield, and we call it the quasifield obtained when (V, Γ) is assigned a labelling with v as unit line relative to the coordinate axes V_X and V_Y.

Every coordinatization of π by a quasifield is obtainable by a labelling relative to some unit line and point, and a pair of X and Y axis. (N.B. The statement is intended to imply that the isomorphism onto W is immaterial, once the unit point and all three axes are fixed: it is pointless to make other variations in the choice of Ξ as this will not yield coordinatizations by any new quasifields.)

5.6.2 Properties Of Coordinatization.

Theorem 5.6.1 *Let $\pi = (V, \Gamma)$ be a spread, coordinatized by a quasifield $Q_e = (W, +, \circ)$, $e \in W^*$ is the identity. Thus there is a linear bijection*

$$\Psi : V \to W \oplus W$$

such that Ψ is also an isomorphism from the spread π onto $\pi(Q)$. Let $\mathbf{u} = (u_1, u_2)$ denote the unit point, so $\Psi(\mathbf{u}) = (e, e)$. Then

1. *If A is a subspread of V*[1]*, that contains the coordinate frame $\mathbf{e}$, the x-axis and the y-axis then $\Psi(A)$ is a subquasifield A_Q of Q_e; thus A is coordinatized by the A_Q relative to the 'same frame', as used on π to yield $\pi(Q)$; the labelling map for A is the restriction of $\Xi : U \to W$ to $A \cap U$.*
   Conversely, if R is a subquasifield of Q then $R = A_Q$, where A is a subspread of type just described.

2. *Suppose $\alpha \in GL(W, +)$. Then α is an automorphism of $(Q, +, \circ)$ iff the map $\tilde{\alpha} : (x, y) \mapsto (x^\alpha, y^\alpha)$ of $\pi(Q)$ is a collineation of the plane $\pi(Q)$ that fixes (e, e). Now $Fix(\alpha)$ is a subquasifield A of Q, and $Fix(\tilde{\alpha})$ is the subplane $\pi(A)$ of $\pi(Q)$.*

3. *If a group $G \leq GL(W, +)$ is in $Aut(Q, +, \circ)$, and A denotes the subquasifield $Fix(G)$, then G is permutation isomorphic to the (clearly*

[1] In the sense that it is an additive subgroup and the components meeting it non-trivially define a spread on it.

faithful) action of the collineation group $\tilde{G}$ retricted to any line that it fixes. Conversely, any collineation group acting on $\pi(Q)$ and fixing the unit point and the axes must be of type $\tilde{G}$, and such groups are planar, in fact, their fixed points define the subplane $\pi(A)$, where $A = Fix(G)$.

Thus, subquasifields of a quasifield Q, and subplanes of $\pi(Q)$ containing the unit point, are linked by a natural one-one correspondence. Similarly, there is a natural correspondence between subgroups of $Aut(Q)$ and planar collineation groups of $\pi(Q)$ that fix the two axis and the unit point, and the correspondence is such that the action of the collineation on any fixed component is isomorphic as an additive group to the action of the corresponding subgroup of $Aut(Q)$ on Q.

Of course, using the coordinatizing isomorphism, we can extend these links in the obvious way to encompass subgroups and subplanes of any spread coordinatized Q. These connections are freely used in the literature, without explicit reference, and we shall normally follow this practice. However, even at the cost of being repetitive, we shall consider all this explictly in the following section, without referring to the above analysis, for the very important case associated with rational partial spreads.

5.7 Coordinatizing Rational Partial Spreads.

Given a spread $(V, \mathcal{S})$, we regard a subspace $A \leq V$ as being a *subspread* of $(V, \mathcal{S})$ if the components $\sigma \in \mathcal{S}$ that meet A non-trivially induce a spread on A. More generally:

Definition 5.7.1 *Let $(V, \mathcal{S})$ be a [partial] spread and suppose A is a non-zero additive subspace of $(V, +)$. Thus*

$$\mathcal{S}(A) := \{s \in \mathcal{S} \mid s \cap A \neq 0\}$$

denotes the set of components in $\mathcal{S}$ that meet A non-trivially. The subspace A is called a subspread *of the [partial] spread $(V, \mathcal{S})$ if*

$$\mathcal{S}_A = \{s \cap A \mid s \in \mathcal{S}(A)\}$$

is the set of components of a spread on A.

In general, if A is a subspread, of a partial spread $(V, \mathcal{S})$, then $\mathcal{S}(A)$ is the partial spread determined *by the subspace A, and A is said to be a subspace* across *the partial spread $\mathcal{S}(A)$.*

That the above definition is expressed using different notation in definition 6.3.1 ahead.

Partial spreads *all* whose components lie across some subspread are called rational partial spreads.

Definition 5.7.2 *Let (V, Γ) be a partial spread and let A be a non-zero additive subspace of V such that A is a subspread of (V, Γ) and additionally:*

$$\gamma \in \Gamma \Longrightarrow \gamma \cap V \neq \mathbf{O};$$

thus A is a subspread (or a 'subplane') across Γ. *The partial spread (V, Γ) is called a* **rational** *partial spread if Γ has at least one subspread across it. If, additionally, (V, Δ) is a Desarguesian spread such that $\Delta \supset \Gamma$ then (V, Γ) is called a rational Desarguesian partial spread.*

Note that essentially the same definition, but in different terminolgy is covered by definitions 6.3.3 and 6.3.1 ahead.

