# Chapter 2

# The Bruck-Bose Projective Representation Of Spreads.

In this chapter, we shall be discussing a model of translation planes, due to Bruck and Bose, which mainly uses projective spaces, rather than vector spaces, so we obtain what amounts to a projective version of the results of André discussed above. However, the Bruck-Bose model and the André model are 'equivalent' only in the sense that vector spaces and projective spaces are 'equivalent'.

## 2.1 Foundational Structures In Finite Geometries: A Review.

In the first chapter, see page 2, we introduced the basic notion of an incidence structure, although so far the only incidence structures we have considered explicitly have been affine planes. To consider projective versions of spread theory, we shall need to consider Desarguesian spaces — affine and projective — and also arbitrary projective planes because they correspond to the 'closure' of arbitrary affine planes. In this lecture, we shall review these concepts and introduce some notational devices useful for the study of translation planes.

All these concepts are closely related to generalizations of affine planes called nets: later we shall study these too.

**Definition 2.1.1** *Let $\mathcal{N} := (P, L, C, I)$ be a quadruple, where $P$, $L$, $C$, and $I$ are pairwise disjoint sets consisting of* POINTS, LINES, PARALLEL CLASSES, *and* INCIDENCE, *respectively, and where $I \subset P \times L$; so $(P, L, I)$ is an incidence structure in the usual sense. Then $\mathcal{N}$ is a* NET *if*

1. *$C$ is a partition of the lineset $L$, based on an equivalence relation called* PARALLELISM, *and the members of $C$ are called* PARALLEL CLASSES.
2. *Each point is incident with exactly one line of each parallel class.*
3. *Given a point $p$ and a line $A$ such that $p$ and $A$ are not incident, there is a unique line $B$ parallel to $A$ which is incident with $p$.*
4. *Two lines from distinct parallel classes have a unique common incident point.*

*If there are $n$ points per line and $k = |C|$ parallel classes, the net is said to have* ORDER $n$ *and* DEGREE $k$

It follows immediately:

**Remark 2.1.2**

1. *Every affine plane $\mathcal{A}$ is a net.*
2. *Let $D \subset C$, where $C$ is the set of parallel classes of any net $\mathcal{A}$. Then the points of $\mathcal{A}$ and the lines covered by the members of $D$ form a net — a subnet of $\mathcal{A}$ — provided $D$ is appropriately non-degenerate, e.g.* $|D| \geq 3$.
3. *An affine plane of order $n$ is a net of order $n$ with degree $n + 1$, and every net with these parameters is an affne plane of order $n$.*
4. *Let $\mathcal{M}$ be a partial spread on a vector space $V$. Then the net with pointset $V$ whose lines are additive cosets of the members of $\mathcal{M}$ form a net; this net is called the net of the partial spread $\mathcal{M}$, and which we denote by $\Pi_{\mathcal{M}}$: the parallel classes may be identified with the members of $\mathcal{M}$: so if $\mathcal{M}$ is a spread then the net $\Pi_{\mathcal{M}}$ coincides with the translation plane $\Pi_{\mathcal{M}}$. (See exercise 2.1.5 for details).*

The PROJECTIVE CLOSURE $\overline{\mathcal{N}}$ of a net $\mathcal{N} = (P, L, C, I)$ is the *incidence structure* obtained by adjoining to its pointset the set of its parallel classes $C$ and lineset $L \cup \{\ell_\infty\}$ as its lineset and with natural incidence, i.e., the new line $\ell_\infty$ is adjacent to all the parallel classes only and every line in $L$ is incident with its parallel class. When $\mathcal{N}$ is an affine plane then its projective closure is defined to be a projective plane. We adopt a more explicit and homogenious version of this defintion.

**Definition 2.1.3** *A projective plane $\pi$ is an incidence sructure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ with the following properties:*

1. *Given two distinct points $P$, $Q$ of $\mathcal{P}$ , there exists a unique line $p$ such that $(P,p)$ and $(Q,p) \in \mathcal{I}$;*

2. *Given two distinct lines $p$, $q$ of $\mathcal{L}$ , there exists a unique point $P$ such that $(P,p)$ and $(P,q) \in \mathcal{I}$;*

3. *There exist four points no three of which are incident with the same line.*

Incidence is clearly set-theoretic, so we continue with the notational devices for projective planes that were introduced earlier for set-theoretic incidence structure, see page 2. The notion of a central collineation differs slightly for projective planes from the corresponding definition for an affine plane.

