Chapter 3

Combinatorics Of Spreads: Nets and Packings.

In this chapter, we introduce some packing problems related to translation planes, via their spreads, so what we are concerned with might be called the combinatorics of spreads. The process of derivation, a powerful tool for constructing new affine and projective planes, is essentially a packing problem: points covered by certain sets of lines are replaced by sets of subplanes covering the same points, to yield a new plane. In the context of spreads in projective spaces, derivations are closely associated with reguli, and Desarguesian spreads may be combinatorially characterised in terms of the reguli they contain. Reguli and other partial spreads are also closely related to nets and combinatorial structures called packings that are associated with the construction of exceptionally interesting translation planes. The aim of this chapter is to explore these combinatorial tools, particularly in the context of translation planes.

3.1 Reguli and Regular Spreads.

We begin this lecture with a brief review of the classical concept of a regulus in $PG(3,K)$; these reguli provide the most important tool for constructing linespreads and hence two-dimensional translation planes. The overall aim of the lecture is to extend the theory of reguli in $PG(3,q)$ to reguli in arbitrary projective spaces $\Sigma = PG(V,K)$. The section ends with the Bruck-Bose characterization of Desarguesian spreads in terms of reguli.

A line t is called a transversal to a set of pairwise skew lines Λ, in any projective space, if t meets every line of Λ. In $PG(3, K)$, K a field, the points of a hyperbolic quadric can be written as a union of a set of mutually skew lines Λ and also as the union of all the lines in Λ', the set of transversals to the lineset Λ. In fact, it turns out that Λ and Λ' are linesets such that each is precisely the the set of transversals of the other; moreover every line of each set is covered by every line of the other. The line complexes Λ and Λ' are said to be mutually opposite reguli.

Notice that if Σ is a linespread in $PG(3, q)$ that contains a regulus Λ then replacing Λ in Σ by its opposite regulus

$$\Sigma' = (\Sigma \setminus \Lambda) \cup \Lambda',$$

yields a new spread, said to be *derived* from Λ. One can go further: look for a set of k pairwise disjoint reguli in a spread and replace some or all of them yielding in all 2^k distinct spreads, although some of them may be isomorphic. All of this reflects the fact that reguli play an indispensible rôle in the construction and analysis of translation planes. For the rest of the lecture our discussion of reguli includes not just arbitrary odd-dimensional projective spaces $PG(2n - 1, K)$, but also the infinite-dimensional case — arguably, these are always odd (and even!) dimensional.

We begin by defining a transveral to a collection of *subspaces* Θ to be any line that meets all the lines of Θ, but we shall also insist that any transversal is *covered* by Θ, modifying our earlier usage of the term:

Definition 3.1.1 *Let Θ be a collection of pairwise skew subspaces of any projective space Σ. A line ℓ of Σ is called a* TRANSVERSAL *to Θ if ℓ meets every subspace in the collection Θ and every point of ℓ lies in some member of Θ.*

Note that this is still not the most general useful form of a transversal. We could have introduced the notion of a pseudo-transversal to take care of the case when Σ consists of additive subspaces of $\Sigma = PG(V, K)$, rather than K-subspaces. However, to focus on the essentials, we shall stick with the above definition.

We now turn to the general definition of a regulus. The motivating example, as indicated above, is a collection R of pairwise skew lines, in some $PG(3, K)$, that are covered by the set of all lines that are transversals to R. In the general case R is still required to be a partial spread of the given projective space $\Sigma = PG(V, K)$. So we need to reslove what a partial spread is to mean in the context of infinite-dimensional spaces.

There are two reasonable ways of defining R, a pairwise skew collection of subspaces of Σ, to be a partial spread: both are motivated by the need to make the components have 'half' the dimension of V, in the infinite-dimensional case. The more general method is to assume that all the members are isomorphic to some X, where $V = X \oplus X$; the alternative is to regard R as a partial spread if Σ is a direct sum of any two distinct members of R, for $|R| > 2$. We shall follow the latter path since it leads to tidier and less technical-sounding results; we shall leave it to the interested reader to develop more general results that apply to 'X-partial spreads'.

Definition 3.1.2 *Let Σ be a projective space and Γ any collection of at least three pairwise-skew subspaces. Then Γ is a called a partial spread if to each triple (x, U, V), where $U, V \in \Gamma$ are distinct and do not contain x, there corresponds a unique line ell of Σ such that $x \in \ell$ and ℓ meets X and Y*

We can define a regulus in the general case.

Definition 3.1.3 *Let Σ be any projective space and suppose R is a partial spread in Σ that has at least three components. Then R is a* REGULUS *of Σ if the following hold:*

1. *If a line t of Σ meets three members of R then t is a transversal of R, see definition 3.1.1 above;*
2. *the points covered by R coincide with the points covered by the transversals to R.*

We now provide the alternative definition of a regulus, indicated above, based on the possibility of the alternative definition of a partial spread.

Definition 3.1.4 *Let Σ be a projective space associated with a direct sum vector space $W = X \oplus X$, where X is any vector space over a skewfield K. Suppose R is a collection of pairwise skew subspaces of Σ each of which is K-isomorphic to X. Then R is an X-*REGULUS *of Σ if the following hold:*

1. *If a line t of Σ meets three members of R then t is a transversal to R, see definition 3.1.1 above;*
2. *the points of R are covered by the transversals to R.*

Exercise 3.1.5 *If R is an X-regulus in Σ, in the notation of definition 3.1.4, is W always the direct sum of every pair of distinct members of R, that is, is every X-regulus a regulus in the 'standard' sense of definition 3.1.3?*

As already mentioned, we shall work with reguli, in the sense of definition 3.1.3, rather than wih X-reguli; extending results concerning reguli to X-reguli is left to the interested reader.

Exercise 3.1.6 *Suppose R is a collection of $q+1$ distinct subspaces $PG(2n-1,q)$ such that every member of R has projective dimension $n-1$ and that R is covered by all transversal across it. (1) Are the members of R pairwise skew? (2) Is R a regulus?*

We now proceed to a complete description of all reguli in an arbitrary projective space $PG(V,K)$, K a field. The prototype for all such reguli is the *scalar regulus*, and $V = W \oplus W$, W any K-space; the components of the scalar regulus are $y = xk$, $k \in K$, together with $Y = \mathbf{O} \oplus W$. It will turn out that all reguli are essentially of this type. If K above is permitted to be non-commutative skewfield then, as we shall see, a regulus cannot exist in $PG(V,K)$.

However, the absence of reguli, when K is a non-commutative skew field, is true only in a technical sense: in this case all the '$y = xk$' still turn out to be additive subgroups of $V = W \oplus W$, and although they are not always K-spaces they still define a partial spread (when V is veiwed as a vector space over the prime field) that are covered by pairwise skew lines of $PG(V,K)$ that one might call transversals. We shall refer to such structures as (scalar) pseudo-reguli and incorporate them in our analysis; they arise in the classification of subplane covered nets, a fundamental result in the theory of nets and derivation.

To provide a uniform treatment of left and right vector spaces, and also to take into account that skewfields become unavoidable in our analysis, we express '$y = xk$' as $y = (x)k$, $(x)k$ indicating the action induced by $k \in K$ on $x \in V$.

