Chapter 4

Quasifields And Their Variants.

Quasifields coordinatize translation planes. In the finite case, these are basically non-associative division rings but possibly missing a distributive law and a multiplicative identity. Here we consider, alternative approaches to the definition, and the problems that arise in the infinite case.

4.1 Quasigroups and Loops.

A binary system $(X, \circ)$ is a *quasigroup* if:

$$a, b, c \in X \Longrightarrow \exists! x, y \in X \ni a \circ x = c \wedge y \circ b = c,$$

or

"Two in $x \circ y = z$ fixes Third."

If a two-sided multiplicative identity exists in a quasigroup then it is a *loop*, thus, loops additionally satisfy:

$$\exists e \in X \ni \forall x \in X : x \circ e = e = e \circ x.$$

Exercise 4.1.1 *Let $(X, \circ)$ denote a quasigroup.*

1. *A loop has a unique identity e, and every one-sided identity is two-sided and hence must coincide with e.*

2. *If $(X, \circ)$ is a finite loop with identity e and $Y \subset X$ is a non-empty set closed under $\circ$, then $(Y, \circ)$ is a loop iff $e \in Y$.*

3. *Show that the finiteness hypothesis is essential above: consider the case when X is a group.*

4. *Let Y be a set and suppose $C : X \to Y$, $A : X \to Y$ and $B : X \to Y$ denote bijections. Then define $(Y, *)$ by:*

$$\forall x, y \in X : (x \circ y)C = (xA) * (yB).$$

Show that $(Y, *)$ is a loop.

5. *Define the cartesian product of a family of quasigroups and hence demonstrate the ubiquity of quasigroups and non-associative loops. In particular resolve the following question:*
Is there a non-associative loop of order n for all integers $n > 2$?

Now if $(X, \circ)$ and $(Y, \circ)$ are related by a triple of bijections $\mu = (A, B, C)$ then the triple is called an *isotopism* from $(X, \circ)$ to $(Y, \circ)$; the latter is called an isotope of the former: isotopism is an equivalence relation. The set of isotopisms from $(X, \circ)$ to itself are called its autotopisms. Compostion of isotopisms are defined in the natural way, and under this defintion the autotopisms of a quasigroup $(X, \circ)$ form a group: its *autotopism group*. The *automorphism* group of $(X, \circ)$, in the usual sense, are just the autotopisms satisfying $A = B = C$; similarly the isomorphisms from one quasigroup to another are just the isotopisms with all three components in agreement.

Exercise 4.1.2

1. *Assume $(X, \circ)$ is a quasigroup. Choose $e \in X$ and define the binary operation $*$ on X by:*

$$\forall x, y \in X : x \circ y = (x \circ e) * (e \circ y).$$

Show that $(X, *)$ is a loop with identity $e \circ e$.

2. *Show that every quasigroup is isotopic to a loop.*

3. *Show that every loop admits autotopisms that are not automorphisms.*

4. *Show that every quasigroup $(X, \circ)$ is isotopic to a quasigroup $(X, *)$ such that the two quasigroups are* non-*isomorphic.*

5. *Every finite group G is the automorphism group of a finite abelian group, e.g., G lies in infinitely many $GL(n, q)$. The question arises: Is every finite group an autotopism/automorphism group of at least one non-associative loop? [What if the non-associative requirement is dropped?]*
   Suggestion: G can be viewed as a planar group of some free plane, and this can easily be chosen so that the fixed plane can be coordinatized by a ternary ring with non-associative multiplication.

4.2 Translation Algebras and Quasifields.

In this section, we consider certain choices for the definition of a quasifield — the systems coordinatizing affine translation planes. For example, some translation planes have simpler representations when they are coordinatized by certain 'quasifields' with the multiplicative identity missing — prequasifields. Also, the simple axioms that characterise finite quasifields and prequasifields, do not yield translation planes in the infinite case — so the structures that satisfy the natural axioms for finite quasifields have sometimes been called 'weak' quasifields [18]. To put things in perspective we shall make a brief examination of the most general such systems in this section: these are 'weak-pre-quasifields', but we prefer to call them simply translation algebras, and we define [pre]-quasifields as the translation algebras that coordinatize translation *planes*, rather than more general *combinatorial* structures. The reader is invited to complete the 'André theory' for translation structures that is hinted at here.