**Definition 2.1.4** *Let $g$ be a collineation of a projective plane $\pi$ that fixes all the points of a line $\ell$ and all the lines through a point $P$. Then $g$ is a* CENTRAL COLLINEATION *with* AXIS *$\ell$ and* CENTER *$P$; $g$ is a* TRANSLATION *(resp.* HOMOLOGY *) if $P \in \ell$ (resp. $P \notin \ell$).*

Exploiting the point-line duality for projective planes it is clear that a central collineation may be equivalently be defined to be one that fixes all the points (lines) on a line (point). Note also that only the trivial collineation is a both an elation and a homology.

We have already indicated, remark 2.1.2, how the 'closure' of a net when applied to an affine plane yields a projective plane. For a projective plane the reverse also holds. The details of all this discussed in the following exercise.

**Exercise 2.1.5** *Let $\pi$ be a projective plane. Choose any line $\ell_\infty$ and form the incidence structure $\pi^{\ell_\infty}$ of 'points' those points of $\pi$ which are not on $\ell_\infty$ and lines of $\pi$ not equal to $\ell_\infty$. Incidence is defined as inherited from the incidence of $\pi$. $\pi^{\ell_\infty}$ is called the affine restriction of $\pi$ with respect to $\ell_\infty$.*

1. *Show that* $\pi^{\ell_\infty}$ *is an affine plane.*

2. *Conversely, if* $\alpha$ *is an affine plane we may define a projective plane* $\alpha^+$ *as follows: The points of* $\alpha^+$ *are the points of* $\alpha$ *and the parallel classes of* $\alpha$ *and the lines of* $\alpha^+$ *are the lines of* $\alpha$ *and the set of parallel classes of* $\alpha$. *The 'points' of* $\alpha^+$ *which are parallel classes of* $\alpha$ *are called the set of 'infinite points' and the line of* $\alpha^+$ *which is the set of parallel classes of* $\alpha$ *is called the 'line at infinity* $\ell_\infty$*' of* $\alpha^+$. *(We shall also refer to* $\ell_\infty$ *as the 'line at infinity' of the affine plane* $\alpha$*).*

   *Show that* $\alpha^+$ *is a projective plane.* $\alpha^+$ *is called the projective closure of* $\alpha$.

3. *Let* $\alpha$ *be an affine plane and* $\pi$ *and* $\rho$ *two projective planes extending* $\alpha$ *with respect to the adjoinment of lines* $p$ *and* $q$ *of* $\pi$ *and* $\rho$ *respectively.*

   (a) *Show that there is an isomorphism from* $\rho$ *to* $\pi$ *which carries* $q$ *to* $p$.

   (b) *Show that* $\pi^p \cong \rho^q \cong \alpha$.

4. *Let* $\alpha$ *be an affine plane with collineation group* $G$. *Let* $\alpha^+$ *denote the projective closure of* $\alpha$, *and let* $\ell_\infty$ *be the line at infinity. Let* $G^+$ *denote the collineation group of* $\alpha^+$. *Show that* $G$ *is isomorphic to the subgroup* $G^+_{\ell_\infty}$, *the global stabilizer of* $\ell_\infty$

We shall normally consider translation planes $\pi$ as affine planes although, occasionally , we shall refer to the line at infinity of $\pi$ to mean the line adjoined to $\pi$ to produce the projective closure $\pi^+$. Similarly, we will use interchangeably the terms 'infinite point' and parallel class.

In the remainder of our review of foundational matters, we consider some of the fundamental concepts related to affine and projective spaces.

**Definition 2.1.6** *Let* $V$ *be a vector space over a skewfield* $K$. *The corresponding* AFFINE SPACE $AG(V,K)$ *is the collection of all the* $K$*-subspaces* $W \leq V$ *together with their translates:*

$$AG(V,K) := \{c + W \mid c \in V, W \leq V\}.$$

*The members of* $AG(V,K)$ *are called the affine subspaces of* $V$, *and an affine subspace* $c + W$ *is regarded as having same dimension as* $W$, *when viewed as a vector subspace of* $V$. *The zero-dimensional subspaces are called points, so*