Definition 3.1.7 *Let $\Sigma := PG(V,K)$ be a projective space over a skewfield K such that $V = W \oplus W$, where W is a K-space.*
Then for any $w \in W$, $(w)k$ denotes wk (resp kw) depending on whether W is taken to be a right (resp. left) K-space and $y = (x)k$, for $k \in K$ denotes the additive subgroup $\{(w,(w)\kappa \mid \kappa \in K\}$ of $V = W \oplus W$

The collection S of subspaces of the K-space V given by:

$$S = \{Y\} \cup \{`y = (x)k' \mid k \in K\},$$

where $Y = \mathbf{O} \oplus W$, is called the W-coordinatized SCALAR PSEUDO-REGULUS *in $PG(V, K)$. The members of S are called its* COMPONENTS. *S is called a* SCALAR *regulus if it turns out to be regulus in Σ.*

For all $w \in W^*$, the lines of Σ of form let

$$T_w := \{(wk_1, wk_2) \mid k_1, k_2 \in K\},$$

and define the STANDARD COVER *of the scalar pseudo-regulus S, by*

$$\tau = \{T_w \mid w \in W\}.$$

Note that from our point of view it turns out to be quite harmless to ignore the dependence of some of the above notation on W; we assume a fixed W as our starting point: we avoid references to 'W-defined' objects.

We now show that in projective spaces over a skewfield K, the scalar pseudo-regulus is a regulus iff K is a field, and when this is case, the standard cover, definition 3.1.7, turns out to be the set of its transversals. In the more general situation, when K is non-commutative, virtually the same conclusions would apply if the definition of a transversal were to be appropriately relaxed.

Theorem 3.1.8 *(Scalar Pseudo-Reguli.) Let S be the scalar pseudo-regulus associated with $V = W \oplus W$, where W is a vector space over a skewfield K. Then*

1. *S is an additive partial spread, with ambient space $(V, +)$.*

2. *The components of S are K-subspaces iff K is field.*

3. *The standard cover τ is a collection of pairwise-skew lines of $PG(V, K)$ such that $\cup\tau = \cup S$, with both sides viewed as subspaces of V.*

4. *K is a field iff the pseudo-regulus S is a regulus and the standard cover, definition 3.1.7, is its set of transversals.*

Proof: (1) Let A and B denote any two distinct components of S; the main case is when they are, respectively, $y = (x)a$ and $y = (x)b$, for distinct $a, b \in K^*$. Now these two spaces have trivial intersection, so we have a partial spread provided $A + B = V$. For convenience, write (x, y), $x, y \in W$ to denote $x \oplus y$. Now $(x, y) \in A \oplus B$ holds iff

$$\exists u, v \in W \ni: (x, y) = (u, (u)a) + ((v, (v)b),$$

and this can easily be solved for u and v. Thus S is an additive spreadset.
(2) Consider a non-zero $w \oplus (w)k \in y = (x)k$. Now for $l \in K$,

$$(w \oplus (w)k)l = ((w)l \oplus ((w)k)l = (w)l \oplus ((w)l)l^{-1}kl,$$

thus $y = (x)k$ is left invariant under K iff k is centralized by K.
(3) Since $T_w = T_{w'}$ holds iff w and w' generate the same rank-one K-space it follows that τ is a collection of pairwise-skew lines of Σ.

The subspace

$$T_w := \{((w)k_1, (w)k_2) \mid k_1, k_2 \in K\}$$

meets Y when $k_2 = 0$, and meets $X := W \oplus vecO$ when $k_1 = 0$. It meets every other component $y = (x)k$ of S at $(w, (w)k)$. Moreover T_w is covered by the components of S because $((w)k_1, (w)k_2)$, for $k_1 \neq 0$, may be expressed as $(wk_1, wk_1\frac{k_2}{k1})$, for $k_1 \neq 0$, meets the component $y = (x)k$, $k := \frac{k_2}{k1}$, and it of course meets Y as well. If $s \in V^*$ is in some $y = (x)k$ then $s = w \oplus (w)k$, $w \in W^*$, and this lies in T_w. So $\cup\tau$ and $\cup S$ coincide as subsets of V.
(4) This follows from the above cases. ∎

We now proceed towards showing that all reguli may be identified with the scalar reguli, that is, scalar pseudo-reguli over a commutative field. We shall not consider here the more general problem of providing a geometric characterization of all pseudoreguli.

Lemma 3.1.9 *Let S be the scalar regulus in $PG(V = W \oplus W, K)$, K a field. Suppose R is any regulus that shares the components $Y = \mathbf{O} \oplus W$, $X = \mathbf{O} \oplus W$ and at least one other component. Then $R = S$.*

Proof: Let $\rho \in R - \{X, Y\}$. So V is a direct sum of any two distinct members of the triad $\{X, Y, \rho\}$, hence, by linear algebra, there is a unique linear bijection $M_\rho : W \to W$ such that

$$\rho := \{(w, wM_\rho) \mid w \in W\}.$$

Since every transversal t of S meets at least three components of R, t must also be a transversal of R, by definition 3.1.3(def:regl). But, by theorem 3.1.8, the transversals of S are of form

$$T_w := \{(wk_1, wk_2) \mid k_1, k_2 \in K\},$$

and this meets ρ non-trivially iff for some $k_1 \in K^*$ there corresponds a $k_2 \in K$ such that $wk_1M = wk_2$, and this implies that M leaves invariant the rank-one space wK, and this to holds for all $w \in W$ iff M is projectively trivial and hence of form $y = (x)m$, for some $m \in K^*$. Thus R includes all the components of S and hence must coincide with S: if R had more components then the transversals of S would fail to be transversals of R. ∎

The following theorem asserts that any regulus R over a field may be identified with the scalar regulus S; in fact R may be coordinatized by S so that any three components of R may be identifed with the three standard components of S, viz., X, Y and the unit line.

Theorem 3.1.10 *(Standard Coordinates For Reguli.) Let $V = W \oplus W$, where W is a vector space over a field K, and let Σ be the associated projective space $PG(V, K)$. Let S denote the scalar regulus in Σ, relative to W. Then given any regulus R of Σ, and an ordered triple of three distinct components (A, B, C) of R, there is a nonsingular bijection $g \in GL(V, K)$ that maps the triple (A, B, C) onto (X, Y, Z), and the regulus R onto the scalar regulus S; here X, Y and Z are the 'standard components' of S in the usual sense:*

$$X = W \oplus \mathbf{O}, \quad Y = \mathbf{O} \oplus W, \quad \text{and} \quad Z = \{(w, w) \mid w \in W\}.$$

Proof: It is a simple exercise in linear algebra to see that the group $GL(V, K)$ is transitive on the set of all ordered triples (A, B, C) such that V is a direct sum of any two members of the triple. Thus choosing (A, B, C) to be three distinct components of R there is a linear bijection g of V such that g maps (A, B, C) onto (X, Y, Z), and now the regulus $g(R)$ satisfies the conditions of lemma 3.1.9 above, hence $g(R)$ is the scalar regulus. ∎

The following corollary is immediate:

Corollary 3.1.11 *If a projective space Σ, over a field K, contains three mutually skew K-subspaces A, B and C such that any two sum to Σ, then the three subpaces are components of a unique regulus in Σ.*

In the context of a projective space $\Sigma = PG(V, K)$, the concept of a spread and partial spreads only make sense if $V = W \oplus W$ for some K-space W. Hence we shall tacitly assume that Σ has this form, when we refer to its partial spreads.

Definition 3.1.12 *Let Σ be a projective space over a field. A spread of Σ is called* REGULAR *if the unique regulus containing any three mutually distinct spread components is contained within the spread.*

Every spread over $GF(2)$ is regular:

Remark 3.1.13 *Let $K = GF(2)$ and suppose V is any vector space over K. Then every spread S in $PG(V, K)$ is regular.*

Proof: Since, c.f. corollary 3.1.11, the regulus R determined by any three distinct components $a, b, c \in S$ coincides with $R \subset S$. ∎
It will become evident that there are many non-isomorphic translation planes of even order $2^n > 8$, and these may be identified with mutually non-isomorphic spreads in $PG(2n-1, 2)$.