If Q is a quasifield and R is a subquasifield then the spread $\pi(Q)$, coordinatized by Q, has a subspread that may be identified with $\pi(R)$ and, by definition, the partial spread Γ determined by $\pi(R)$ is rational, with $\pi(R)$ across it. The converse is also true: any rational spread $\Gamma \subset \mathcal{S}$, contained in a spread $(V, \mathcal{S})$, may be 'coordinatized' by a subquasifield R of a quasifield Q coordinatizing $(V, \mathcal{S})$. We now verify this elementary, but fundamental, property of rational partial spreads; it reflects the fact that *subplanes $\mathcal{A}_0$, of any affine plane $\mathcal{A}$, are coordinatized in the classical sense by some subternary ring T_0 of a ternary ring coordinatizing $\mathcal{A}$.*

Remark 5.7.3 *Let $\mathcal{S}$ be a spreadset defined on a vector space T. Suppose that $\pi_{\mathcal{S}} := (T \oplus T, \Sigma)$, the spread coordinatized by $\mathcal{S}$, contains a rational partial spread $\Gamma \subseteq \Sigma$ such that Γ contains the standard components $X = T \oplus \mathbf{O}$, $Y = \mathbf{O} \oplus T$ and $I = \{(t,t) \mid t \in T\}$. Let $U \leq T \oplus T$ be any subspread of $\pi_{\mathcal{S}}$ that lies across Γ. Then*

$$R := \{r \in T \mid r \oplus r \in I \cap U\},$$

is a subspace of T such that $U = R \oplus R$, and for each $e \in R^*$, the quasifield $Q_e = (T, +, \circ)$, coordinatizing the spreadset $\mathcal{S}$, contains the system $G_e = (R, +, \circ)$ as a subquasifield and the standard isomorphism from $\pi(Q_e)$ onto $\pi_{\mathcal{S}}$:

$$\Psi : \pi(Q_e) \to \pi(T \oplus T, \mathcal{S}),$$

identifies $\pi(G_e)$ with the subspread $\pi(R \oplus R, \Gamma)$ of $\pi(T \oplus T, \mathcal{S})$; thus $\pi(G_e)$ represents a standard coordinatization of (U, Γ), relative to $e \in R^*$, by the quasifield G_e.

Conversely, given a spread (V, Σ) coordinatized by a subquasifield $Q = (T, +, \circ)$, such that $Q_0 := (R, +, \circ)$ is a subquasifield (so they share the multiplicative identity), then the components $y = x \circ r$, $r \in R$, along with $Y := \mathbf{O} \oplus T$, defines a rational partial subspread of (V, Σ), across $\pi(Q_0)$.

Proof: The converse part is a matter of unravelling the terminology, so we only consider '$\Rightarrow$'. Since U meets the three standard components, it is evident that R is a subspace of the vector space T, and $\{r \oplus r \mid r \in R\}$ is a component of U. Thus, any line $x = r$, for $r \in R$, is a line of the translation plane associated with U and hence $x = r$ meets X in U, and this clearly implies that $X_R := R \oplus \mathbf{O}$ is a component of the spread U. Similarly, $Y_R := \mathbf{O} \oplus R$ is also a component of of U and this means $U = X_R \oplus Y_R = R \oplus R$, in particular $R \oplus R$ is an additive subspace of $T \oplus T$.
We now show that the elments of Γ, other than Y are of form $y = x \circ r$, for some $r \in R$. First observe that any member $\gamma \neq Y$, of the spread $\pi_{\mathcal{S}}$, has form $y = x \circ g$ for some $g \in T$. Now choosing $x = e$ shows $(e, g) \in \gamma)$. Hence, since Γ is the partial spread determined by $U = R \oplus R$, it follows that all members of $\Gamma \setminus \{Y\}$ are of form $y = x \circ r$, for some $r \in R$, and conversely that all such components $y = x \circ r$, $r \in R$, lie Γ.
But for $r, c \in R$, $y = x \circ r$ and $x = \circ c$ are two non-parallel lines of the affine plane associated with U, so their intersection point $(c, c \circ g) \in R \oplus R$, hence R is closed under the binary operation $\circ$. But since Q_e has no zero divisors, it follows that R^* is a subloop of $(T, \circ)$,both with the same identity e. Thus we have established that $(R, +, \circ)$ is such that $(R, +)$ is an additive group with a zero, $(R^*, \circ)$ is a loop and left or right multiplication by zero always yields zero, since $(T, +, \circ)$ is a quasifield. Thus we clearly have a zero-linked system $(R, +, \circ)$, see definition 4.2.1 satisfying the right distributive law and hence, by proposition 4.2.3, $(R, +, \circ)$ is a translation algebra. But the associated incidence structure $\pi(R, +, \circ)$ is, by hypothesis, an affine translation plane, and now, by proposition 4.2.4, $(R, +, \circ)$ is a quasifield. ■