*$V$ itself is the set of all affine points, the one-dimensional subspaces are the affine lines and the two dimensional subspaces are the affine planes, etc.*

*The translation group of $AG(V, K)$ consists of all the bijections of $V$ that hae the form $\tau_v : x \mapsto x + v$, for $v \in V$, and two subspaces are called parallel if they lie in the same orbit of the translation group.*

*An incidence structure is* CONSIDERED *an affine space if it is isomorphic to the subincidence structure corresponding to the points and lines of $AG(V, K)$, for some vector space $V$ over a skewfield $K$.*

It is not hard to characterise the subspaces of an affine space $AG(V, K)$ in terms of its point-line incidence structure (and its collineation group), and also to determine completely the $K$ vector space $V$. Thus an incidence structure cannot be isomorphic to the incidence structure of more than one affine space. Hence we shall let the context determine whether we are considering a 'standard' affine plane $AG(V, K)$, or an incidence structure isomorphic to that of an affine space.

The fundamental connections between affine and projective planes, developed in exercise 2.1.5, have straightforward analogues relating affine and projective spaces. For example, projective spaces could be introduced by adding on the equivalence classes of affine spaces as 'infinite' subspaces. However, as in the planar case, we choose to introduce this 'closure' of an affine space by giving a more homogeneous version of the definition.

**Definition 2.1.7** *Let $W$ be any $K$-vector space where $K$ is a skewfield. The* PROJECTIVE SPACE *$PG(W, K)$ is the lattice of vector spaces where incidence is inherited from that of $W$.*

*Let $A$ be any $K$-vector subspace of $W$. Then $A$ and $PG(A, K)$ are both regarded as being the 'same' projective subspaces of $PG(W, K)$, and the [projective] dimension of $A$ is $a - 1$ where $a$ is the rank of $A$ as a $K$-vector space; so $PG(W, K)$ has dimension $dim_K W - 1$.*

*The [projective]* POINTS *of $PG(W, K)$ are the subspaces with projective dimension zero, the* LINES *are the subspaces that have projective dimension one, the* PLANES *have projective dimension two and the* HYPERPLANES *$H$ are the subspaces of $PG(W, K)$ that are maximal in $W$: so hyperplanes $H$ are vector subspaces of $W$ that have codimension one in $W$.*

*An incidence structure is* CONSIDERED *a projective space if it is isomorphic to the subincidence structure corresponding to the points and lines of $PG(W, K)$, for some vector space $V$ over a skewfield $K$.*

**Remarks 2.1.8** *A projective space $PG(W, K)$ has all it subspaces determined by the incidence structure of its points and lines: a set $S$ of projective points is a subspace iff $S$ contains the points of every lines that meets it in at least two points.*

However, it still remains to exclude the possibility that projective spaces that are isomorphic as incidence structures arise from non-isomorphic vector spaces, possibly even defined over different fields. We do this by first constructing the associated affine planes.

**Definition 2.1.9** *Let $PG(W, K)$ be a projective space associated with a vector space $W$ defined over a skewfield $K$. Let $V$ be any hyperplane of $W$. Then $PG(W, K)^V$ is the incidence structure whose points are the projective points in $PG(W, K) - PG(V, K)$ and whose lines are all the sets of points of type $\ell^* = \ell \setminus \{L\}$, where $\ell$ is any line not in $V$ that meets $V$ in the projective point $L$.*

We now establish the equivalence between affine and projective spaces, generalising the corresponding result for planes.

One approach to this would be to follow the procedure of exercise 2.1.5: define parallel classes for the lines of $AG(V, K)$, and show that the associated projective closure is the incidence structure of a projective space. But the latter incidence structure needs to be axiomatically recognisable, as in the planar case. Since at this stage these axioms are not available (for dimension $> 2$), we shall follow an alternative approach based on the method of homogeneous coordinates, but adapted for the infinite-dimensional case.