The following theorem, due to Bruck and Bose [5], implies that in every other case all finite regular spreads of the same order are isomorphic. The proof introduces powerful computational techniques that will be systematically conisdered in later chapters. The theorem may be stated more generally, with appropriate modifications, so as to include the infinite case.

Theorem 3.1.14 *A finite spread in $PG(2k-1, q)$ and $q \neq 2$ is regular if and only if the associated translation plane is Desarguesian.*

Proof: We will prove this only in the case $PG(3, K)$, $K = GF(q)$, but the proof remains valid in general.

Let S be a spread in $PG(3, q)$. Choose any three lines of S and write the plane vectorially with points (x, y) where x and y are 2-vectors over K and $x = 0, y = 0, y = x$ are components. Then the regulus defined by the three components has as its components $x = 0$ and $y = xu$ for all u in K. Let

$$y = x \begin{bmatrix} g(t,u) & f(t,u) \\ t & u \end{bmatrix} := M_{t,u} := M$$

be any component of the spread with the choice of three components as $x = 0, y = 0, y = x$. Change bases by

$$\begin{bmatrix} I_2 & 0 \\ 0 & M^{-1} \end{bmatrix}$$

and note that the unique regulus containing $x = 0, y = 0$ and $y = xM$ after the basis change also contains $y = x$ and hence must have the form $x = 0, y = 0, y = xk$ for all k in K. Hence, we have that $y = xMk$ must be in the spread, whenever $y = xM$ is in the spread. This implies that $g(tw, uw) = g(t, u)w$ and $f(tw, uw) = f(t, u)w$ for all $u, t, w \in K$.

Now choose $x = 0$, $y = xM_{s,v}$ and $y = xM_{t,u}$ and determine the regulus containing these three components. Change bases by $\begin{bmatrix} I_2 & -M_{s,v} \\ 0 & I_2 \end{bmatrix}$ to rewrite the spread in the form $x = 0, y = x(M_{k,w} - M_{s,v}) = N$. Use the previous basis change with $\begin{bmatrix} I_2 & 0 \\ 0 & N^{-1} \end{bmatrix}$ to realize the standard form of the regulus containing the three indicated components. Now reverse the basis changes to obtain that $x = 0$ and $y = x\left((M_{t,u} - M_{s,v})w + M_{s,v}\right)$ are components for all $t, u, s, v, w \in K$, provided $(t, u) \neq (s, v)$. In particular, if $t = s$ but $u \neq v$ then this implies that the matrix

$$\begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

defines a collineation for all u in K. Similarly, the previous argument shows that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

defines a collineation for all $w \neq 0$ of K. Hence, we obtain $g(t, u) + w = g(t, u + w)$ for all u, t, w in K so that if follows that $g(t, u) = tg(1, 0) + u$ and similarly $f(t, u) = f(t, u + w)$ so that $f(t, u) = f(t, 0) = tf(1, 0)$. Hence, the spread has the following form, for some constants f and g in K:

$$x = 0, y = x \begin{bmatrix} tg + u & uf \\ t & u \end{bmatrix} \forall t, u \in K.$$

■

Exercise 3.1.15 *Show that the matrices in the spread define a field isomorphic to $GF(q^2)$.*

Hence, the spread consists of all 1-dimensional $GF(q^2)$-spaces within a 2-dimensional $GF(q^2)$-vector space. That is, the spread is Desarguesian. ■

3.2 Derivation.

We have seen that a regulus R in $PG(3,K)$, K a field, is covered by its opposite regulus R'. If S is a spread of $PG(3,K)$ that contains R then $(S \setminus R) \cup R' = S'$ is also a spread called the spread 'derived' from S.

We consider this more generally, but only for finite spreads.

Definition 3.2.1 *If S is a spread in a projective space $\Sigma \simeq PG(2k-1,q)$ and R is a partial spread of S such that R is a regulus in some $PG(3,h)$ where $h^2 = q^k$ then we shall say that R is a 'derivable partial spread' of S. The corresponding affine structure in the associated translation plane is called a 'derivable net'.*

Exercise 3.2.2 *Let π be a translation plane with an associated spread in $PG(3,K)$, $K \equiv GF(q)$. Show that a basis for the vector space can be chosen so that any derivable net D has the spread set*

$$x = 0, y = x \begin{bmatrix} u^\sigma & 0 \\ 0 & u \end{bmatrix} \text{ for all } u \text{ in } K \text{ and } \sigma \text{ in } Gal(K).$$

Exercise 3.2.3 *Consider the spread $x = 0, y = x \begin{bmatrix} u^\sigma & \gamma t^\rho \\ t & u \end{bmatrix}$ for all u, t in $K \simeq GF(q)$, q odd and σ, ρ in $Gal(K)$ and γ is a nonsquare in $K - \{0\}$. Find at least $2q$ derivable nets in the associated translation plane. Show that if neither σ nor ρ is 1 that none of the derivable nets is a regulus in $PG(3,K)$. For each derivable net D, find a field K_D isomorphic to K such that D defines a regulus in $PG(3,K_D)$.*

Theorem 3.2.4 *The number of regular spreads in $PG(3,q)$ is*

$$q^4(q^3-1)(q-1)/2.$$

Proof: Each regular spread defines a field extension of K, $K[t] \cong GF(q^2)$. By the theorem of André, each two Desarguesian affine planes are isomorphic by an element of $\Gamma L(4,K)$. The full collineation group which fixes the zero vector of a given Desarguesian affine plane is clearly $\Gamma L(2, K[t])$, $K[t] \cong GF(q^2)$). Hence, the number of regular spreads is

$$N := \frac{|\Gamma L(4,q)|}{|\Gamma L(2,q)|} = q^4(q^3-1)(q-1)/2,$$

and now it is a simple exercise to verify that $N = q^4(q^3-1)(q-1)/2$. ■

Remark 3.2.5 *The number of reguli in any regular spread, contained $PG(3, q)$, is given by*

$$\frac{\binom{q^2+1}{3}}{\binom{q+1}{3}} = q(q^2+1).$$

Proof: Exercise.■

Theorem 3.2.6 *Let R be any regulus in $PG(3,q)$ and let N_R denote the corresponding net of order q^2 and degree $q+1$. Let ℓ be any line of $PG(3,q)$ so that $R \cup \{\ell\}$ is a partial spread. Then there exists a unique regular spread containing $R \cup \{\ell\}$.*

Proof: Let $K = GF(q)$ Represent R is standard form:

$$x = 0, \quad y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K.$$

Let ℓ be represented in the form

$$y = x \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

It is immediate that $bc \neq 0$. Furthermore, the difference of these matrices must be non-singular so that

$$\det \begin{bmatrix} a-u & b \\ c & d-u \end{bmatrix} = (a-u)(d-u) - bc = u^2 - (a+d)u - bc \neq 0 \forall u \in K.$$

Hence, the polynomial $x^2 - (a+d)x - bc$ is irreducible over K. Write $d - u = v, b = gt_0$ and then $e = a - d = ft_0$. Now consider the set of matrices

$$\left\{ \begin{bmatrix} ft+v & gt \\ t & v \end{bmatrix} \mid v, t \in K \right\}.$$

We have noted previously that this set forms a field isomorphic to $GF(q^2)$ so that there is a unique Desarguesian (regular) spread defined by this field of matrices. Hence, there is a unique regular spread containing $R \cup \{\ell\}$. ■
In the next theorem, we shall need to appeal to the following elementary fact:

Remark 3.2.7 *The number of polynomials x^2+fx+g for g and f in $GF(q)$ which are $GF(q)$-irreducible is $q(q-1)/2$.*

Proof: Exercise. ∎

Theorem 3.2.8 *Any regulus R in $PG(3,q)$ can be embedded in exactly $q(q-1)/2$ regular spreads.*

Proof: Represent R in the standard form $x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all u in $GF(q)$. Any regular spread containing R corresponds to a Desarguesian affine plane and hence a corresponding quadratic field extension of $GF(q)$. The theorem follows by remark 3.2.7.∎

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Corollary 3.2.9 *There are exactly $q^4(q^3-1)(q^2+1)$ reguli in $PG(3,q)$.*

Proof: Consider the incidence structure of reguli and regular spreads and count the incidence pairs (flags). Let k denote the number of reguli in $PG(3,q)$. Then the number of Desarguesian spreads times the number of reguli in each Desarguesian spread is equal to the number of reguli times the number of Desarguesian spreads containing a given regulus.