If $(K, +, \circ)$ is a skewfield then the associated incidence structure is an affine Desarguesian plane $\Pi(K)$, whose points are the members of $K \oplus K$ and whose lines are all sets of points that are of form $y = x \circ m + c$ or $x = k$, for $m, c, k \in K$. More generally, one might consider structures of type $(Q, +, \circ)$ such that the associated incidence structures $\Pi(Q)$, obtained as above, are non-Desarguesian affine planes. Affine planes coordinatized by *cartesian groups* are of the form $\Pi(Q)$, where $(Q, +)$ is a group.

Our interest in such systems is restricted to the case when $(Q, +)$ is an abelian group: this will allow us to deal simultaneously with the notions of prequasifields, weak quasifields, pre-weak quasifields..., which become unavoidable in the study of translation planes: many translation planes have their simplest forms when they are expressed in terms of *pre*-quasifields, and the associated objects in the infinite case are 'weak'.

Now if $(Q,+)$ is an abelian group then the additive group $Q \oplus Q$ admits a natural translation group τ, consisting of all bijections

$$\begin{aligned}\tau_{(a,b)} : Q \oplus Q &\rightarrow Q \oplus Q \\ (x,y) &\mapsto (x,y)+(a,b),\end{aligned}$$

for $a, b \in Q$. Thus τ is regular on the points of $\Pi(Q)$, when $(Q,+,\circ)$ is such that the additive group $(Q,+)$ is abelian. Our interest in $(Q,+,\circ)$ is restricted to the case when τ is, additionally, a collineation group of the incidence structure, and $x \circ 0 = 0 \circ x = 0$, where 0 is the identity of the additive group $(Q,+)$.

In the finite case, this simply turns out to mean that $\Pi(Q)$ is a translation plane, and eventually it will be shown that all finite translation planes are of this type. In the infinite case, $(Q,+,\circ)$ becomes a 'weak' pre-quasifield: the incidence structure $\Pi(Q)$ may fall short of being an affine plane, although still admitting the transitive translation group τ.

Definition 4.2.1 *$Q = (V,+,\circ)$ is called a zero-linked system if:*

1. *$(V,+)$ is an abelian group with neutral element 0;*
2. *$V^* = V - \{0\}$ is a quasigroup;*
3. *$0 \circ x = 0 = x \circ 0 \forall x \in V$,*

The set-theoretic incidence structure $\Pi(Q)$, coordinatized by Q, is defined to have $V \oplus V$ as its points, and its lines are the subsets of $V \oplus V$ that may be expressed in the form

$$\forall m, b \in K : y = x \circ m + b := \{(x, x \circ m + b) \mid x \in V\},$$

or

$$\forall k \in K : x = k := \{(k,y) \mid y \in V\}.$$

The zero-linked system $Q = (V,+,\circ)$ is called a translation algebra *if additionally the translation group of the additive group $V \oplus V$, viz:*

$$\tau := \{\tau_{a,b} : (x,y) \mapsto (x+a, y+b) \mid (a,b) \in V \oplus V\}$$

is a collineation group of $\Pi(Q)$.

A translation algebra is called a pre-quasifield *if $\Pi(Q)$ is an affine plane (and hence an affine translation plane).*

Exercise 4.2.2 *Let $(Q, +, \circ)$ be any zero-linked structure.*

1. *Show that the group*

$$\Theta := \{(x, y) \mapsto (x, y + b) \mid b \in Q\}$$

   is a translation group of $\Pi(Q)$.

2. *Give examples of finite $(Q, +, \circ)$ such that $\Pi(Q)$ is not a translation plane.* Consider coordinatizing a dual translation plane.

3. *Are all zero-linked systems translation algebras?*

The following proposition means that a *finite* translation algebra is the same things as a *finite* prequasifield.In the infinite case a translation algebra is the same thing as a 'weak (pre)quasifield' in the sense of Hughes and Piper. Thus, translation algebras are introduced (temporarily) to refer to the same objects that have been given different names in the finite and infinite situations.

Proposition 4.2.3 *Let $(Q, +, \circ)$ be a zero-linked system. Then it is a translation algebra iff the right distributive law holds:*

$$\forall a, b, c \in Q : (a + b) \circ c = a \circ c + b \circ c.$$

Proof: Assume the translations $\tau_{a,b} : (x, y) \mapsto (x + a, y + b)$, of $Q \oplus Q$, permute the lines of $\Pi(Q)$. So

$$\begin{aligned} y = x \circ m + c &\mapsto y = x \circ m' + c' \\ \Rightarrow \{(x + a, x \circ m + c + b) \mid x \in Q\} &= \{(x + a, (x + a)m' + c') \mid x \in Q\} \\ \Rightarrow x \circ m + c + b &= (x + a) \circ m' + c' \\ \text{So by } x \leftarrow 0 : -a \circ m' + c + b &= c' \\ \text{So} : x \circ m + c + b &= (x + a) \circ m' + (-a \circ m' + c + b) \\ \Rightarrow x \circ m + a \circ m' &= (x + a) \circ m' \end{aligned}$$

and the result follows because all translations must be permitted. The converse, that the right distributive law implies that τ is a collineation group of $\Pi(Q)$, is just as easy. ∎

The following proposition gives the standard condition for a translation algebra, finite or infinite, to be a prequasifield in the usual sense of the term.