This method has the advantage of providing a concrete link $\Lambda : AG(V, K)^+ \to PG(V^+, K)$ between the projective closure (which we shall define) of the affine space $AG(V, K)$ and the projective space defined over $V^+ = V \times K$ a rank one extension of $V$. Basically $\Lambda$ is the unique extension of the affine-space isomorphism $v \mapsto (v, 1)$, from $AG(V, K)$ to $PG(V^+, K)^{H_\infty}$, where $H_\infty = V \times 0$, such that the 'slope' $(W)$ of a coset $c + W$ maps under $\Lambda$ to $W \times 0$, in the hyperplane $H_\infty$. We now summarize all this and a few related properties:

**Theorem 2.1.10 (Homogeneous Coordinates.)** *Let $V$ be a vector space over a skewfield $K$; so the direct product $V^+ := V \times K$, viewed as a $K$-space, contains hyperplane*

$$H_\infty := (V) := \{(v, 0) \mid v \in V\} \cong PG(V, K).$$

*Define the copies* $(V) := \{(v) \mid v \in V\}$ *and* $V_0 := \{(v, 0) \mid v \in V\} (= H_\infty)$ *of* $V$*, and let* $(W)$ *and* $W_0$ *be the natural image of any subspace* $W \leq V$ *in* $(V)$ *and* $V_0$ *respectively.*

*Let* $AG(V, K)^\circ := AG(V, K) \setminus V$ *denote set the set of all the affine subspaces of* $AG(V, K)$ *with* the affine points excluded. *Define the* GRADIENT *or* SLOPE MAP*:*

$$\begin{array}{rcl} \nabla : AG(V, K)^\circ & \to & PG((V), K) \\ c + W & \mapsto & (W). \end{array}$$

*Then the following hold.*

1. $\nabla(AG(V, K)^\circ) = PG((V), K)$*; the image* $\nabla(c + W) = (W)$ *is called the* SLOPE *of the affine subspace* $c + W$*, for* $c \in V$*,* $\mathbf{0} \neq W \leq V$*. The projective space* $PG((V), K)$ *is the* HYPERPLANE AT INFINITY *for* $AG(V, K)$*.*

2. *Define the structure* $AG(V, K)^+$ *consisting of* POINTS *and* SUBSPACES *where, the point set is defined by*

$$\mathbf{P} := [AG(V, K)] \cup [\nabla(AG(V, K)^\circ)],$$

   *and the subspaces of* $AG(V, K)^+$ *are (1) the members of* $\mathbf{P}$*; (2) the subspaces of the projective space* $PG((V), K)$*; and (3) subsets of* $\mathbf{P}$ *that may be expressed in the form:*

$$(c + W)^+ := (c + W) \cup \{(W)\},$$

   *where* $W$ *is any non-trivial vector subspace of* $V$ *and* $c \in V$*. The subspace* $(c + W)^+$ *is called the (projective)* CLOSURE *of* $c + W$ *(and does not depend on the choice of the coset representitive* $c$*);* $(W)$ *is the* SLOPE *or* GRADIENT *of* $c + W$*.*

   *Then* $AG(V, K)^+$ *is a lattice, relative to containment, and the closure of any affine subspace* $c + W$ *is the smallest lattice element containing all the points in it.*

3. *There is a unique lattice isomorphism*

$$\Lambda : AG(V, K)^+ \to PG(V^+, K),$$

*such that its restriction to the points of $AG(V,K)$ defines the following isomorphism $\lambda$ of affine spaces:*

$$\begin{aligned}\lambda : AG(V,K) &\to PG(V^+,K)^{H_\infty}\\ v &\mapsto (v,1).\end{aligned}$$

*$\Lambda$ maps the closure of every affine subspace $c+W$ of $AG(V,K)$, $W \neq \mathbf{0}$, into the subspace of $PG(V^+,K)$ that meets $H_\infty$ in $W_0$: that is, $\Lambda$ maps the slope $(W)$ of any affine subspace of $AG(V,K)$ into its 'copy' $W \times 0$ in the hyperplane $H_\infty \leq PG(V^+,K)$.*

4. *Explicitly, $\Lambda$ is an isomorphism from the projective space $AG(V,K)^+$ onto the projective space $PG(V^+,K)$ given by:*

$$\begin{array}{ccc} AG(V,K)^+ & \to & PG(V^+,K) \\ (W+c)^+ & \mapsto & (W,0) \oplus (c,1) \\ (W) & \mapsto W \times 0. & \end{array}$$