Hence,

$$k = \frac{(q^4(q^3-1)(q-1)/2)q(q^2+1)}{(q(q-1)/2)} = q^4(q^3-1)(q^2+1).$$

∎

Corollary 3.2.10 *Let R be a regulus in $PG(3,q)$. Then, the order of the collineation group of the corresponding regulus net N_R which fixes an affine point is $(q(q^2-1))^2(q-1)r$ where $q=p^r$ and p is a prime.*

Proof: Since any two Desarguesian spreads are isomorphic and since any Desarguesian affine plane admits a collineation group which fixes the zero vector and acts triply transitive on the line at infinity, it follows that

$$|\Gamma L(4,q)_{N_R}| = \frac{q^6(q^4-1)(q^3-1)(q^2-1)(q-1)r}{q^4(q^3-1)(q^2+1)) = (q(q^2-1))^2(q-1)r}.$$

■

We shall see shortly that the collineation group of a regulus net which fixes an affine point is isomorphic to $GL(2,q)\Gamma L(2,q)$ where the product is a central product of intersection the subgroup of order $q-1$ of scalar matrices.

3.3 Direct Products of Affine Planes and Packings.

In $PG(3,q)$ linespreads have size (q^2+1), and the total number of lines is exactly $(q^2+1)(q^2+q+1)$. Thus one might ask for a collection $\mathcal{C}$ of (q^2+q+1) spreads such that every line belongs to (exactly) one spread in the collection $\mathcal{C}$; one might even ask that all the members in $\mathcal{C}$ be regular. Such packings will be used in this section to construct perhaps the two most intriguing translation planes: the Lorimer-Rahilly plane of order 16 and its transpose the Johnson-Walker plane: these are the only known translation planes admitting $GL(3,2)$. The concept of a net product will be introduced partly as an aid to the above, and also because of potential aplications in wider contexts; net products are helpful in constructing nets with interesting properties.

Definition 3.3.1 *Let Σ be a projective space relative to a left K-vector space $X \oplus X$.*

A PACKING (PARALLELISM) *of Σ is a set of spreads which are disjoint with respect to subspaces K-isomorphic to X and such that the union of subspaces isomorphic to X of the set of spreads is the set of all K-subspaces isomorphic to X. The packing Σ is* REGULAR *if the psreads in it are all regular spreads.*

For example, a packing of $PG(3,q)$ is a set of $1+q+q^2$ spreads of q^2+1 lines each. In particular, a regular packing in $PG(3,q)$ gives rise to a set of $1+q+q^2$ *Desarguesian* spreads of order q^2.

In the following, we shall require the concept of the direct product of nets and affine planes. The notion of net was introduced in definition 2.1.1.

Definition 3.3.2 *Let $\pi_1 = (P_1, L_1, C_1, I_1)$ and $\pi_2 = (P_2, L_2, C_1, I_2)$ be two translation planes. Let σ be a $1-1$ correspondence from the set C_1 of parallel classes of π_1 and the set C_2 of parallel classes of π_1. We form the direct product $\pi_1 \times_\sigma \pi_2$ as follows:*

The 'points' are the elements of the cross product $P_1 \times P_2$.

Let ℓ_1 be a line of L_1 so that $\ell_1\sigma$ is a line of L_2. If ℓ_2 is any line parallel to $\ell_1\sigma$, then the set of points of $P_1 \times P_2$ incident with $\ell_1 \times \ell_2$ is a 'line' of the direct product incidence structure.

Note that the construction does not use finiteness. If σ is an isomorphism, we use the term 'regular direct product'.

Exercise 3.3.3 *Show that if the planes are of order n then $\pi_1 \times_\sigma \pi_2$ is a net of order n^2 and degree $n+1$.*

Theorem 3.3.4 *Let T_1 and T_2 denote the translation groups of π_1 and π_2 respectively. Then $T_1 \times T_2$ is a translation group of $\pi_1 \times_\sigma \pi_2$.*

Proof: Define the action of (g_1, g_2) on (a_1, a_2) for a_i in P_i for $i = 1,2$ by $(a_1, a_2)(g_1, g_2) = (a_1 g_1, a_2 g_2)$. Let ℓ_1 be a line of L_1 and ℓ_2 a line parallel to $\ell_1\sigma$. Then $\ell_1 g_1$ is parallel to ℓ_1 and $\ell_2 g_2$ is parallel to ℓ_2 and to $\ell_1\sigma$. Then $\ell_1 g_1 \times \ell_2\, g_2$ is a line of $\pi_1 \times_\sigma \pi_2$. To show that (g_1, g_2) is a translation, simply note that (g_1, g_2) fixes each parallel class but fixes no affine point.

Definition 3.3.5 *Let $\Sigma \cong PG(2k-1, q)$. A $(k-1)$-regulus $R_{(k-1)}$ is a set of $q+1$ $(k-1)$-dimensional projective subspaces which are mutually skew such that any line of Σ which intersects any three necessarily intersects all elements of $R_{(k-1)}$.*

Note that a regulus in $PG(3,q)$ is a 1-regulus.

Theorem 3.3.6 *If π_1 and π_2 are Desarguesian affine planes of order q and σ is an isomorphism of π_1 onto π_2 then*

(1) there is a collineation group isomorphic to $GL(2,q)\Gamma L(2,q)$ acting on $\pi_1 \times_\sigma \pi_2$ and

(2) $\pi_1 \times_\sigma \pi_2$ is a derivable net.

(3) If π_1 is a Desarguesian affine plane whose spread S_1 is in $PG(3,q)$ then $\pi_1 \times \pi_1$ is a derivable net with partial spread in $PG(7,q)$ which contains a 2-regulus.

Proof: We identify π_1 and π_2 and without loss of generality, we let $\sigma = 1$. We note that $\Gamma L(2,q)$ is a collineation group of π_1.

Exercise 3.3.7 *For h in $\Gamma L(2,q)$ show that (h,h) is a collineation group of $\pi_1 \times \pi_1$.*

Now for $\alpha, \beta, \gamma, \delta \in GF(q)$ such that $\alpha\delta - \beta\gamma \neq 0$, we define $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ acting on (a_1, a_2) to be $(a_1\alpha + a_2\gamma, a_1\beta + a_2)$ where the indicated multiplication is scalar multiplication. Let L_1 is a line represented in the form $y_1 = x_1\sigma + \rho$, it is easy to verify that $L_1\alpha$ is $y_1 = x_1\sigma + \alpha\rho$. It follows that $L_1 \times L_2$ maps to $(L_1\alpha + L_2\gamma) \times (L_1\beta + L_2\delta)$ and $(L_1\alpha + L_2\gamma)$ is parallel to $(L_1\beta + L_2\delta)$. Note that it follows that there is a group isomorphic to $GL(2, q)$ which fixes each line of the net incident with $(0, 0)$. Hence, $GL(2, q)\Gamma L(2, q)$ is a collineation group of the net. This proves (1).