Proposition 4.2.4 *Let $(Q, +, \circ)$ be a translation algebra. Then $\Pi(Q)$ is a translation plane if and only if:*

$$\forall a, c, d \in Q, a \neq c(\exists! x \ni x \circ a - x \circ c = d)$$

Proof: We need to check that the incidence structure $\Pi(Q)$ is an affine plane iff the condition holds. We verify that for any translation algebra, distinct points (a, b) and (c, d) of $\Pi(Q)$ lie on a unique line. When $a \neq c$, then the equations:

$$\begin{aligned} a \circ m + b &= n \\ c \circ m + d &= n, \end{aligned}$$

together with the right distributive law and the quasigroup property of Q^*, enable m and n to be uniquely determined since we have: $(a - c) * m = -(b - d)$. And if $a = c$ then '$x = c$' is the only common line. So two points meet, and clearly parallel lines, meaning those with the 'same slope', do not meet. Hence for $\Pi(Q)$ to be an affine plane everything clearly depends on whether or not the lines '$y = x \circ a + b$' and '$y = x \circ c + d$' meet for $a \neq c$. But these lines meet at points whose X-coordinates x satisfy:

$$x \circ a + b = x \circ c + d$$

and this equation has a unique solution iff:

$$x \circ a - x \circ c = d - b,$$

and this is the given condition. The result follows. ■

Corollary 4.2.5 *Finite translation algebras and distributive translation algebras always coordinatize translation planes.*

Proof: Using the notation of proposition 4.2.4 above, the map $\theta : x \mapsto x \circ a - x \circ c$ is an additive map, and its kernel corresponds to x satisfying $x \circ a = x \circ c$, contradicting the quasigroup hypothesis on $W^*, \circ)$, unless θ is injective. So in the finite case θ is certainly bijective. In the general case, when $\circ$ is distributive, the distributive law yields the identity $-u \circ v = u \circ (-v)$ and hence also $\theta(x) = x \circ (a - c)$. So distributivity implies that θ is bijective since $(W^*, \circ)$ is a multiplicative loop. Thus in both cases, finite or distributive, θ

is bijective whenever $a \neq c$. Hence proposition 4.2.4 yields the desired result. ■

Thus the concept of a translation algebra coincides with that of a pre-quasifield (structures that coordinatize affine translation planes) in the *finite case* or when *both the distributive law* holds.

4.3 Schur's Lemma, Slope Maps and Kern.

In this lecture we introduce some tools and concepts essential for the study of spreads and translation planes. We begin by recalling Schur's lemma, a result that plays a central part in spread theory. We shall use it in a moment to show that all translation algebras are built on vector spaces.

Result 4.3.1 *[Schur's Lemma.] If V and W are irreducible modules and $\Phi : V \longrightarrow W$ is a non-trivial linear map from V to W then Φ is a bijective isomorphism.*

Proof: The kernel of Φ is trivial because V is irreducible and Φ is surjective because its image is a submodule of W. ■

We have met the concept of slopesets (or slope maps) of a spread. We now turn to slope maps of a translation algebra. We shall eventually see that slope maps associated with a translation algebra and those associated with a spread are essentially identical concepts.

Definition 4.3.2 (Slope Maps) *Let $Q = (W, +, \circ)$ be a translation algebra. Then the endomorphisms of $(W, +)$ of form: $\tau_a : x \mapsto x \circ a$ are its* slope maps. *$\tau = \{\tau_a \mid a \in W\}$ is the* slope-set *of the translation algebra Q.*

We can now apply Schur's lemma to show that translation algebras, of all types, are built on vector spaces and that their non-zero slope maps are *non-singular* relative to the vector structure.