**Proof:** For convenience assume all vector spaces are taken as right $K$ spaces. (1) is trivial, it is really only concerned with introducing definitions. (2) is a straightforward verification. (3) is essentially part of the next case: (4). Here the main point is to realise that if $W+c$ is a coset of a subspace of $W$ of $V$ then in the lattice $PG(V^+,K)$:

$$[(W,0) \oplus (c,1)K] = (c+W,1) \cup (W \times 0),$$

where $[X]$ denotes the set of projective points in $X \in PG(V^+,K)$, and that $\Lambda$ maps the the affine subspace $c+W$ of $AG(V,K)$ onto $(c+W,1)$, and its closure $(W)$ onto $W \times 0 \leq H_\infty$. The proof follows easily. ∎

The above theorem contains within it the equivalence between projective and affine spaces, specifically, that $PG(V^+,K)^V \cong AG(V,K)$ whenever $V$ has codimension one in $V$:

**Corollary 2.1.11 (The Theorem of Veblen, [39].)** *Suppose $V^+$ is a vector space over a skewfield $K$ of rank $> 1$; thus $V^+ = V \oplus < c >$, for subspaces $V$ and $< c >$ that have resp. codimension and dimension one in $V$.*

*Form the projective space $AG(V,K)^+$, the closure of $AG(V,K)$, obtained by defining points at infinity to be the parallel classes of the lines and with* *each line assigned an extra point, viz., its parallel class. Let $PG(V^+, K)$ be the incidence structure associated with the lattice of $K$-subspaces of $V^+$. Then we have the following incidence structure isomorphisms:*

1. $AG(V, K)^+ \cong PG(V^+, K)$;
2. $AG(V, K) \cong PG(V^+, K)^\infty$.

It is worth stressing:

**Remark 2.1.12** *The affine space $AG(V, K)$ has the same dimension as $V$ whereas $PG(W, K)$ has dimension the dimension of a hyperplane $H(W)$ of $W$; there is an affine space isomorphism:*

$$PG(W, K)^\infty \cong AG(H(W), K).$$

## 2.2 Projective Space Representations: Bruck-Bose Theory.

In this lecture, we shall be discussing a model of translation planes, due to Bruck and Bose, which mainly uses projective spaces, rather than vector spaces, so we obtain what amounts to a projective version of the results of André discussed above. However, the Bruck-Bose model and the André model are 'equivalent' only in the sense that vector spaces and projective spaces are 'equivalent'.

We first introduce the projective space version of an André-type spread; this is essentially a restatement of the usual definition of a spread in projective space terminology.

**Definition 2.2.1** *Let $\Sigma = PG(V, K)$ be an arbitrary projective space, associated a vector space $V$ over a skewfield $K$, and let $\mathcal{P}$ denote a collection of [at least two] mutually skew subspaces of $\Sigma$. Then $\mathcal{P}$ is called a* PROJECTIVE PARTIAL SPREAD *such that given any two distinct subspaces $L, M \in \mathcal{P}$ and any point $p \in \Sigma$ not on $L$ or $M$, there is a unique line $\ell$ which contains $p$ and intersects both $L$ and $M$.*

*If furthermore the points of $\mathcal{P}$ form a cover of the points of $\Sigma$ then $\mathcal{P}$ is called a* PROJECTIVE SPREAD.

It is immediate that a projective [partial] spread in $PG(V, K)$ is just a vector space [partial] spread of $K$-subspaces of the $K$-vector space $V$, and conversely that every space [partial] spread consisting of $K$-subspaces of $V$ is a [partial] [projective] spread in $PG(V, K)$: the existence of '$\ell$' ensures that two distinct subspaces always direct-sum to the whole space, and hence when at least three components are present, all the components have 'half' the dimension of the associated vector space.

Thus 'ordinary' and 'projective' [partial] spreads are essentially the same objects but viewed from different perspectives; we normally do not distinguish between them. Hence, a spread is defined by its context either vectorially or projectively. Accordingly, we shall not repeat for projective spaces all the terminology that we introduced for ordinary spreads; when interpreting spreads in projective spaces, we shall sometimes use the term '*projective* spread'.

Before moving on, we consider as an exercise a more general, but putatively equivalent form, for the definition of a [partial] spread: instead of requiring the direct sum condition could we replace it by the weaker-to-state condition that if $V = X \oplus X$ then a collection of pairwise skew subspaces isomorphic to $X$, as projective $K$-space, form a partial spread?

The following example shows that the indicated generalization does not characterise partial spreads, satisfying the standard definition.