Now $(p, 0)$ for all points p of π_1 is a subplane isomorphic to π_1. Furthermore, $GL(2, q)$ acts transitively on the points of each line thru $(0, 0)$. Hence, the net is covered by subplanes isomorphic to π_1. This is enough to ensure that the net is a derivable net. However, if we represent π_1 by the components $y_1 = x_1\alpha$ and $x_1 = 0$ and π_2 as $y_2 = x_2\alpha$ and $x_2 = 0$ then the points of the direct product have the form $((x_1, y_1), (x_2, y_2))$. Rerepresenting the points in the form (x_1, x_2, y_1, y_2) takes the lines $(y_1 = x_1\alpha) \times (y_2 = x_2\alpha)$ to the form $y = x\alpha$ where (x_1, x_2) and $y = (y_1, y_2)$.

Thus, the direct product net may be coordinatized by a net defined by $y = x\alpha$, $x = 0$ which is clearly a regulus in $PG(3, q)$. This proves (1).

Now assume that π_1 is defined by a regular spread in $PG(3, q)$ so that the order of π_1 is q^2. Then if the associated field if $GF(q)[t] \cong GF(q^2)$, the previous argument shows that there is a net of the form $y = x\alpha$, $x = 0$ for all α in $GF(q)$. Hence, this defines a 2-regulus in $PG(7, q)$. This proves (2) and (3).

We now consider the direct product of two Desarguesian affine planes whose corresponding regular spreads are in the same $PG(3, q)$.

Proposition 3.3.8 *Let S_1 and S_2 be distinct regular spreads in $PG(3, q)$, let π_1 and π_2 denote the Desarguesian affine planes corresponding to S_1 and S_2 respectively.*

Form $\pi_1 \times \pi_1 = D_1$ and $\pi_2 \times \pi_2 = D_2$.

Then $D_1 \cap D_2$ (the intersection of components) is a 2-regulus R_2 and $D_1 \cup D_2$ is a partial spread in $PG(7, q)$ of $2(q^2 - q) + 1 + q$ components. Hence, $N_{(D_1 \cup D_2)}$ is a translation net (admits a translation group transitive on its points) of order q^4 and degree $2(q^2 - q) + 1 + q$.

Proof: We note that D_1 may be coordinatized by a quadratic field extension of $K \cong GF(q)$ say $K[t_1]$. Similarly, D_2 may be coordinatized by a quadratic field extension $K[t_2]$ of K. If S_1 and S_2 are distinct, it follows that $K[t_1] \cap K[t_2] = K$. Each derivable net has exactly $1 + q^2$ components as π_i is a Desarguesian affine plane of order q^2 for $i = 1, 2$.

Theorem 3.3.9 *Let $\mathcal{P}$ be a regular packing of $1+q+q^2$ spreads in $PG(3,q)$. Let the corresponding Desarguesian translation planes be denoted by π_i for $i = 1, 2, ..., 1 + q + q^2$.*

(1) Then $\cup_{i=1}^{(1+q+q^2)} \pi_i \times \pi_i$ is a translation plane of order q^4 whose spread is in $PG(7, q)$.

(2) The spread consists of $1 + q + q^2$ derivable nets each containing a 2-regulus R_2.

(3) The collineation group of the translation plane contains $GL(2, q)$ in its translation complement . Furthermore, $GL(2, q)$ is generated by central collineations and leaves each derivable net invariant.

Proof: From the preceding, it remains to show that $GL(2, q)$ is a collineation group of the translation plane.

We note that the full group of each derivable net that stabilizes the zero vector is $GL(2, K[t_i])\Gamma L(2, K[t_i])$ where $K[t_i]$ is the quadratic field extension of $K \cong GF(q)$ which coordinatizes π_i and $\pi_i \times \pi_i$.

Clearly, $\cap_1^{1+q+q^2} GL(2, K[t_i])\Gamma L(2, K[t_i]) \cong GL(2, q)\Gamma L(2, q)$. However, only the group isomorphic to $GL(2, q)$ generated by the scalar mappings as noted above are collineations of the translation plane (with the possible exception of the collineations induced by field automorphisms).

3.3.1 A regular parallelism in $PG(3, 2)$.

Let S_1 be any regular spread in $PG(3, 2)$ we shall construct a parallelism as follows: let C be a cyclic group of order $2^3 - 1 = 1 + 2 + 2^2 = 7$ in $PG(4, 2)$ which fixes three components of S_1 then $\cup_C S_1\sigma$ is a regular parallelism.

Choose any point X of $PG(3, 2)$. There are exactly seven lines containing X and the seven involutions fixing the lines pointwise respectively generate an elementary Abelian group of order 3 (a 3-dimensional $GF(2)$-vector space) A which is a normal subgroup of $PGL(3, 2)_X$. The group induced on A turns out to be isomorphic to $SL(3, 2)$ (see e.g. Walker [40]) which is also isomorphic to $PSL(2, 7)$.

The stabilizer of each line L_i containing X is isomorphic to S_{4i} and the alternating group A_{4i} fixing L_i fixes it pointwise. For each each element σ of order three in A_{4i}, there is a unique line M_i skew to L_i which is σ invariant. It turns out that

$$\{L_i \cap M_i S_{4i} \mid i = 1, 2, ..., 7\}$$

is a spread and

$$\cup_1^7 \{L_i \cap M_i S_{4i} \mid i = 1, 2, ..., 7\}$$

is a regular parallelism of $PG(3,2)$.

Corollary 3.3.10 *Corresponding to the regular parallelism of $PG(3,2)$ is a translation plane of order 16 with kernel $GF(2)$. The plane admits a collineation group isomorphic to $SL(2,2) \times Z_7$. The full collineation group is $PSL(2,7) \times S_3$.*

Now essentially the same construction on the dual space of V_4 produces another translation plane of order 16 from a corresponding regular parallelism. Actually, this may be given a more general construction.

3.3.2 Transpose.

Let $V_{2k} = V$ be a $2k$-dimensional left vector space over a skew field K and let V^* denote the dual space of linear functionals. Choose a basis $\{e_i | i = 1, 2, ..., 2k\}$ of V and let $\{f_i | i = 1, 2, ...2k\}$ denote the dual basis of V^*, so

$$f_j(e_i) = \delta_{ij} \text{ for all } i, j = 1, 2, ..., 2k.$$

Define

$$f\alpha(x) := f(x)\alpha \forall f \in V^*, \alpha \in K,$$

so now V^* becomes a $2k$-dimensional right vector space over K.

Represent vectors of V by

$$(x, y) \equiv (x_1, x_2, ..., x_k, y_1, y_2, ...y_k) \equiv \sum_1^k x_i e_i + \sum_{k+1}^{2k} y_i e_i$$

and represent vectors of V^* by

$$(z, w) \equiv (z_1, z_2, ..., z_k, w_1, w_2, ..., w_k) \equiv \sum_1^k f_i z_i + \sum_{k+1}^{2k} f_i w_i.$$

Define the annihilator mapping $\perp$ as follows:

$$W^{\perp} = \{f \in V^* \mid f(w) = 0 \forall w \in W\},$$

where W is a subspace of V. In terms of the basis then $(z_1, z_2, ..., z_k, w_1, w_2, ..., w_k)$ annihilates $(x_1, x_2, ..., x_k, y_1, y_2, ...y_k)$ if and only if

$$(x_1, x_2, ..., x_k, y_1, y_2, ...y_k) = (z_1, z_2, ..., z_k, w_1, w_2, ..., w_k)^t = 0,$$

where t denotes the transpose matrix.