Lemma 4.3.3 (Kern Endomorphisms.) *Let $Q = (V, +, \circ)$ be a translation algebra; so its slopeset τ consists of a subset of $Hom(V, +)$ such that τ_a, for all $a \in V^*$, are bijective members of $Hom(V, +)$. Let K be the centralizer of τ in $Hom(V, +)$. Then the following apply:*

1. *K is a skewfield whose non-zero elements are all bijections in $Hom(V, +)$;*

2. *τ consists of K-linear maps of V when V is viewed as a vector space over K: thus, $\theta \in \tau$ implies:*

$$vk\theta = v\theta k \forall k \in K, \theta \in \tau.$$

Proof: The quasigroup condition on V^* shows that τ^* generate a group acting transitively on V^*, and so the group $< \tau^* >$ is irreducible. Now apply Schur, lemma 4.3.1. ■
The skewfield K of the lemma will be called the external *kern* of the translation algebra:

Definition 4.3.4 (External Kern.) *Let τ be the set of slope maps of a translation algebra $Q = (V, +, \circ)$. Then the the centralizer of τ in $Hom(V, +)$ is the [external] KERN of Q, and also of τ; these are denoted by $kern(Q)$ and $kern(\tau)$ resp.*

The following remarks follow from lemma 4.3.3 and the definition of the kern of a translation algebra. It might be helpful to remind the reader that all prequasifields are translation algebras and in the finite case both concepts coincide.

Remarks 4.3.5 *Let K be the kern of a translation algebra $(Q, +, \circ)$. Then the following hold.*

1. *The additive group $Q \oplus Q$ becomes a vector space relative to the operation:*

$$k(x, y) := (x^k, y^k) \forall k \in K, x, y \in Q.$$

This is always taken as the STANDARD kern action on $Q \oplus Q$.

2. *The standard action of K^* on $Q \oplus Q$ induces faithfully a group of collineations of $\Pi(Q)$ that fixes $(0, 0)$ and all the lines through it. Conversely every additive bijection of $Q \oplus Q$ that fixes every line through the origin $(0, 0)$ is of form $(x, y) \mapsto (x^k, y^k)$, $k \in K^*$.*

Thus the above remark shows that the concept of kern homolgies, associated with a translation plane, carries over to a considerable extent to $\Pi(Q)$, where Q is a translation algebra.

Exercise 4.3.6 *To what extent does the André theory of spreads and translation planes carry over to $\Pi(Q)$, the incidence structure associated with translation structures? For example, resolve the following questions:*

1. *Is the full group of 'dilations' of* $\Pi(Q)$ *just the group* τK^**?*

2. *Does every collineation* σ *fixing the origin an element of* $\Gamma L(Q \oplus Q, K^*)$*, the group of non-singular semilinear maps of the* K*-space* $Q \oplus Q$.

The obvious approach to the above exercise is to try and imitate the André theory. However, since we only deal with translation structures that are quasifields, and thus by *definition* $\Pi(Q)$ is a translation plane, we already have available the complete answer to such questions by André theory.

We now establish a simple result of fundamental importance: the kern of any quasifield $(Q, +, \circ)$, as opposed to a *pre*-quasifield, may be defined in two equivalent ways — as the centralizer of the slope maps of $(Q, +, \circ)$ in $Hom(Q, +)$, as done earlier, definition 4.3.4, and as the sub[skew] field of $(Q, +, \circ)$ consisting of the elements in the left nucleus $N_\ell(Q)$ that distribute from the left — the internal kern.

Definition 4.3.7 (Internal Kern.) *Let* $Q = (V, , +, \circ)$ *be a translation algebra with multiplicative identity* e. *Then the* INTERNAL *kern* $\kappa(Q)$ *of* Q *is:*

$$\{k \in Q \mid \forall x, y \in V : (k \circ (x + y) = k \circ x + k \circ y) \wedge (k \circ x) \circ y = k(\circ x \circ y)\} .$$

The following result establishes the equivalence of the external and the internal kern, c.f., definition4.3.4 and definition 4.3.7.

Proposition 4.3.8 *Let* $(Q, +, \circ)$ *be a translation algebra that has a multiplicative identity* e *and let* $\kappa(Q)$ *be its internal kern, c.f., definition 4.3.7. To each* $k \in \kappa(K)$ *assign the map:* $\overline{k} : x \mapsto k \circ x$. *Then*

$$End(Q, +) \geq \overline{\kappa(Q)} = kern(Q),$$

where the RHS is the [external] kern, c.f., definition 4.3.4.