**Example 2.2.2** *Let $W$ be a vector space over any skewfield $K$, with an infinite $K$-basis $(e_1, e_2, \ldots)$. Now on $V = W \oplus W$ take any spread $\mathcal{S}$ that includes $X := W \oplus \mathbf{0}$, $Y := \mathbf{0} \oplus W$ and $Z := \{w \oplus w \mid w \in W\}$. Now let $H_1$, $H_2$ and $H_3$ be hyperplanes of the three components $X$, $Y$ and $Z$, respectively, obtained when $(0, e_1)$, $(e_1, 0)$ and $e_1$ are deleted. Then*

$$\mathcal{H} := (\mathcal{S} \setminus \{X, Y, Z\}) \cup \{H_1, H_2, H_3\}$$

*is a collection of pairwise disjoint $K$-subspaces of $V$ each of which are isomorphic to $W$, and $V = W \oplus W$. However, $V$* CANNOT *always be expressed as the direct sum of any two members of $\mathcal{H}$.*

The example shows that $\mathcal{H}$ is a partial spread on $V = W \oplus W$, in the sense that all its members are pairwise disjoint and 'half-dimensional'; however $\mathcal{H}$ is not a partial spread, according to the standard meaning, since the direct sum condition is required to hold.

However, the example does not settle the question when $\mathcal{H}$ is a 'spread' in the sense that all its components form a coveing of $PG(V, K)$. We leave this matter for the reader to resolve:

**Exercise 2.2.3** *Let be $V = W \oplus W$ a vector space, over a skewfield, such that every point is covered by exactly one $K$-subspace from a family of such subspaces $\mathcal{H}$, such that every $H \in \mathcal{H}$ is $\cong W$ as a $K$-space. Is it the case that $\mathcal{H}$ is a spread, i.e., is $V$ the direct sum of every pair of distinct members of $\mathcal{S}$.?!*

Note that the answer is clearly in the affirmative if the projective space being considered is finite-dimensional.

We now turn to the Bruck-Bose model of a spread: it is closely related to the projective version of André's defintion 12.4.12 above, but it enables the *projective plane* associated with a translation plane to be viewed as an incidence substructure of a projective space.

If $\mathcal{S}$ is a spread of $K$-subspaces, of a vector space $V$ over a skewfield $K$, then the affine translation plane $\Pi_{\mathcal{S}}$ has $V$ as its points and the lines of $\Pi_{\mathcal{S}}$ are the additive cosets of the components of $\mathcal{S}$. Thus the lines of the translation planes are the set of all the affine subspaces of $AG(V, K)$ that are parallel to the members of $\mathcal{S}$. Thus in $AG(V, K)^+$ the subspaces of $AG(V, K)$ that are the lines of the translation plane $\Pi_{\mathcal{S}}$ have as their closure the set of subspaces

$$(\mathcal{S}) := \{(S) \mid S \in \mathcal{S}\},$$

on the hyperplane at infinity $(V)$.

But each $(S) \in (\mathcal{S})$ may also be regarded as the point at infinity of the lines of $\Pi_{\mathcal{S}}$ that are parallel to $(S)$, and $(\mathcal{S})$ as the line at infinity, c.f., exercise 2.1.5. Thus we have established:

**Theorem 2.2.4 (Embedding Translation Planes in Projective Spaces.)** *Let $V$ be a vector space over a skewfield $K$ and $\mathcal{S}$ a spread of $K$-subspace of $V$. Then the projective closure of the translation plane $\Pi_{\mathcal{S}}$, with pointset $V$ and lines the cosets of $S \in \mathcal{S}$, is just the projective closure $\Pi^+$ of $\Pi_{\mathcal{S}}$ in $AG(V, K)^+$, when the points and lines of $\Pi_{\mathcal{S}}$ are regarded as affine subspaces of $AG(V, K)$.*

*More explictly, the hyperplane at infinity of $\Pi_{\mathcal{S}}$ in $\Pi^+$ is the subspace $(V)$, a 'copy' of $V$, associated with the projective space $PG((V), K) \cong PG(V, K)$; the infinite points are the members $(S) \in (\mathcal{S})$, the finite points are members of $V$ and the closure of the line $c + S$ is $(c + S) \cup S$.*