Now let S be a spread in $PG(V, K)$ then $\{T^{\perp}; T \in S\} = S^*$ is a set of $k-1$-dimensional projective subspaces of $PG(V^*, K)$ such that each hyperplane of the projective space contains exactly one element of S^*.

Definition 3.3.11 *Let $W = Z \oplus Z$ be a L-vector space where L is a skewfield. A dual spread of $PG(W, L)$ is a set S of mutually skew subspaces each L-isomorphic to Z such that every hyperplane contains exactly one subspace of S.*

Hence, S^* is a dual spread of $PG(V^*, K)$ if and only if S is a spread in $PG(V, K)$.

Exercise 3.3.12 *Show that if $\{(x, xA)\}$ is a spread component of S then $\{(x, xA)^{\perp}\} = \{(z, -zA^{-t})$.*

Exercise 3.3.13 *Show that interchanging $x = 0$ and $y = 0$ by a basis change $(x, y) \longmapsto (-y, x)$ maps a partial spread set $\{A \mid A \in \mathcal{M}\}$ onto the partial spread set $\{-A^{-1} \mid A \in \mathcal{M}\}$.*

Hence, we obtain:

Theorem 3.3.14 *Let S be a spread in $PG(V, K)$ for V a $2k$-dimensional left vector space over a skewfield K. Then there is a dual spread S^* in $PG(V^*, K)$ where V^* denotes the dual space of V such that if $\{(x, xA)$ for $A \in \mathcal{M}\}$ is a spread set for S then $\{(x, xA^t)$ for $A \in \mathcal{M}\}$ is a dual spread set for S^*.*

Exercise 3.3.15 *Show that any spread in $PG(2k-1, q)$ is also a dual spread and conversely any dual spread is a spread.*

Given any infinite skewfield K, there is a spread which is not a dual spread due to the work of Bruen and Fisher [6] and Bernardi [4].

Corollary 3.3.16 *Let S be a spread in $PG(2k-1, K)$ for K a field which is a dual spread.*

If $\{x = 0, y = xA$ for $A \in \mathcal{M}\}$ is a spread representation in the associated vector space then $\{ x = 0, y = xA^t$ for $A \in \mathcal{M}\}$ is also a spread called the transposed spread S^t.

Exercise 3.3.17 *Show that the full collineation group of a transposed spread is isomorphic to the group of the transposed spread.*

Exercise 3.3.18 *Show that the transposed partial spread of a derivable net is a derivable net.*

Previously, p 63, we have given an example of a regular parallelism in $PG(3, 2)$ and hence an associated translation plane π. There is a corresponding transposed plane π^t with the property that the spread for π^t still consists of seven derivable nets sharing a 2- regulus in $PG(7, 2)$. It follows that there is a corresponding regular parallelism which we might called the transposed parallelism.

The plane corresponding to the original parallelism is called the *Lorimer-Rahilly* plane of order 16 as it was initially found independently by Lorimer and Rahilly. Similarly, the transposed plane is called the *Johnson-Walker* plane of order 16 as it was determined by Walker using group theory and by Johnson using derivation of the semifield planes of order 16.

Remark 3.3.19 *There are exactly three regular parallelisms of even order; two in $PG(3, 2)$ and one in $PG(3, 8)$. The corresponding translation planes of order q^4 with spreads in $PG(7, q)$ all admit the collineation group $SL(2, q) \times Z_{1+q+q^2}$. Jha and Johnson [20] have shown that translation planes with such collineation groups must correspond to regular packings in $PG(3, q)$.*

There is exactly one known regular parallelism of odd order which is in $PG(3, 5)$ and is due to A. Prince ([36]). The collineation group has not yet been fully determined.

3.4 Introduction to Quadrics and Unitals.

In this section we introduce some standard concepts and tools from linear algebra and projective spaces that have proven to be useful in translation plane theory. As an application, a theorem of Buekenhout, establishing the existence of unitals in translation planes associated with linespreads, is proved using the Bruck-Bose representation of translation planes. The reader might consider skipping this section as nothing in the sequel depends upon it.

Definition 3.4.1 *A correlation of any vector space is an incidence reversing bijection. Let V_n denote a correlation of a n-dimensional K-vector space where K is a field. So, a correlation will map a vector to a hyperplane.*

We represent a vector as a n-tuple $(x_1, x_2, ..., x_n)$ and since a hyperplane is given in terms of a linear equation, $a_1x_1+a_2x_2+...+a_nx_n = 0$, we represent a hyperplane by $(a_1, a_2, ..., a_k)^t$ where t denotes the transpose matrix operation. Hence, a vector X is incident with a hyperplane Y^t if and only if $XY^t = 0$.

We define the following mapping: Let A be any nonsingular $k \times k$ matrix over K and σ any automorphism of K. If $X = (x_1, x_2, ..., x_n)$ define $X^\sigma = (x_1^\sigma, x_2^\sigma, ..., x_n^\sigma)$.

Define $\delta_{A,\sigma}$ as follows: $\delta_A(X) = AX^{t\sigma}$. Furthermore, the induced mapping on Y^t is $\delta_{A,\sigma}(Y^t) = Y^\sigma A^{-1}$.

We shall be interested in 'polarities' which are defined as correlations of order 2 acting on the corresponding projective space.

Exercise 3.4.2 *Show that δ_A is a correlation.*

Remark 3.4.3 *It can be shown that all correlations on a finite dimensional vector space over a field K can be represented in the form $\delta_{A,\sigma}$ for some matrix A and automorphism σ .*

Proposition 3.4.4 *A correlation $\delta_{A,\sigma}$ is a polarity if and only if $\sigma^2 = 1$ and $A^{\sigma t} = kA$ for some k in K such that $k^{\sigma+1} = 1$.*

Proof: $\delta_{A,\sigma}^2(X) = \delta(AX^{\sigma t}) = (AX^{\sigma t})^{t\sigma}A^{-1}$. In order to induce the identity mapping on the projective space, if follows that this latter equation is kX for some nonzero k of F. Hence, a polarity is obtained if and only if $X^{\sigma^2} = X$ for all X and $A^{\sigma t} = kA$.■

Exercise 3.4.5 *Show that $k^{\sigma+1} = 1$.*

Definition 3.4.6 *A polarity δ is said to be 'orthogonal', 'symplectic', or 'unitary' accordingly as $(\sigma, k) = (1, 1), (1, -1)$ and $(\neq 1, k)$.*

A subspace W of V_n is said to 'totally isotropic', 'isotropic', or 'non-isotropic' if and only if $W \cap W^\delta = W, \neq 0$, or 0 respectively. If W is a 1-dimensional subspace (point in the projective space) then a totally isotropic 1-space is said to be 'absolute'.

Correlations are related to sesquilinear forms:

Definition 3.4.7 *Let V be a vector space over a skewfield K. A mapping s from $V \times V$ into K is called a sesquilinear form if and only if*

$$s(x + x', y + y') = s(x, y) + s(x', y) + s(x, y') + s(x', y')$$

and

$$s(\alpha x, \beta y) = \alpha s(x, y)\beta^\sigma$$

where σ is an automorphism of K. A sesquilinear form is said to be non-degenerate if and only if $s(x, y) = 0$ for all y in V implies that $x = 0$ and $s(x, y) = 0$ for all x in V implies that $y = 0$.

It turns out that correlations may always be defined from nondegenerate sesquilinear forms as follows:

$$W^\delta = \{x \in V \mid s(x, w) = 0 \forall w \in W\}.$$

Conversely, given any correlation, there is an associated non-degenerate sesquilinear form which gives rise to it as above.