Proof: It is straightforward to verify that the elements of $\kappa(Q)$ are additive maps of Q and that they centralize the slopemaps of the quasifield Q and hence, by definition, $\kappa(Q)$ is contained in $kern(Q)$. We verify the converse. Suppose $\alpha \in kern(Q)$ and let $e^\alpha = a$. We must demonstrate that a satisfies the defining identities for $\kappa(Q)$. Since α centralizes the slope maps of Q we have:

$$\forall x, m \in Q : (x \circ m)^\alpha = (x^\alpha) \circ m,$$

so

$$\forall x, m \in Q : (x \circ m)^{\alpha} = (x^{\alpha}) \circ m,$$

and choosing $x = e$ yields:

$$\forall m \in Q : m^{\alpha} = a \circ m$$

so

$$\forall x, m \in Q : a \circ (x \circ m) = (a \circ x) \circ m,$$

so $a \in N_{\ell}(Q, \circ)$. Moreover, the requirement that $\alpha \in Hom(Q, +)$, now easily yields the required distributive law:
$a \circ (x + y) = a \circ x + a \circ y$. ■
In view of the above theorem, we shall eventually cease to distinguish between the internal and external kern. Note that by lemma 4.3.3, and definition 4.3.4, the external kern is always a skewfield and hence by the proposition above the same holds for the internal kern. Thus we have established:

Remark 4.3.9 *Let $(Q, +, \circ)$ be any translation algebra with a multiplicative identity. Then internal and external kern of $(Q, +, \circ)$, are isomorphic skewfields.*

Appendix: Quasi-Quasifields

We pause to mention another system, distinct from a translation algebra, that in the finite case reduces to a pre-quasifield, as does translation algebras. These structures are called quasi quasifields, and in the infinite case, quasifibrations are either spreads or maximal partial spreads, see [19]; thus they arise naturally in investigations involving transation nets.

The essential difference between the two structures, translation algebras and quasi-quasifields, lies in the fact that the one-half of quasigroup condition, '$a \circ \boxed{x} = b$' need not hold for quasi-quasifields, but holds for translation algebras, while the distributive-equation

$$(a + b) \circ \boxed{x} = a \circ \boxed{x} + b \circ \boxed{x},$$

has a unique solution for x in quasi-quasifields but may fail for infinite translation algebras.

Definition 4.3.10 *A triple $(Q, +.\circ)$ is called a quasi-quasifield if:*

1. *$(Q, +)$ is an abelian group: so 0 denotes the additive identity;*

2. *$\forall x : x \circ 0 = 0 \circ x = 0$;*

3. *$(Q, \circ)$ has a left identity e: so $e \circ x = x$ for $x \in Q$;*

4. *The right distributive law holds:*

$$\forall x, a, b \in Q : (a + b) \circ x = a \circ x + b \circ x;$$

5. *For $a, b, c \in Q$, $a \neq c$, the equation $x \circ a = x \circ b + c$ has a unique solution for x.*

The maps $T_a : x \mapsto xa$, for $a \in Q$. are called the slope maps of the quasi qausifield, and τ_Q, the set of all slope maps, is called the slope set: it is clearly a subset of $Hom(Q, +)$. The centralizer of τ in $Hom(Q, +)$ is a ring K called the [outer] kern of the quasi-quasifield.

Remark 4.3.11

1. *The slopeset τ is a sharply one-transitive set on Q, equivalently, in $(Q^*, \circ)$ every equation $x \circ a = b$ has a unique solution for $x \in Q^*$, when $a, b \in Q^*$: so the 'right-loop law' holds.*

2. *The outer kern K of a quasi-quasifield Q is a skew field, and the slopemaps of Q are linear maps of $(Q, +)$, when this additive group is regarded as a vector space over K under its standard action.*

3. *The difference $T_a - T_b$ is non-singular when $a, b \in \tau$ are distinct.*

4. *A finite quasi-quasifield is a quasifield.*

Proof: Case (1): Apply condition 4.3.10(5) with $b = 0$. Case (2): the previous case enables a Schur argument to be applied, see lemma 4.3.3. Now applying the condition 4.3.10(5) again yields Case (3). Case (4) follows by noting that if for $a \neq 0$: $a \circ x = a \circ y$ then for $x \neq y$ we have $T_x - T_y$ is singular, contrary to case (3); hence $x \mapsto a \circ x$ is injective and thus in the finite case it is bijective. ∎

Thus a finite translation algebra and a finite quasi-quasifield are just pre-quasifields. In the infinite case they lead to different structures: a translation algebra may have the condition 4.3.10(5) missing, but the multiplication is required to yield a quasigroup, so $a \circ x = c$ has a solution for x when $a \neq 0$: this need not hold in an infinite quasi-quasifield. The structure associated with quasi-quasifields are called quasifibrations.