Since $AG(V,K)^+$ is isomorphic to a projective space $PG(V^+,K)$, where $V^+$ as a hyperplane, the theorem implies that *any translation plane associated with a spread is a subincidence structure of a projective space*; here a SUBINCIDENCE STRUCTURE $\mathcal{J}$ of a projective space $\mathcal{P}$ means that points and lines of $\mathcal{J}$ are selected from $\mathcal{P}$, viewed as a lattice, and incidence is containment (treated symmetrically). More explicitly

**Corollary 2.2.5** *Every projective translation plane $\Pi_{\mathcal{S}}$ is isomorphic to an incidence substructure of $PG(V^{\cdot}K)$, such that the affine points of $\Pi_{\mathcal{S}}$ are the points of the affine space $PG(V^+,K)^H$, $H$ a hyperplane is the line at infinity, the points at infinity are the components of a projective spread $\mathcal{S}_0 \cong \mathcal{S}$ in $H$, and all the other lines are the subspaces meeting $H$ in a member of $\mathcal{S}$.*

We summarize what we have done. Any spread $(V,\mathcal{S})$ defines a translation plane $\Pi$ in $AG(V,K)$ whose lines are the cosets of the members of $\mathcal{S}$. The projective closure $\Pi^+$ of $\Pi$ lies in the projective closure space $AG(V,K)^+$, the closure of $AG(V,K)$, and the line at infinity $H_\infty$ of $\Pi^+$ is the hyperplane at infinity of $AG(V,K)$; $H_\infty$ has a copy $\mathcal{S}_0$ of $\mathcal{S}$ such that all the lines parallel to $S \in \mathcal{S}$ have as there slope the corresponding $S_0 \in \mathcal{S}_0$. Hence, since every translation plane arises from a spread we conclude that *every translation plane is a subincidence structure of a projective space.*

We have seen that there is a natural isomorphism between the closure of affine spaces $AG(V,K)^+$ and the associated projective space lattice $PG(V^+,K)$, $V^+/V \cong K$, based on homogeneous coordinates. Thus theorem 2.2.4 above, that embeds an affine plane $\pi$ into its projective closure $AG(V,K)^+$, may be used to define a generic embedding of a projective translation plane in $PG(V^+,K)$ in terms of a projective spread $\mathcal{S}$ in $PG(V,K)$ that defines the plane $\pi$. This is the Bruck-Bose model, and it follows immediately from theorem 2.2.4.

**Theorem 2.2.6 (The Bruck-Bose Construction.)** *Let $\mathcal{S}$ be a projective spread in $\Sigma \cong PG(W,K)$ where $W$ is a $K$-vector space. Embed $PG(W,K)$ in a projective space $\Sigma^+$ so that $PG(W,K)$ is a hyperplane of $\Sigma^+$.*
*Define the incidence structure, defined by inclusion, whose point-set $\mathcal{P}$ is the set of projective points $\mathcal{P} := \Sigma^+ \setminus \Sigma$ and whose line-set $\mathcal{L}$ includes the hyperplane $\Sigma$, the 'infinite line', and the other members of $\mathcal{L}$, the 'fiinte lines', are the projective subspaces of $\Sigma^+$ that contain some component of $\mathcal{S}$ as a hyperplane.*

*Then the incidence structure with points* $(\mathcal{P})$*, and lines* $(\mathcal{L})$ *and with incidence defined by inclusion is a projective translation plane* $\pi$*. The translation axis is* $\Sigma$ *and* $\pi$ *is isomorphic to the affine translation plane on the ambient vector space* $V$ *of* $\mathcal{S}$*, whose lines are the cosets of the members of* $\mathcal{S}$*.*

*The isomorphism may be chosen so that the lines parallel to* $S \in \mathcal{S}$ *maps to the point* $S \in \mathcal{S}$*, i.e. itself when regarded as a point on the 'line'* $\Sigma \in \mathcal{L}$*.*
$\pi$

The above theorem, due to Bruck and Bose, may be regarded as the projective version of André's fundamental theorem of translation plane. Although, the original Bruck-Bose version considered only finite dimensional projective spaces, it was their intent to represent a translation plane projectively and within a projective space. It will become apparent that this viewpoint is extremely useful when considering construction processes within projective planes. Moreover, objects which might be considered "geometric" in some sense might be more conveniently visualized within a projective space as opposed to within a vector space where the projective line is essentially missing. For example, the notion of duality cannot easily be expressed using vector space spreads whereas a dual translation plane has an elegant representation using the projective space projective spreads.