An orthogonal polarity corresponds to a symmetric, bilinear form ($\sigma = 1$) and $s(x, y) = s(y, x)$. A symplectic polarity corresponds to a skew-symmetric bilinear form where $s(x, y) = -s(x, y)$ (for characteristic two, $s(x, x) \neq 0$ for some x is required), and a unitary polarity corresponds to a Hermitian form where $s(x, y) = s(y, x)^\sigma$ for some automorhism σ of order two.

Definition 3.4.8 *A quadratic form Q is a mapping of V into K such that $Q(\alpha x) = \alpha^2 Q(x)$ and $Q(x+y) = Q(x)+Q(y)+s(x, y)$ where s is a symmetric bilinear form. A quadric is the set of points x in the associated projective space such that $Q(x) = 0$. If the characteristic is not two then the form is nondegenerate if and only if $s(x, y) = 0$ for all y in V if and only if $x = 0$. If the characteristic is two then Q is nondegenerate if and only if $Q(w) \neq 0$ when $s(w, x) = 0$ for all x.*

The set $\{v; Q(v) = 0$ and $s(v, y) = 0$ for all y in $V\}$ is the set of singular points.

It turns out that when K is not of characteristic two then the set of absolute points of the associated symmetric bilinear form is the set of points of the quadric.

Moreover, in any case, all maximal subspaces contained in a nondegenerate quadric have the same rank (as do all maximal totally isotropic subspaces of a polarity) which is called the index.

Definition 3.4.9 *An ovoid in $PG(3,q)$ is a set of q^2+1 points such that no three are collinear and for any point P the set of tangent lines forms a plane (hyperplane).*

If Q is a nondegenerate quadric in $PG(3,q)$ of rank 1 then the quadric is an ovoid.

Now let π be a Desarguesian projective plane of order q^2 considered as $PG(2,q^2)$. Let σ denote the involutory automorphism of the associated field $F \cong GF(q^2)$ coordinatizing π and defined by $z^\sigma = z^q$ for all z in F.

Let V_3 denote the associated 3-dimensional vector space whose lattice of subspaces define $PG(2,q^2)$. Let $A = I_3$ and consider the unitary polarity $\delta_{I,\sigma}$.

The major facts about unitary polarities in V_3 are as follows: Let $\Sigma = PG(2,q^2)$.

Theorem 3.4.10 *A unitary polarity of V_3 over $GF(q^2)$ has q^3+1 absolute points and $q^4-q^3+q^2$ non-isotropic lines in Σ.*

Assuming that the polarity is $\delta_{I,\sigma}$, a point represented by (x,y,z) is absolute if and only if $x^{\sigma+1}+y^{\sigma+1}+z^{\sigma+1}=0$.

Exercise 3.4.11 *Prove part (1) assuming $\delta_{I,\sigma}$ represents the polarity.*

Theorem 3.4.12 *(1) Each non-isotropic line contains $q+1$ absolute points and every two absolute points are incident with a unique non-isotropic line.*

(2) There are exactly q^2 non-absolute lines on any absolute point. Hence, there is a unique absolute line incident with any point.

Definition 3.4.13 *A $t-(v,k,\lambda)$-design is an incidence structure of 'points', 'blocks', and 'incidence' where there are v points, k points per block and any set of t distinct points is incident with exactly λ blocks.*

A 'unital' is defined to be a $2-(q^3+1,q+1,1)$-design.

Hence, we see that the absolute points and non-absolute lines in $\Sigma \cong PG(2, q^2)$ form a unital called the classical unital. However, there are unitals which are not classical some of which cannot be embedded into projective planes. But, if unitals are embedded into projective planes where the blocks are lines, they share the regularity conditions exhibited in the previous theorem.

Theorem 3.4.14 *Let π^+ denote a projective plane of order q. Assume that π^+ contains a unital $\mathcal{U}$ as a $2-(q^3+1, q+1, 1)$-design.*

Then

(1) each point P of $\mathcal{U}$ lies on exactly q^2 lines of $\mathcal{U}$ which we call 'secant lines'. The remaining line incident with P intersects $\mathcal{U}$ in exactly P and is called a 'tangent line'.

(2) Each line of π^+ is either a secant line of a tangent line. That is, each line of the plane either intersects $\mathcal{U}$ in one of $q+1$ points and ,in the latter case, is a line of the design.

(3) Each point Q of $\pi^+ - \mathcal{U}$ is incident with exactly $q+1$ tangent lines and $q^2 - q$ secant lines. The $q+1$ intersections of the tangents of Q with $\mathcal{U}$ are called the feet of Q. When the unital is classical, the line (hyperplane) $\delta_{I,\sigma}(Q)$ is non-isotropic so intersects $\mathcal{U}$ in exactly $q+1$ points which implies that the feet of Q are collinear in the classical situation.

Proof: We count the flags (point of $\mathcal{U}$, line (block) of $\mathcal{U}$) and note that the number of points of $\mathcal{U}$ times the number B of blocks per point $= (q^3+1)B =$ the number U of lines of $\mathcal{U}$ times the number of points of $\mathcal{U}$ per line $= U(q+1)$. Given any point P and any of the q^3 remaining points Q of $\mathcal{U}$, there is a unique line of the unital containing P and Q. Hence, there are exactly q^3/q lines incident with P which are lines of the unital. Hence, it follows that $B = q^3$ so that $U = q^4 - q^3 + q^2$. Since there are exactly $q^4 + q^2 + 1$ lines of the projective plane and there are $q^3 + 1$ tangent lines by the above argument, this accounts for all of the lines of the plane and proves (1) and (2).

Exercise 3.4.15 *Prove part (3).*

The motivation for inducing unitals at this time is to employ the Bruck-Bose representation to show there exist unitals in any translation plane of order q^2 with spread in $PG(3, q)$.

The reader is referred to Buekenhout [7] for further and more complete details.

Proposition 3.4.16 *Let π be an affine Desarguesian translation plane of order q^2 with spread S in $PG(3,q)$. and let $\mathcal{U}$ be a classical unital embedded in the projective plane π^+.*

Realize π and π^+ in $PG(4,q)$ using the Bruck-Bose representation.

We note that the points on ℓ_∞ are represented as the lines of S in the hyperplane $PG(3,q)$.

Let $A(\mathcal{U}) = \mathcal{U} \cap \pi$. Futhermore, let

$$\Delta(\mathcal{U}) := A(\mathcal{U}) \cup \{\textit{points on lines of } S \textit{ corresponding to infinite points of } \pi\}.$$

1. *If ℓ_∞ is a tangent line to the unital then,in $PG(4,q)$, $\mid \Delta(\mathcal{U}) \mid = q^3+q+1$ and*

2. *if ℓ_∞ is a secant line to the unital then, in $PG(3,q)$, $|\Delta(\mathcal{U})| = q^3 - q + (q+1)^2$.*

Exercise 3.4.17 *Prove the above proposition.*

Definition 3.4.18 *In situation (1), the unital is said to be 'parabolic' and in situation (2), 'hyperbolic'.*

The main theorem of Buekenhout is

Theorem 3.4.19 *$\Delta(\mathcal{U})$ is a quadric in $PG(4,q)$.*

(1) If $\mathcal{U}$ is parabolic then $\Delta(\mathcal{U})$ has one singular point p and is the union of all lines joining p to the points of some 3-dimensional ovoid of $AG(4,q)$ with one point at infinity.

(2) If $\mathcal{U}$ is hyperbolic then $\Delta(\mathcal{U})$ is non-singular.

Proof: We shall sketch the proof of (1). The proof of (2) is similar. Consider the regular spread

$$x = 0, y = x \begin{bmatrix} u+tg & tf \\ t & u \end{bmatrix} \forall u, t \in K \cong GF(q),$$

in $PG(3,q)$. Note that $x^2 - xg + f$ is a K-irreducible polynomial. By results from the algebraic tract, we extend K to a field $K[e]$ such that $e^2 = eg - f$ and multiplication in $K[e] \cong GF(q^2)$ is given as follows:

$$(t^*e + u^*)(te + u) = (t^*, u^*) \begin{bmatrix} u+tg & tf \\ t & u \end{bmatrix}$$

written over $\{e, 1\}$ for all t^*, u^*, t, u of K.

Let σ denote the automorphism of $K[e]$ given by $x^\sigma = x^q$.

We consider the classical unital $\mathcal{U}$ in the associated Desarguesian projective plane $PG(2, K[e] = F \cong GF(q^2))$ whose points are given homogeneously by (x, y, z) for x, y, z in F and $(x, y, z) \neq (0, 0, 0)$.

We choose $z = 0$ to be the line at infinity ℓ_∞ and $z = 1$ to denote the affine point of π. Furthermore, we identify $(x, y, 1)$ and (x, y). We choose $(x, 1, 0) = (x)$ and $(0, 1, 0) = (\infty)$ on the line at infinity. We choose the unique point on ℓ_∞ of $\mathcal{U}$ as $(\infty) = (0, 1, 0)$. We choose a matrix for the unitary polarity so that $(0, 1, 0)$ is an absolute point. In particular, the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ provides the form as $\{(x, y, z); x^{\sigma+1} + zy^\sigma + yz^\sigma = 0\}$. Hence, with our notation, we have $\{(x, y); x^{\sigma+1} + y^\sigma + y = 0\} \cup \{(\infty)\} = \mathcal{U}$.

Now to form $\Delta(\mathcal{U})$. We note that using the Bruck-Bose model, $x = 0$ $= (x_1, x_2)$ is a set of $q + 1$ points of $\Delta(\mathcal{U})$. Since $x^2 - xg + f$ is irreducible, it follows that $x_1^2 f - x_1 x_2 g + x_2^2 = 0$ is equivalent to $(x_1, x_2) = 0$.

Exercise 3.4.20 *Show that* $e^\sigma = -e + g$ *and* $e^{\sigma+1} = -f$. *Letting* $x = x_1 e + x_2$ *and* $y = y_1 e + y_2$ *show that*

$$x^{\sigma+1} + y^\sigma + y = 0 = -(x_1^2 f - x_1 x_2 g + x_2^2 + y_1 g + 2y_2).$$

Exercise 3.4.21 Embed the affine space $AG(4, q)$ into $PG(4, q)$ as follows:

$$(x_1, x_2, y_1, y_2) \longmapsto (x_1, x_2, y_1, y_2, z)$$

and consider the points of $PG(4, q)$ as the 1-dimensional subspaces of a 5-dimensional K-vector space. Show that

$$x_1^2 f - x_1 x_2 g + x_2^2 + y_1 g + 2y_2 = 0$$

if and only if

$$x_1^2 f - x_1 x_2 g + x_2^2 + z y_1 g + 2 z y_2 = 0.$$

Note that the intersection with the infinite points when $z = 0$ is $(x_1, x_2) = 0$ which is $\{(0, 0, 1, \alpha), (0, 0, 0, 1, 0); \alpha \in GF(q)\}$.

Hence, the above equation defines a quadric defining $\Delta(\mathcal{U})$.

Exercise 3.4.22 *Show the quadric above is degenerate. Show the unique singular point is* $(0, 0, -2, g, 0) = p$.

Now choose the hyperplane defined by $y_2 = 0$ and note that intersection with $\Delta(\mathcal{U})$ is given by

$$\{(x_1, x_2, y_1, z); x_1^2 f - x_1 x_2 g + x_2^2 + z y_1 g = 0.$$

Exercise 3.4.23 *The above quadric in the hyperplane isomorphic to* $PG(3, q)$ *is nondegenerate and of index* 1. *Show this when* q *is odd.*

Hence, all points of $\Delta(\mathcal{U})$ lie on lines of p, there are exactly $q^2 + 1$ points of an ovoid of H in $AG(4, q)$ and exactly one infinite point $(0, 0, 1, 0, 0)$ of H. Since each line is a 2-dimensional K-vector space and $\Delta(\mathcal{U})$ is a quadric, it follows that there are exactly $q + 1$ points of $\Delta(\mathcal{U})$ on each line thru p. Hence, this accounts for the $q^3 + q + 1$ points as $(q^2 + 1)q + 1$ points on lines thru p. Hence, there is an ovoid $\mathcal{O}$ in $PG(3, K)$ such that $\Delta(\mathcal{U})$ lies on $p\mathcal{O}$.

Now, it turns out that $\Delta(\mathcal{U})$ induces a unital in any translation plane with spread in $PG(3, K)$.

Theorem 3.4.24 *Let* ρ *be any translation plane of order* q^2 *with spread in* $PG(3, q)$ *then* ρ *contains a parabolic unital.*

Proof: The idea of the proof is to show that $\Delta(\mathcal{U})$ remains a unital in ρ.

If (∞) is the tangency point, we may assume that $x = 0$ (L) is a line common to ρ and the Desarguesian affine plane π. We identify the points of π and ρ so that we may consider $\Delta(\mathcal{U})$ as a set of points in ρ^+ (the projective extension of ρ). We assert that the lines of ρ^+ which join pairs of points of $\Delta(\mathcal{U})$ is a $2-(q^3 + 1, q + 1, 1)$-design ; a unital. It remains only to show that the lines of ρ^+ joining pairs of such points intersect $\Delta(\mathcal{U})$ in exactly $q + 1$ points.

First consider a line of ρ incident with (∞). Any such line becomes a 2-dimensional projective subspace which intersects the hyperplane at infinity in $x = 0$ which consists of $q + 1$ points of $\Delta(\mathcal{U})$.

Suppose a, b are points of $\Delta(\mathcal{U})$ which are in π so in ρ. The line ab is a plane of $AG(4, K)$ and $\Delta(\mathcal{U})$ is a quadric. Assume that ab is not on (∞). Hence, the projective extension $ab^+ \cap \Delta(\mathcal{U}) = C$ is a quadric possibily degenerate. In the former case, a nondegenerate quadric in a projective plane $PG(2, q)$ is a conic of $q + 1$ points. In the latter case, it is possible that C is the union of two lines of $PG(4, K)$. If C contains a line of $PG(4, K)$ then it contains a line ℓ of $AG(4, K)$ which is contained in a line ρ_ℓ of the translation plane ρ. But, the projective extension of ℓ contains a point of $\Delta(\mathcal{U})$ so that ρ_ℓ must be incident with (∞). This completes the proof.

We have noted that any regulus in $PG(3, K)$ can be embedded in a regular spread. The same idea as above shows that any translation plane with spread S in $PG(3, K)$ such that S contains a regulus in $PG(3, K)$ forces the existence of a hyperbolic unital in such translation planes.

Theorem 3.4.25 *Let ρ be a translation plane with spread S in $PG(3, K)$. If S contains a regulus then ρ^+ contains a hyperbolic unital (ℓ_∞ is a secant line to the unital).*

There are many questions and problems that might be mentioned with regard to translation planes admitting unitals. However, here is a general problem.

Let π denote a translation plane with spread in $PG(3, q)$ that admits a unital. When is the unital a Buekenhout unital?

Finally, we point out that the construction given can be generalized and need not depend upon a classical unital.